7. Prove that the *gamma function*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad z \in \mathbb{C}, \quad \mathrm{Re} z > 0,$$

is the solution of the difference equation $\Gamma(z+1) = z\Gamma(z)$
[*Hint*: integrate by parts.]

$$t^z \qquad e^{-t}$$

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z dt \qquad z t^{z-1} \qquad -e^{-t}$$

$$= -e^{-t} t^z \Big)_0^\infty + \int_0^\infty z t^{z-1} e^{-t} dt$$

$$= z\,\Gamma(z)$$

9. Consider the following family of one-step methods depending on the real parameter α

$$u_{n+1} = u_n + h[(1-\frac{\alpha}{2})f(x_n, u_n) + \frac{\alpha}{2}f(x_{n+1}, u_{n+1})].$$

Study their consistency as a function of α; then, take $\alpha = 1$ and use the corresponding method to solve the Cauchy problem

$$\begin{cases} y'(x) = -10y(x), & x > 0, \\ y(0) = 1. \end{cases}$$

Determine the values of h in correspondance of which the method is absolutely stable.
[*Solution*: the family of methods is consistent for any value of α. The method of highest order (equal to two) is obtained for $\alpha = 1$ and coincides with the Crank-Nicolson method.]

Taylor expension: $y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2} y''(x_n) \cdots$

$$= y(x_n) + h f(x_n) + \frac{h^2}{2}(f_x + f f_y) + O(h^3)$$

$$\Rightarrow f(x_{n+1}, y_{n+1}) = f(x_n, y_n) + h f(x_n, y_n) + \frac{h^2}{2}(f_x + f f_y)(x_n, y_n) + O(h^3)$$

$$\Rightarrow \tau_{n+1} = h f(x_n) + \frac{h^2}{2}(f_x + f f_y)(x_n, y_n) - h[(1-\frac{\alpha}{2})f(x_n) + \frac{\alpha}{2} f(x_{n+1}, y_{n+1})] + O(h^3)$$

$$= \frac{\alpha}{2} h f(x_n) + \frac{h^2}{2}(f_x + f f_y)(x_n, y_n) - \frac{\alpha}{2} h (f(x_n) + h f(x_n)) + O(h^3)$$

$$= \frac{h^2}{2}(1-\alpha) f(x_n) + O(h^3) = \begin{cases} O(h^3) & \alpha = 1 \\ O(h^2) & \text{o.w.} \end{cases}$$

take $\alpha = 1$

$$\Rightarrow u_{n+1} = u_n + \frac{h}{2}(f(x_n, u_n) + f(x_{n+1}, u_{n+1})) = u_n + \frac{h}{2}(-10u_n - 10u_{n+1})$$

$$\Rightarrow u_{n+1} = \frac{1-5h}{1+5h} u_n \Rightarrow \left|\frac{1-5h}{1+5h}\right| < 1 \Rightarrow h > 0$$

thus absolutely stable when $h > 0$