1. (5)

let $h=\frac{2}{n}, N=\frac{n}{2}$

$\Rightarrow x_0=-1=-hN,\ x_1=-1+h=-h(N-1),\ x_2=-h(N-2)\ \cdots,\ x_N=-h(N-N)=0$

$\cdots, x_n=-h(N-n)=hN=1$

$\Rightarrow W_{n+1}(x)=\prod_{i=0}^{n}(x-x_i)=\prod_{i=0}^{n}(x+(N-i)h)$

take $x=rh$ with $N-1<r<N$

$\Rightarrow W_{n+1}(rh)=\prod_{i=0}^{n}(rh+(N-i)h)=h^{n-1}\prod_{i=0}^{n-2}(r+(N-i))\cdot(x-x_{n-1})(x-x_n)$

$\because \prod_{i=0}^{n-2}((N-1)+N-i)\le\prod_{i=0}^{n-2}(r+(N-i))\le\prod_{i=0}^{n-2}(N+N-i)$

$\Rightarrow \prod_{i=0}^{n-2}(n-1-i)\le\prod_{i=0}^{n-2}(r+(N-i))\le\prod_{i=0}^{n-2}(n-i)$

$\Rightarrow (n-1)!\le\prod_{i=0}^{n-2}(r+(N-i))\le n!$

$\therefore h^{n-1}(n-1)!\,|(x-x_{n-1})(x-x_n)|\le|W_{n+1}(x)|\le n!\,h^{n-1}|(x-x_{n-1})(x-x_n)|$

(6)

by (5) $W_{n+1}(x)=\prod_{i=0}^{n}(x+(N-i)h)$ , $h=\frac{2}{n}$ , $N=\frac{n}{2}$

$$\Rightarrow \left|\frac{W_{n+1}(x+h)}{W_{n+1}(x)}\right| = \left|\frac{\prod_{i=0}^{n}(x+(N-i+1)h)}{\prod_{i=0}^{n}(x+(N-i)h)}\right| = \left|\frac{(x+(N+1)h)\cancel{(x+Nh)}\cdots\cancel{(x+(-N+1)h)}}{\cancel{(x+Nh)}\cancel{(x+(N-1)h)}\cdots(x-Nh)}\right|$$

$$= \left|\frac{x+(N+1)h}{x-Nh}\right| = \left|\frac{x+1+\overset{>1}{h}}{x-1}\right|_{\in(0,1)} > 1 \ , \ x\in(0,x_{n-1})$$

$\Rightarrow |W_{n+1}(x+h)|>|W_{n+1}(x)|$ , $x\in(0,x_{n-1})$ $\Rightarrow$ increasing

Notice that $W_{n+1}$ is even function

$\therefore$ $|W_{n+1}(x-h)|>|W_{n+1}(x)|$ , $x\in(x_1,0)$ $\Rightarrow$ decreasing

Thus maximum not in $(x_1,x_{n-1})$

$\therefore$ $|W_{n+1}|$ is maximum if $x\in(x_{n-1},x_n)$

(8)

Consider Taylor polynomial $T_n(x)=\sum_{i=0}^{n}\frac{f^{(i)}(x_0)}{i!}(x-x_0)^i$

$$T_n^{(k)}(x)=\sum_{i=k}^{n}\frac{f^{(i)}(x_0)}{i!}\,i(i-1)(i-2)\cdots(i-k+1)(x-x_0)^{i-k} \ , \ k\le n$$

$$\Rightarrow T_n^{(k)}(x_0)=f^{(k)}(x_0) \Rightarrow T_n(x)=(H f)(x)=\sum_{i=0}^{n}\frac{f^{(i)}(x_0)}{i!}(x-x_0)^i$$

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Since $W_{n+1}(x) = \frac{1}{2^{n-1}}(x^2-1)U_{n-1}(x)$ and $w_i = \frac{1}{W'_{n+1}(x_i)}$

$$\Rightarrow W'_{n+1}(x) = \frac{1}{2^n} x U_{n-1}(x) + \frac{1}{2^{n-1}}(x^2-1)U'_{n-1}(x),$$

$$= \frac{1}{2^{n-2}} x U_{n-1}(x) + \frac{1}{2^{n-1}}(x^2-1)\left(\frac{n\sin\theta\cos(n\theta) - \sin(n\theta)\cos\theta}{(\sin\theta)^2} \cdot \frac{-1}{\sqrt{1-x^2}}\right), \quad \cos\theta = x$$

$$= \frac{x}{2^{n-2}} U_{n-1}(x) + \frac{1}{2^{n-1}}\sqrt{1-x^2}\left(\frac{n\cos(n\theta)}{\sin\theta} - \frac{\sin(n\theta)\cos\theta}{\sin^2\theta}\right)$$

$x_i = \cos(\frac{i}{n}\pi)$

$$\Rightarrow W'_{n+1}(x_i) = \frac{\cos(\frac{i}{n}\pi)}{2^{n-2}} \cdot \frac{\sin(i\pi)}{\sin(\frac{i}{n}\pi)} + \frac{\sin(\frac{i}{n}\pi)}{2^{n-1}}\left(\frac{n\cos(i\pi)}{\sin(\frac{i}{n}\pi)} - \frac{\sin(i\pi)\cos(\frac{i}{n}\pi)}{\sin^2(\frac{i}{n}\pi)}\right)$$

$$\Rightarrow w_i = \frac{1}{W'_{n+1}(x_i)} = \begin{cases} \frac{2^{n-1}}{n}(-1)^i, & \text{for } i = 1,2,3\cdots n-1 \\ \dfrac{1}{\frac{n}{2^{n-2}} + \frac{n}{2^{n-1}} - \frac{n}{2^{n-1}}}, & i = 0 \\ \dfrac{1}{\left(\frac{n}{2^{n-2}} + \frac{n}{2^{n-1}} - \frac{n}{2^{n-1}}\right)(-1)^n}, & i = n \end{cases}$$

$$\frac{\sin n\pi}{\sin\pi} = \frac{n\cos n\pi}{\cos\pi} = -n(-1)^n$$

Rescaling with $\frac{2^{n-1}}{n}$

$$\Rightarrow \frac{n}{2^{n-1}} \cdot w_i = \begin{cases} (-1)^i, & i = 1,2\cdots n-1 \\ \frac{1}{2}, & i = 0 \\ (\frac{1}{2})(-1)^n, & i = n \end{cases}$$