1. Prove that Heun's method has order 2 with respect to h.
[*Hint*: notice that $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h) = E_1 + E_2$, where

$$E_1 = \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

and

$$E_2 = \frac{h}{2}\{[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]\},$$

where E_1 is the error due to numerical integration with the trapezoidal method and E_2 can be bounded by the error due to using the forward Euler method.]

$$y_{n+1} = y_n + \frac{h}{2}\{f_n + f(t_{n+1}, y_n + hf_n)\}$$

$$\tau_{n+1} = \frac{y_{n+1} - y_n}{h} - \frac{1}{2}\{f_n + f(t_{n+1}, y_n + hf_n)\}$$ Truncation error

$$\Rightarrow h\tau_{n+1} = y_{n+1} - y_n - \frac{h}{2}\{f_n + f(t_{n+1}, y_n + hf_n)\}$$

$$= \underbrace{\int_{t_n}^{t_{n+1}} f(s, y_n)\,ds - \frac{h}{2}\{f_n + f(t_{n+1}, y_{n+1})\}}_{E_1} + \underbrace{\frac{h}{2}\{f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf_n)\}}_{E_2}$$

E_1 is $O(h^3)$ by trapezoidal rule

$\because$ $y_{n+1} = y_n + hf(t_n, y_n) + O(h^2)$ by forward Euler

$\Rightarrow f(t_{n+1}, y_{n+1}) = f(t_{n+1}, y_n + hf(t_n, y_n)) + O(h^2)$

$\therefore$ $E_2 = \frac{h}{2}\{f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n)) + O(h^2)\} \sim O(h^3)$

thus $\tau = \frac{E_1 + E_2}{h}$ is order 2

2. Prove that the Crank-Nicoloson method has order 2 with respect to h.
[*Solution*: using (9.12) we get, for a suitable ξ_n in (t_n, t_{n+1})

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] - \frac{h^3}{12} f''(\xi_n, y(\xi_n))$$

or, equivalently,

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] - \frac{h^2}{12} f''(\xi_n, y(\xi_n)). \qquad (11.90)$$

Therefore, relation (11.9) coincides with (11.90) up to an infinitesimal of order 2 with respect to h, provided that $f \in C^2(I)$.]

$$y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1})$$

$$\Rightarrow \tau_{n+1} = \frac{1}{h}\left[(y_{n+1} - y_n) - \frac{h}{2}(f_n + f_{n+1})\right]$$

$$= \frac{1}{h}\left[\int_{t_n}^{t_{n+1}} f(s)\,ds - \frac{h}{2}(f_n + f_{n+1})\right]$$

$$= \frac{1}{h}\left[\frac{h^3}{12} f''(\xi)\right], \quad \xi \in [t_n, t_{n+1}] \text{ by trapezoidal rule.}$$

$\therefore$ Crank-Nicoloson method has order 2.