1.

① If Σ is symmetric matrix

then $\Sigma = Q\lambda Q^T$, Q is orthogonal, λ is diagonal $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$

$$\Rightarrow x^T\Sigma^{-1}x = (Qx)^T\lambda^{-1}(Qx) = \sum_{i=1}^{k}\frac{(Qx)^2}{\lambda_i}$$

$\frac{(Q(x-u))^2}{\lambda_i} = u^2$, $\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{2\pi}$

$$\Rightarrow \int_{\mathbb{R}^k} f(x)dx = \frac{1}{\sqrt{(2\pi)^k|\Sigma|}}\int_{\mathbb{R}^k} e^{-\frac{1}{2}\sum_{i=1}^{k}\frac{(Q(x-u))^2}{\lambda_i}} = \frac{1}{\sqrt{(2\pi)^k|\Sigma|}}\prod_{i=1}^{k}\sqrt{\lambda_i}\cdot\sqrt{2\pi}$$

$$= \frac{1}{\sqrt{(2\pi)^k|\Sigma|}}(2\pi)^{\frac{k}{2}}\underbrace{\sqrt{\prod_{i=1}^{k}\lambda_i}}_{=|\Sigma|} = 1$$

② If Σ not symmetric

ex: $\Sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \Sigma^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $|\Sigma| = 1$

$$x^T\Sigma^{-1}x = (x_1\ x_2)\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1\ x_2)\begin{pmatrix} x_1 - x_2 \\ x_2 \end{pmatrix} = x_1^2 - x_1x_2 + x_2^2$$

$$||$$

$$(x^T\Sigma^{-1}x)^T = (x_1\ x_2)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1\ x_2)\begin{pmatrix} x_1 \\ -x_1 + x_2 \end{pmatrix} = x_1^2 - x_1x_2 + x_2^2$$

$$\Rightarrow x^T\Sigma^{-1}x = x^T\frac{\Sigma^{-1} + (\Sigma^T)^{-1}}{2}x = x^TS^{-1}x$$, S^{-1} is symmetric $\begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}$

$$\Rightarrow \int_{\mathbb{R}^2} f(x) = \int_{\mathbb{R}^2}\frac{1}{\sqrt{(2\pi)^k|\Sigma|}}e^{\frac{1}{2}(x-u)^T\Sigma^{-1}(x-u)}dx = \int_{\mathbb{R}^2}\frac{1}{\sqrt{(2\pi)^k}}e^{\frac{1}{2}(x-u)^TS^{-1}(x-u)}dx$$

$$\neq \int_{\mathbb{R}^2} \frac{1}{\sqrt{(2\pi)^k |S|}} e^{-\frac{1}{2}(x-u)^T S^{-1}(x-u)} dx = 1 \quad \text{by } ① \quad (\because |S| \neq 1)$$

$\therefore \Sigma$ is symmetric.

2.(a)

$$tr(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ji}$$

$$\Rightarrow \frac{\partial}{\partial A} tr(AB) = \begin{pmatrix} \frac{\partial A_{11}B_{11}}{\partial A_{11}} & \frac{\partial A_{12}B_{21}}{\partial A_{12}} & \cdots & \frac{\partial A_{1n}B_{n1}}{\partial A_{1n}} \\ \vdots & \ddots & & \vdots \\ \frac{\partial A_{n1}B_{1n}}{\partial A_{n1}} & \cdots & & \frac{\partial A_{nn}B_{nn}}{\partial A_{nn}} \end{pmatrix} \quad \left(\because \frac{\partial A_{ij}B_{ji}}{\partial A_{kl}} = 0,\ kl \neq ij\right)$$

$$= \begin{pmatrix} B_{11} & B_{21} & \cdots & B_{n1} \\ \vdots & \ddots & & \\ B_{1n} & & & B_{nn} \end{pmatrix} = B^T$$

(b)

$$x^T A x = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} A_{ij} x_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j A_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (xx^T)_{ji} A_{ij} = tr(xx^T A)$$

(c)

$$L(u, \sigma^2) = \prod_{i=1}^{n} f(x_i \mid u, \Sigma) = \frac{1}{\left((2\pi)^{\frac{k}{2}} |\Sigma|^{\frac{1}{2}}\right)^n} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-u)^T \Sigma^{-1}(x_i-u)}, \quad x_i\ u \in \mathbb{R}^k$$

$$\Rightarrow \ln L = \frac{-nk}{2} \ln(2\pi) - \frac{n}{2} \ln(|\Sigma|) - \frac{1}{2} \sum_{i=1}^{n} (x_i-u)^T \Sigma^{-1} (x_i-u)$$

$$\frac{\partial}{\partial u}(\ln L) = -\frac{1}{2}\frac{\partial}{\partial u}\sum_{i=1}^{n}\frac{(Q(x_i-u))^2}{\lambda_i} = -\frac{1}{2}\sum_{i=1}^{n}\frac{2}{\lambda_i}Q^2(x_i-u) = \sum_{i=1}^{n}-\Sigma^{-1}(x_i-u) = 0$$

$$\Rightarrow \quad n\Sigma^{-1}u = \sum_{i=1}^{n}\Sigma^{-1}x_i \quad \Rightarrow \quad u = \frac{1}{n}\sum_{i=1}^{n}x_i$$

$$\frac{\partial \ln L}{\partial \Sigma^{-1}} = \frac{\partial}{\partial \Sigma^{-1}}\frac{n}{2}\ln|\Sigma^{-1}| - \frac{1}{2}\frac{\partial}{\partial \Sigma^{-1}}\sum_{i=1}^{n}(x_i-u)^T\Sigma^{-1}(x-u)$$

$$= \underline{\frac{n}{2}\Sigma^{-1}} \ ? - \frac{1}{2}\frac{\partial}{\partial \Sigma^{-1}}\sum_{i=1}^{n}tr\left((x_i-u)(x-u)^T\Sigma^{-1}\right) \quad \text{by (b)}$$

$$= \frac{n}{2}\Sigma^{-1} - \frac{1}{2}\frac{\partial}{\partial \Sigma^{-1}}\sum_{i=1}^{n}tr\left(\Sigma^{-1}(x_i-u)(x-u)^T\right)$$

$$= \frac{n}{2}\Sigma^{-1} - \frac{1}{2}\sum_{i=1}^{n}\left((x_i-u)(x_i-u)^T\right)^T \quad \text{by (a)}$$

$$:= 0$$

$$\Rightarrow \quad \Sigma^{-1} = \frac{1}{n}\sum_{i=1}^{n}(x_i-u)(x_i-u)^T$$