1. (5)

let $h=\frac{2}{n}, N=\frac{n}{2}$

$\Rightarrow x_0=-1=-hN, x_1=-1+h=-h(N-1), x_2=-h(N-2) \cdots, x_N=-h(N-N)=0$

$\cdots, x_n=-h(N-n)=hN=1$

$\Rightarrow W_{n+1}(x)=\prod_{i=0}^{n}(x-x_i)=\prod_{i=0}^{n}(x+(N-i)h)$

take $x=rh$ with $N-1<r<N$

$\Rightarrow W_{n+1}(rh)=\prod_{i=0}^{n}(rh+(N-i)h)=h^{n-1}\prod_{i=0}^{n-2}(r+(N-i))\cdot(x-x_{n-1})(x-x_n)$

$\because \prod_{i=0}^{n-2}((N-1)+N-i)\le\prod_{i=0}^{n-2}(r+(N-i))\le\prod_{i=0}^{n-2}(N+N-i)$

$\Rightarrow \prod_{i=0}^{n-2}(n-1-i)\le\prod_{i=0}^{n-2}(r+(N-i))\le\prod_{i=0}^{n-2}(n-i)$

$\Rightarrow \quad (n-1)! \le \prod_{i=0}^{n-2}(r+(N-i))\le n!$

$\therefore h^{n-1}(n-1)!\,|(x-x_{n-1})(x-x_n)|\le|W_{n+1}(x)|\le n!\,h^{n-1}|(x-x_{n-1})(x-x_n)|$

(6)

by (5) $W_{n+1}(x) = \prod_{i=0}^{n} (x + (N-i)h)$, $h = \frac{2}{n}$, $N = \frac{n}{2}$

$$\Rightarrow \left| \frac{W_{n+1}(x+h)}{W_{n+1}(x)} \right| = \left| \frac{\prod_{i=0}^{n} (x + (N-i+1)h)}{\prod_{i=0}^{n} (x + (N-i)h)} \right| = \left| \frac{(x+(N+1)h)\cancel{(x+Nh)}\cdots\cancel{(x+(-N+1)h)}}{\cancel{(x+Nh)}(x+\cancel{(N-1)h})\cdots(x-Nh)} \right|$$

$$= \left| \frac{x+(N+1)h}{x-Nh} \right| = \left| \frac{x+1+h}{\underbrace{x-1}_{\in(0,1)}} \right|^{>1} > 1 \quad , x \in (0, x_{n-1})$$

$\Rightarrow |W_{n+1}(x+h)| > |W_{n+1}(x)|$, $x \in (0, x_{n-1})$ $\Rightarrow$ increasing

Notice that W_{n+1} is even function

$\therefore$ $|W_{n+1}(x-h)| > |W_{n+1}(x)|$, $x \in (x_1, 0)$ $\Rightarrow$ decreasing

Thus maximum not in (x_1, x_{n-1})

$\therefore$ $|W_{n+1}|$ is maximum if $x \in (x_{n-1}, x_n)$

(8)

Consider Taylor polynomial $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i$

$$T^{(k)}_n(x) = \sum_{i=k}^{n} \frac{f^{(i)}(x_0)}{i!} i(i-1)(i-2)\cdots(i-k+1)(x-x_0)^{i-k} \quad , k \leq n$$

$$\Rightarrow T^{(k)}_n(x_0) = f^{(k)}(x_0) \Rightarrow T_n(x) = (H f)(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i$$

2.

Since $W_{n+1}(x) = \frac{1}{2^{n-1}}(x^2-1)U_{n-1}(x)$ and $w_i = \frac{1}{W'_{n+1}(x_i)}$

$$\Rightarrow \quad W'_{n+1}(x) = \frac{1}{2^n} x U_{n-1}(x) + \frac{1}{2^{n-1}}(x^2-1)U'_{n-1}(x),$$

$$= \frac{1}{2^{n-2}} x U_{n-1}(x) + \frac{1}{2^{n-1}}(x^2-1)\left(\frac{n\sin(\theta)\cos(n\theta) - \sin(n\theta)\cos\theta}{(\sin\theta)^2} \cdot \frac{-1}{\sqrt{1-x^2}}\right), \quad \cos\theta = x$$

$$= \frac{x}{2^{n-2}} U_{n-1}(x) + \frac{1}{2^{n-1}}\sqrt{1-x^2}\left(\frac{n\cos(n\theta)}{\sin\theta} - \frac{\sin(n\theta)\cos\theta}{\sin^2\theta}\right)$$

$x_i = \cos(\frac{i}{n}\pi)$

$$\Rightarrow \quad W'_{n+1}(x_i) = \frac{\cos(\frac{i}{n}\pi)}{2^{n-2}} \cdot \frac{\sin(i\pi)}{\sin(\frac{i}{n}\pi)} + \frac{\sin(\frac{i}{n}\pi)}{2^{n-1}}\left(\frac{n\cos(i\pi)}{\sin(\frac{i}{n}\pi)} - \frac{\sin(i\pi)\cos(\frac{i}{n}\pi)}{\sin^2(\frac{i}{n}\pi)}\right)$$

$$\Rightarrow \quad w_i = \frac{1}{W'_{n+1}(x_i)} = \begin{cases} \frac{2^{n-1}}{n}(-1)^i, & \text{for } i = 1,2,3\cdots n-1 \\ \dfrac{1}{\frac{n}{2^{n-2}} + \frac{n}{2^{n-1}} - \frac{n}{2^{n-1}}}, & i=0 \\ \dfrac{1}{\left(\frac{n}{2^{n-2}} + \frac{n}{2^{n-1}} - \frac{n}{2^{n-1}}\right)(-1)^n}, & i=n \end{cases}$$

$$\frac{\sin n\pi}{\sin\pi} = \frac{n\cos n\pi}{\cos\pi} = -n(-1)^n$$

Rescaling with $\frac{2^{n-1}}{n}$

$$\Rightarrow \quad \frac{n}{2^{n-1}} \cdot w_i = \begin{cases} (-1)^i, & i = 1,2\cdots n-1 \\ \frac{1}{2}, & i=0 \\ (\frac{1}{2})(-1)^n, & i=n \end{cases}$$