1. Consider the boundary value problem (12.1)-(12.2) with $f(x) = 1/x$. Using (12.3) prove that $u(x) = -x\log(x)$. This shows that $u \in C^2(0,1)$ but $u(0)$ is not defined and u', u'' do not exist at $x = 0$ ($\Rightarrow$: if $f \in C^0(0,1)$, but not $f \in C^0([0,1])$, then u does not belong to $C^0([0,1])$).

$-u'' = f(x),\ 0<x<1$

$u(0) = u(1) = 0$

$f(x) = \frac{1}{x}$

$$u(x) = x\int_0^1 (1-s) f(s)\,ds - \int_0^x (x-s) f(s)\,ds$$

$$= x\left(\log s - s\right)\Big|_0^1 - \left(x\log s - s\right)\Big|_0^x$$

$$= -x - x\log x + x = -x\log x,\quad x \in (0,1)$$

$\because$ $u'(x) = -\log x - 1$, $u''(x) = -\frac{1}{x}$ exist, $\forall x \in (0,1)$

$\therefore$ $u \in C^2(0,1)$

but $\lim_{x\to 0^+} u'(x) = \infty$, $\lim_{x\to 0^+} u''(x) = -\infty$ not exist. at $x=0$

4. Cerify the summation by parts formula

$$\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j = w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1}(v_{j+1}-v_h)w_{j+1},$$

and show that, for $v_h \in V_h^0$,

$$(L_h v_h, v_h)_h = h^{-1}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2.$$

$$\sum_{j=0}^{n-1}(W_{j+1}-W_j)V_j = \sum_{j=0}^{n-1} W_{j+1}V_j - \sum_{j=0}^{n-1} W_j V_j$$

$$= \sum_{j=1}^{n} W_j V_{j-1} - \sum_{j=1}^{n} W_j V_j + W_n V_n - W_0 V_0$$

$$= \sum_{j=1}^{n} W_j (V_{j-1} - V_j) + W_n V_n - W_0 V_0$$

$$= W_n V_n - W_0 V_0 - \sum_{j=0}^{n-1} W_{j+1}(V_{j+1} - V_j)$$

$$(L_h V_h, V_h)_h = h \sum_{j=1}^{n-1} \frac{1}{h^2}(-V_{j+1} + 2V_j - V_{j-1}) V_j$$

$$= \frac{1}{h}\left(\sum_{j=1}^{n-1} -(V_{j+1}-V_j)V_j + (V_j - V_{j-1})V_j\right)$$

$$= \frac{1}{h}\left(\sum_{j=0}^{n-1} -(V_{j+1}-V_j)V_j + \sum_{j=0}^{n-1}(V_{j+1}-V_j)V_{j+1} + \cancelto{0}{(V_1-V_0)V_0 - (V_n - V_{n-1})V_n}\right.$$

$$= \frac{1}{h}\left(\sum_{j=0}^{n-1}(V_{j+1}-V_j)^2\right)$$

6. Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in (12.4) and G^k is its corresponding discrete counterpart solution of (12.4).
[*Solution*: we prove the result by verifying that $L_h G = he^k$. Indeed, for a fixed x_k the function $G(x_k, s)$ is a straight line on the intervals $[0, x_k]$ and $[x_k, 1]$ so that $L_h G = 0$ at every node x_l with $l = 0, \ldots, k-1$ and $l = k+1, \ldots, n+1$. Finally, a direct computation shows that $(L_h G)(x_k) = 1/h$ which concludes the proof.]

Note that $L_h G^k = e^k$

$$G(x_j, x_k) = \begin{cases} x_k(1-x_j), & 0 \le x_k \le x_j \\ x_j(1-x_k), & x_j \le x_k \le 1 \end{cases}$$

① show that $L_h(hG(x_j, x_k)) = e^k$

$$L_h(hG(x_j, x_k)) = \frac{1}{h}\left(-G(x_{j+1}, x_k) + 2G(x_j, x_k) - G(x_{j-1}, x_k)\right)$$

If $j=k$, then $L_h(hG(x_j,x_k)) = \frac{1}{h}\left(-x_k(1-(x_k+h)) + 2(x_k - x_k^2) - (x_k-h)(1-x_k)\right)$

$$= \frac{h}{h} = 1$$

If $j \neq k$

then
$$L_h(hG(x_j,x_k)) = \begin{cases} \frac{1}{h}\left(-x_k(1-x_j-h) + 2x_k(1-x_j) - x_k(1-x_j+h)\right), & x_j > x_k \\ \frac{1}{h}\left(-(x_j+h)(1-x_k) + 2x_j(1-x_k) - (x_j-h)(1-x_k)\right) & x_j < x_k \end{cases}$$

$$= \begin{cases} \frac{x_k}{h}(-1+x_j+h+2-2x_j-1+x_j-h), & x_j > x_k \\ \frac{1-x_k}{h}(-x_j-h+2x_j-x_j+h), & x_j < x_k \end{cases} = 0$$

$$\therefore L_h G^k = L_h(hG(x_j,x_k)) \Rightarrow G^k = hG(x_j,x_k)$$