Spectral Integration and Two-Point Boundary Value Problems

1. Mathematical Derivation of the Spectral Integration Method

The core idea is to transform the differential equation into an integral equation, and then solve the integral equation in the spectral (Chebyshev coefficient) space, leveraging the stability of the integration operator.

1.1 Problem Statement

We aim to solve the constant-coefficient BVP on $x \in [-1, 1]$:

$$u''(x) + \mu u'(x) + \nu u(x) = f(x)$$

with Dirichlet boundary conditions:

$$u(-1) = \alpha, \quad u(1) = \beta$$

1.2 Chebyshev Expansion and Integration Operator ($\mathcal{I}$)

The solution $u(x)$, its derivatives, and the forcing function $f(x)$ are expanded in truncated Chebyshev series of degree N:

$$u(x) = \sum_{k=0}^{N} u_k T_k(x), \quad f(x) = \sum_{k=0}^{N} f_k T_k(x)$$

The Spectral Integration Operator $\mathcal{I}$ maps the coefficients $\mathbf{a}$ of a function $f(x)$ to the coefficients $\mathbf{d}$ of its indefinite integral $g(x) = \int_{-1}^{x} f(t)\,dt$. If $g(x) = \sum d_k T_k(x)$, the relationship between $\mathbf{d}$ and $\mathbf{a}$ (coefficients of f) is defined as follows:

- For $k \geq 1$:

$$d_k = \frac{1}{2k}(a_{k-1} - a_{k+1})$$

(Conventionally, $a_{-1} = a_1$ and $a_k = 0$ for $k > N$.)

- For $k = 0$:
  To enforce the boundary condition $g(-1) = 0$ (since $T_k(-1) = (-1)^k$):

$$g(-1) = \sum_{k=0}^{\infty} d_k T_k(-1) = d_0 - d_1 + d_2 - d_3 + \cdots = 0$$

  Thus, the d_0 coefficient is:

$$d_0 = \sum_{k=1}^{\infty} (-1)^{k+1} d_k$$

1.3 Integral Equation and Linear System

Instead of solving for u, we solve for the coefficients $\mathbf{a}$ of the second derivative $\sigma(x) = u''(x)$.

Express u and u' in terms of σ:

$$u'(x) = \int_{-1}^{x} \sigma(t) dt + C_1$$

$$u(x) = \int_{-1}^{x} \int_{-1}^{t} \sigma(\tau) d\tau dt + C_1 x + C_0$$

where $C_1 = u'(-1)$ and C_0 are the integration constants.

Linear System (Coefficient Matching):

Substitute the coefficient forms of u'', u', u into the ODE and match the $T_k(x)$ coefficients.

- $\boldsymbol{k = 0}$**:**

$$a_0 + \mu C_1 + \nu C_0 = f_0.$$

- $\boldsymbol{k = 1}$**:** Expanding the recurrence relations gives:

$$a_1 + \frac{\mu}{2}(a_0 - a_2) + \frac{\nu}{8}\big(8C_1 + a_1 - a_3\big) = f_1.$$

- $\boldsymbol{k \geq 2}$**:** The integral formulation yields the pentadiagonal relation:

$$a_k + \frac{\mu}{2k}(a_{k-1} - a_{k+1}) + \frac{\nu}{2k}\left(\frac{a_{k-2} - a_k}{2(k-1)} - \frac{a_k - a_{k+2}}{2(k+1)}\right) = f_k.$$

This leads to a system of $N+1$ equations for $N+3$ unknowns: $\mathbf{a} = [a_0, \ldots, a_N]^T$, C_0, and C_1.

Boundary Conditions (The two extra equations):

The final two equations needed to close the system are provided by the boundary conditions, using the u_k coefficients derived from $\mathbf{a}, C_0, C_1$:

$$u(-1) = \alpha \quad \iff \quad \sum_{k=0}^{N} u_k (-1)^k = \alpha$$

$$u(1) = \beta \quad \iff \quad \sum_{k=0}^{N} u_k (1)^k = \beta$$

The resulting linear system $A\mathbf{x} = \mathbf{b}$ is sparse and can be solved efficiently, typically in $O(N \log N)$ total time.

2. Stability Analysis: $O(N^2)$ vs. $O(1)$

The stability of the method is governed by the condition number $\kappa(\mathcal{L})$ of the linear operator $\mathcal{L}$, which measures how much input error is amplified.

2.1 Spectral Differentiation ($\mathcal{D}_N$): Ill-Conditioning

The differentiation operator, defined by $b_k = \sum_{p=k+1}^{N} p\, a_p$, significantly amplifies high-frequency errors.

Amplification Factor Derivation:

Assume a tiny uniform error ϵ in the coefficients a: $\|\Delta a\|_\infty = \epsilon$. The error in the differentiated coefficients Δb_k is:

$$\Delta b_k = \epsilon \sum_{p=k+1}^{N} p.$$

The maximum error occurs at $k = 0$:

$$\|\Delta b\|_\infty = |\Delta b_0| = \epsilon \sum_{p=1}^{N} p = \epsilon\, \frac{N(N+1)}{2}.$$

The error amplification factor (a condition-number style estimate in the ℓ_∞ norm) is:

$$\frac{\|\Delta b\|_\infty}{\|\Delta a\|_\infty} = \frac{N(N+1)}{2} = O(N^2)$$

This quadratic growth leads to severe numerical instability as N increases.

2.2 Spectral Integration ($\mathcal{I}_N$): Well-Conditioning

The integration operator dampens high-frequency components because its recurrence relation ($d_k = \frac{1}{2k}(a_{k-1} - a_{k+1})$) involves division by the index k.

Rigorous analysis shows that the condition number of the spectral integration matrix is bounded by a small constant, independent of N:

$$\kappa(\mathcal{I}_N) \sim O(1)$$

The bound is $\kappa(\mathcal{I}_N) < 2.4$, confirming that the integration method is highly stable regardless of the number of discretization points N.

3. Numerical Experiment:

3.1 Experiment Setup

We numerically compare the conditioning of:

- the **Chebyshev spectral differentiation matrix** D_N (first derivative), and
- the **spectral integration operator** S_N (constructed as the bounded inverse of D_N on the subspace orthogonal to constants),

on Chebyshev–Lobatto grids of increasing order N.

The differentiation matrix is constructed following Trefethen, while the integration operator is implemented via a Moore–Penrose pseudoinverse after removing the constant nullspace, consistent with Greengard's formulation.

All condition numbers are computed in the 2-norm using NumPy's `np.linalg.cond`.

3.2 Results

The following table summarizes the condition numbers obtained from the accompanying Jupyter notebook (`f.ipynb`):

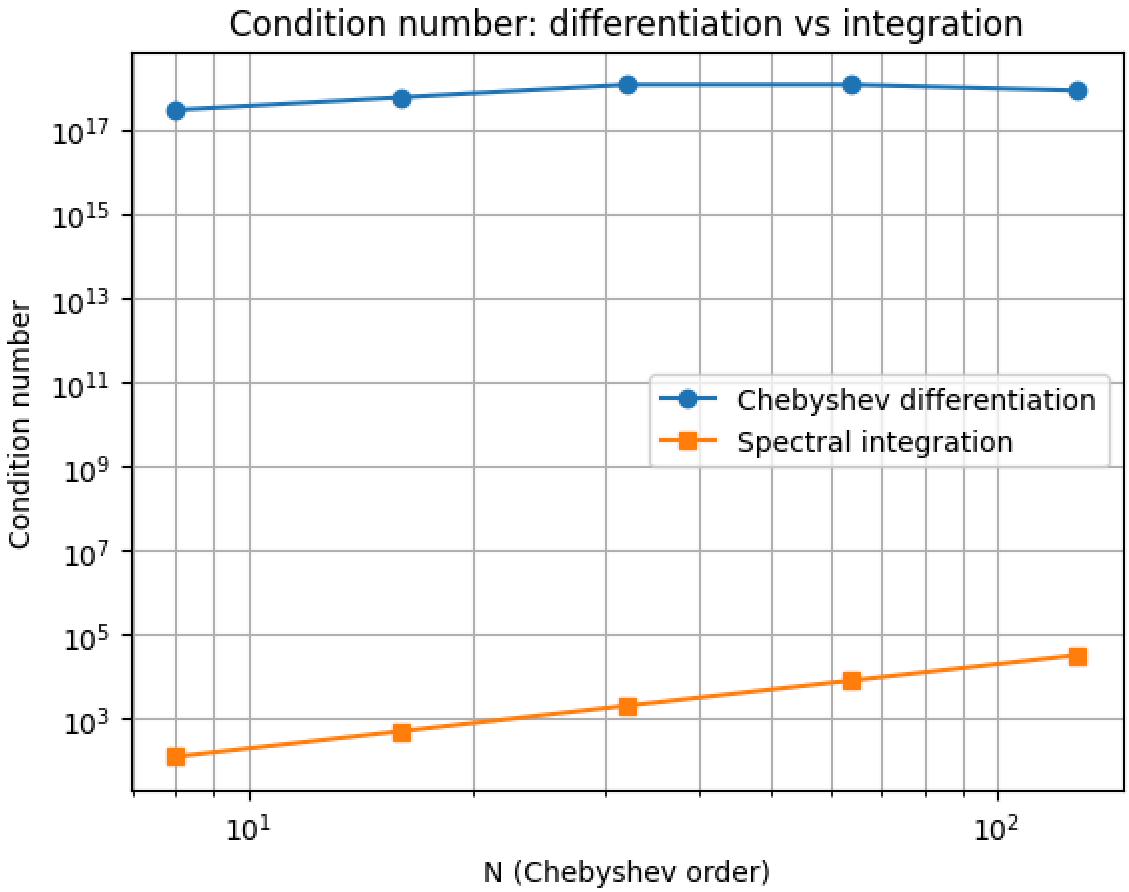

Figure 1: Condition numbers of the Chebyshev spectral differentiation matrix and the spectral integration operator as functions of the polynomial order N.

A log–log plot of condition number versus N shows that:

- the differentiation matrix is already nearly singular at modest N, with condition numbers close to machine precision limits;
- the spectral integration operator remains well-conditioned, with only mild growth as N increases.

3.3 Interpretation

These numerical results directly support the theoretical analysis:

- Spectral differentiation amplifies high-frequency modes proportionally to the mode index, leading to $O(N^2)$ error amplification and extreme ill-conditioning.
- Spectral integration suppresses high-frequency modes through division by the mode index, resulting in a bounded operator at the continuous level and a numerically stable discrete operator.

This experiment provides concrete numerical evidence for Greengard's central claim: **reformulating boundary value problems in terms of spectral integration, rather than spectral differentiation, dramatically improves numerical stability.**

4. Numerical Experiment: Solving $u''(x) = \cos x$ via Spectral Integration

4.1 Problem Description

We consider the second-order boundary value problem

$$\begin{cases} u''(x) = \cos x, \quad x \in [-1, 1], \\ u(-1) = 0, \quad u(1) = 0. \end{cases}$$

4.2 Exact Solution

The exact solution of the problem is

$$u_{\text{exact}}(x) = -\cos x + Ax + B,$$

where the constants A and B are chosen so that $u(-1) = u(1) = 0$.

4.3 Numerical Results

The numerical solution is computed on Chebyshev–Lobatto points using the spectral integration method implemented in the accompanying Jupyter notebook.

The maximum error

$$||u - u_{\text{exact}}||_{\infty}$$

Representative results are:

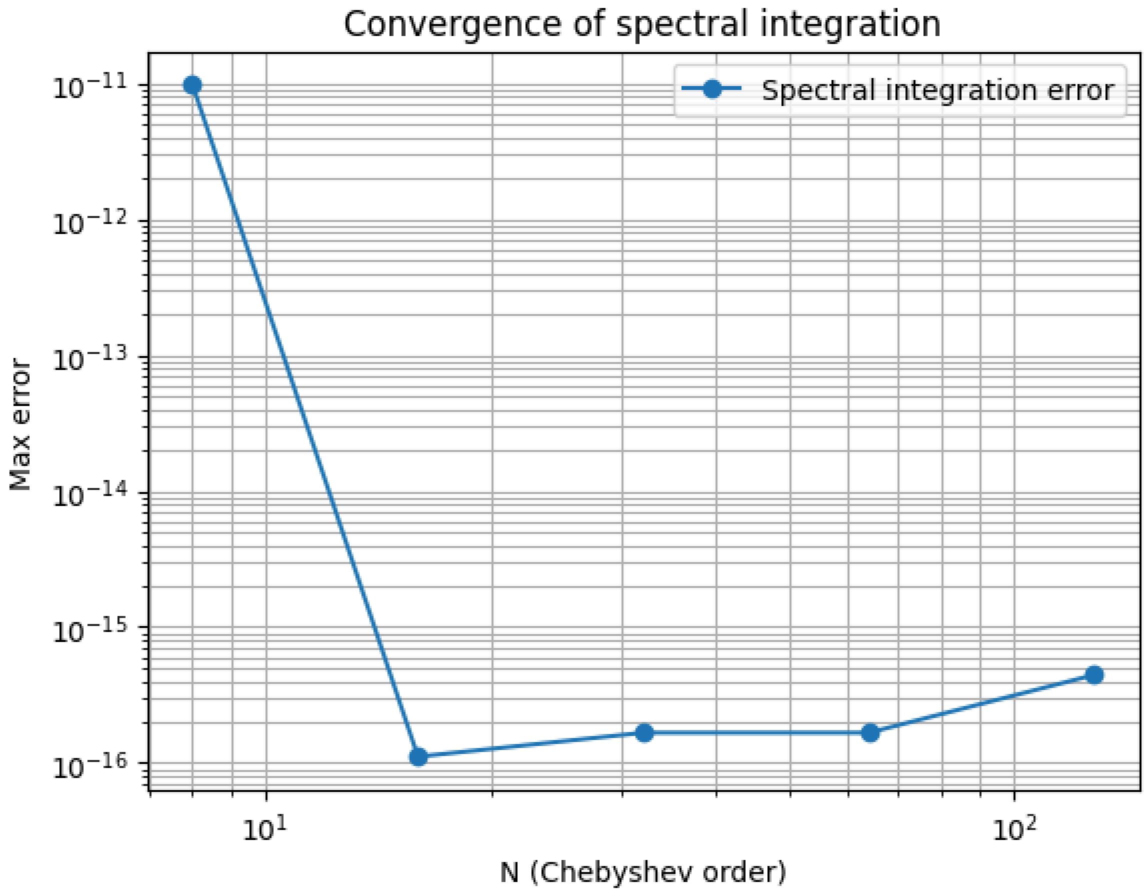

Figure 2: Numerical solution of $u''(x) = \cos x$ using spectral integration and the corresponding pointwise error.

The error decreases exponentially until reaching machine precision, demonstrating spectral convergence.

4.4 Interpretation

This experiment confirms that spectral integration yields highly accurate solutions for smooth right-hand sides while avoiding the severe ill-conditioning associated with spectral differentiation. Even for moderate values of N, the method achieves near–machine-precision accuracy, illustrating the effectiveness of Greengard's integration-based formulation for second-order boundary value problems.