9. Given the following set of data

$$\{f_0 = f(-1) = 1,\ f_1 = f'(-1) = 1,\ f_2 = f'(1) = 2,\ f_3 = f(2) = 1\},$$

prove that the Hermite-Birkoff interpolating polynomial $H_3$ does not exist for them.
[*Solution* : letting $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, one must check that the matrix of the linear system $H_3(x_i) = f_i$ for $i = 0, \ldots, 3$ is singular.]

Let $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$$\Rightarrow \begin{cases} H_3(-1) = -a_3 + a_2 - a_1 + a_0 = 1 \\ H_3'(-1) = 3a_3 - 2a_2 + a_1 + 0 = 1 \\ H_3'(1) = 3a_3 + 2a_2 + a_1 + 0 = 2 \\ H_3(1) = 8a_3 + 4a_2 + 2a_1 + a_0 = 1 \end{cases}$$

matrix $\Rightarrow$
$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -2 & 3 & 4 \\ 0 & 5 & -2 & 3 & 5 \\ 0 & 4 & -2 & 3 & 3 \end{array}\right) = \left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -2 & 3 & 4 \\ 0 & 0 & 8 & -12 & 15 \\ 0 & 0 & 6 & 9 & 13 \end{array}\right)$$

$\because 2a_1 - 3a_0 = \frac{15}{4} \neq \frac{13}{3} \Rightarrow a_1, a_0$ not exist.

$\therefore H_3$ does not exists.

and integrate by parts twice.]

12. Let $f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$; then, consider the following rational approximation

$$r(x) = \frac{a_0 + a_2 x^2 + a_4 x^4}{1 + b_2 x^2}, \tag{8.75}$$

called the *Padé approximation*. Determine the coefficients of $r$ in such a way that

$$f(x) - r(x) = \gamma_8 x^8 + \gamma_{10} x^{10} + \ldots.$$

[*Solution*: $a_0 = 1$, $a_2 = -7/15$, $a_4 = 1/40$, $b_2 = 1/30$.]

$$f(x) - r(x) = \cos x - \frac{a_0 + a_2 x^2 + a_4 x^4}{1 + b_2 x^2} = \gamma_8 x^8 + \gamma_{10} x^{10} \ldots$$

$$\Rightarrow (\cos x)(1 + b_2 x^2) - (a_0 + a_2 x^2 + a_4 x^4)$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots\right) + \left(b_2 x^2 - \frac{b_2}{2!} x^4 + \frac{b_2}{4!} x^6 - \frac{b_2}{6!} x^8 \ldots\right) - (a_0 + a_2 x^2 + a_4 x^4)$$

$$\Rightarrow \begin{cases} a_0 = 1 \\ a_2 = -\frac{1}{2} + b_2 \\ a_4 = \frac{1}{4!} - \frac{1}{2} b_2 \\ \frac{b_2}{4!} = \frac{1}{6!} \end{cases} \Rightarrow \begin{array}{l} a_0 = 1 \\ b_2 = \frac{1}{30} \\ a_2 = -\frac{7}{15} \\ a_4 = \frac{1}{40} \end{array}$$

$$\therefore r(x) = \frac{1 - \frac{7}{15} x^2 + \frac{1}{40} x^4}{1 + \frac{1}{30} x^2}$$