5. Prove the estimate (12.23).

[*Hint*: for each internal node x_j, $j = 1, \ldots, n-1$, integrate by parts (12.21) to get

$$\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}\left[\int_{x_j-h}^{x_j} u''(t)(x_j - h - t)^2\,dt - \int_{x_j}^{x_j+h} u''(t)(x_j + h - t)^2\,dt\right].$$

Then, pass to the squares and sum $\tau_h(x_j)^2$ for $j = 1, \ldots, n-1$. On noting that $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, for any real numbers a, b, c, and applying the Cauchy-Schwarz inequality yields the desired result.]

from 12.21

$$T_h(x_j) = \frac{1}{h^2}\left(\int_{x_j}^{x_j+h} (u'''(t) - u'''(x_j))\frac{(x_j+h-t)^2}{2}\,dt - \int_{x_j-h}^{x_j} (u'''(t) - u'''(x_j))\frac{(x_j-h-t)^2}{2}\,dt\right.$$

$$\overset{IBP}{=} -u''(x_j) - \frac{1}{h^2}\left(\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)^2\,dt - \int_{x_j}^{x_j+h} u''(t)(x_j+h-t)^2\,dt\right.$$

$$\overset{\text{Let}}{=} a_j + b_j + c_j, \quad a_j = -u''(x_j), \; b_j = \frac{1}{h^2}\int_{x_j-h}^{x_j} u'''(t)(x_j-h-t)^2\,dt$$

$$c_j = \frac{-1}{h^2}\int_{x_j}^{x_j+h} u'''(t)(x_j+h-t)^2\,dt$$

$$b_j^2 \le \int_{x_j-h}^{x_j} (u'''(t))^2\,dt \cdot \int_{x_j-h}^{x_j} (x-h-t)^4\,dt = \frac{h}{5}\int_{x_j-h}^{x_j} (u'''(t))^2\,dt$$

$$c_j^2 \le \frac{h}{5}\int_{x_j}^{x_j+h} (u'''(t))^2\,dt$$

$$\Rightarrow \|T_h\|_h^2 = h\sum_{j=1}^{n-1} |T_h(x_j)|^2 = h\sum_{j=1}^{n-1} (a_j+b_j+c_j)^2$$

$$\le h\sum_{j=1}^{n-1} 3(a_j+b_j+c_j) = 3\left(h\sum_{j=1}^{n-1} a_j + \frac{2h^2}{5}\|u''\|_{L(0,1)}^2\right)$$

$$\le 3\left(\|u''\|_h^2 + \|u''\|_{L(0,1)}^2\right) \overset{-u''=f}{=} 3\left(\|u''\|_h^2 + \|f\|_{L(0,1)}^2\right)$$

7. Let $g = 1$ and prove that $T_h g(x_j) = \frac{1}{2} x_j(1 - x_j)$.
[*Solution*: use the definition (12.25) with $g(x_k) = 1$, $k = 1, \ldots, n - 1$ and recall that $G^k(x_j) = hG(x_j, x_k)$ from the exercise above. Then

$$T_h g(x_j) = h \left[\sum_{k=1}^{j} x_k(1 - x_j) + \sum_{k=j+1}^{n-1} x_j(1 - x_k) \right]$$

from which, after straightforward computations, one gets the desired result.]

$g(x_k) = 1$

$$T_h g(x_j) = \sum_{k=1}^{n-1} g(x_k)\, G^k = \sum_{k=1}^{n-1} h\, g(x_k)\, G(x_j, x_k) = h \sum_{k=1}^{n-1} G(x_j, x_k)$$

$$= h\left(\sum_{k=1}^{j} G(x_j, x_k) + \sum_{k=j+1}^{n-1} G(x_j, x_k) \right)$$

$$= h\left(\sum_{k=1}^{j} x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k) \right), \quad G(x_j, x_k) = \begin{cases} x_k(1-x_j), & x_k \le x_j \\ x_j(1-x_k) & x_k > x_j \end{cases}$$

$$= h\left(\sum_{k=1}^{j} kh(1-hj) + \sum_{k=j+1}^{n-1} hj(1-kh) \right)$$

$$= h\left(j\left(\frac{1+j}{2}\right) h(1-hj) + hj\left((n-j-1) - h\left(\frac{n+j}{2}\right)(n-j-1) \right) \right)$$

$$= h^2 j \left(\frac{1}{2}(1+j)(1-hj) + n-j-1 - \frac{1}{2}(n+j)(n-j-1) \right)$$

$$= \frac{h^2 j}{2} \left(\cancel{1} + j - \cancel{hj} - \cancel{hj^2} + 2n - 2j - \cancel{2} - n^2 h \ \cancel{+nh} + \cancel{j^2 h} + \cancel{jh} \right), \quad n = \frac{1}{h}$$

$$= \frac{h^2 j}{2}\left(j + \frac{1}{h} \right) = \frac{hj}{2}(1 + hj) = \frac{1}{2} x_j (1 + x_j)$$

8. Prove Young's inequality (12.40).

$\because \left(\sqrt{\varepsilon}a - \frac{1}{2\sqrt{\varepsilon}}b\right)^2 \geq 0, \quad a, b \in \mathbb{R}, \quad \varepsilon > 0$

$\Rightarrow \varepsilon a^2 - ab + \frac{b^2}{4\varepsilon} \geq 0$

$\Rightarrow ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$

9. Show that $\|v_h\|_h \leq \|v_h\|_{h,\infty} \; \forall v_h \in V_h$.

$\because \|V_h\|_h^2 = h\sum_{k=1}^{n-1} V_h^2(x_k) \leq h\sum_{k=1}^{n-1} \max_{1 \leq k \leq n-1} V_h^2(x_k) = h\sum_{k=1}^{n-1} \|V_h\|_{h,\infty}^2$

$= \frac{1}{n}(n-1)\|V_h\|_{h,\infty}^2 \leq \|V_h\|_{h,\infty}^2$

$\therefore \|V_h\|_h^2 \leq \|V_h\|_{h,\infty}, \; \forall V_h \in V_h$

11. Discretize the fourth-order differential operator $Lu(x) = -u^{(iv)}(x)$ using centered finite differences.
[*Solution*: apply twice the second order centered finite difference operator L_h defined in (12.9).]

$$L_h u_j = -\frac{1}{h^2}(u_{j+1} - 2u_j + u_{j-1})$$

$$\Rightarrow \quad L_h(L_h u_h) = -\frac{1}{h^2}(L_h u_{j+1} - 2L_h u_j + L_h u_{j-1})$$

$$= \frac{1}{h^4}(u_{j+2} - 2u_{j+1} + u_j + u_j - 2u_{j-1} + u_{j-2} - 2u_{j+1} + 4u_j - 2u_{j-1})$$

$$= \frac{1}{h^4}(u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2})$$