第四章 导热问题的数值解法

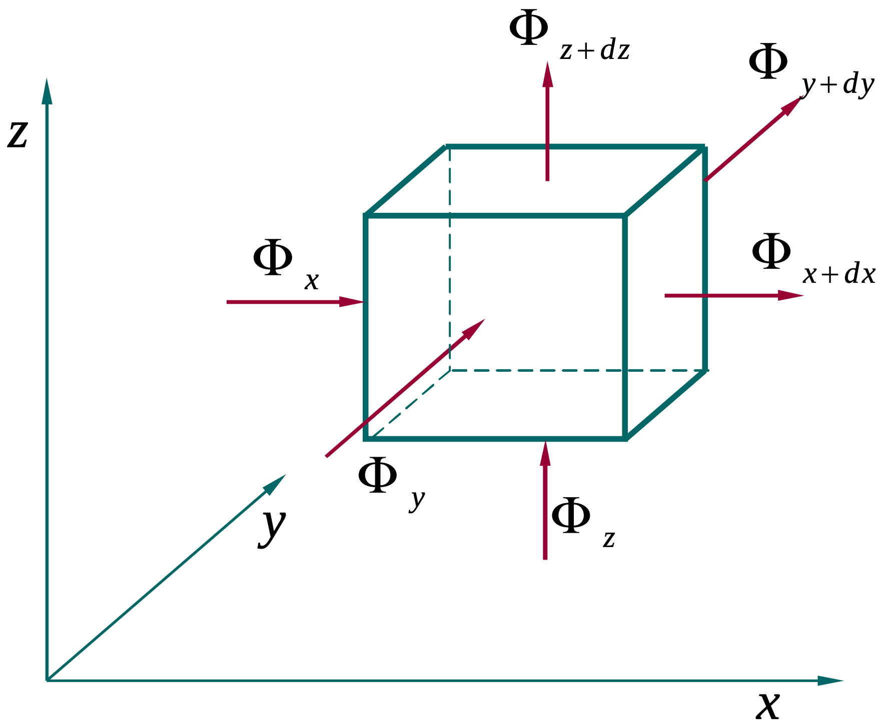
——复杂导热问题的近似解法

热传导方程求解的困难

$$\frac{\partial t}{\partial t}=a\left(\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}\right)$$

- 几何条件复杂的问题
- 边界条件复杂的问题
- 热物性随位置变化的问题
- 热物性随温度变化的问题
- 移动边界的问题

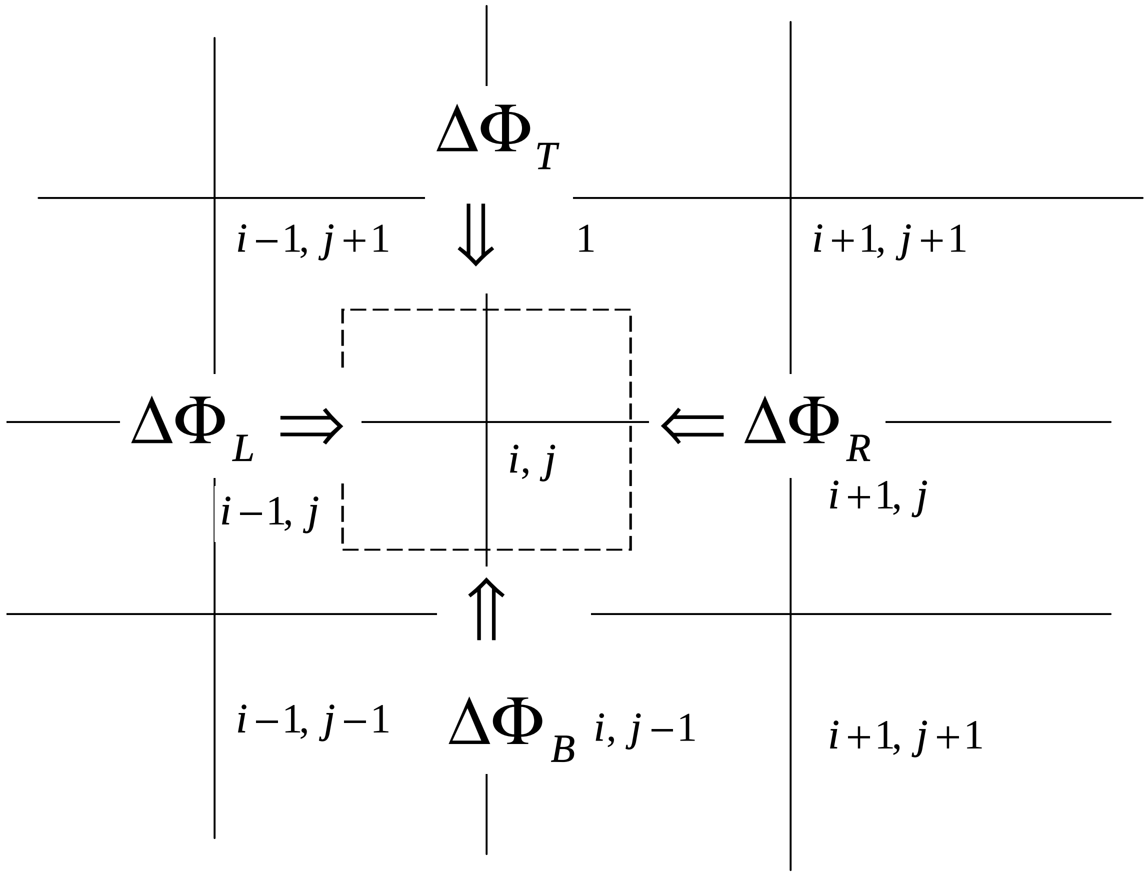
热传导方程的推导过程回顾



$$r\,c\frac{\partial t}{\partial t}=l\left(\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}\right)$$

二维微元体中的能量守恒

稳态无内热源条件下 $\Delta\Phi_L + \Delta\Phi_R + \Delta\Phi_B + \Delta\Phi_T = 0$



$$\Delta\Phi_L = q_L \Delta y = -l \left.\frac{\partial t}{\partial x}\right|_L \Delta y$$

$$\Delta\Phi_R = q_R \Delta y = l \left.\frac{\partial t}{\partial x}\right|_R \Delta y$$

$$\Delta\Phi_B = q_B \Delta y = -l \left.\frac{\partial t}{\partial y}\right|_B \Delta x$$

$$\Delta\Phi_T = q_T \Delta y = l \left.\frac{\partial t}{\partial y}\right|_T \Delta x$$

一阶导数的近似——差商

$$\left.\frac{\partial t}{\partial x}\right|_{i,j} \approx \frac{t_{i,j} - t_{i-1,j}}{\Delta x} \qquad \left.\frac{\partial t}{\partial x}\right|_{i,j} \approx \frac{t_{i+1,j} - t_{i,j}}{\Delta x}$$

$$\left.\frac{\partial t}{\partial y}\right|_{i,j} \approx \frac{t_{i,j} - t_{i,j-1}}{\Delta y} \qquad \left.\frac{\partial t}{\partial y}\right|_{i,j} \approx \frac{t_{i,j+1} - t_{i,j}}{\Delta y}$$

同理，温度对时间的导数为

$$\left.\frac{\partial t}{\partial t}\right|_{i,j}^{(m)} \approx \frac{t_{i,j}^{(m)} - t_{i,j}^{(m-1)}}{\Delta t} \qquad \left.\frac{\partial t}{\partial t}\right|_{i,j}^{(m)} \approx \frac{t_{i,j}^{(m+1)} - t_{i,j}^{(m)}}{\Delta t}$$

微元体能量守恒的近似表达

$$\begin{cases} \Delta\Phi_L = -l\Delta y\left(\dfrac{t_{i,j}-t_{i-1,j}}{\Delta x}\right), & \Delta\Phi_R = l\Delta y\left(\dfrac{t_{i+1,j}-t_{i,j}}{\Delta x}\right) \\ \Delta\Phi_B = -l\Delta x\left(\dfrac{t_{i,j}-t_{i,j-1}}{\Delta y}\right), & \Delta\Phi_T = l\Delta x\left(\dfrac{t_{i,j+1}-t_{i,j}}{\Delta y}\right) \end{cases}$$

$$-l\Delta y\left(\frac{t_{i,j}-t_{i-1,j}}{\Delta x}\right)+l\Delta y\left(\frac{t_{i+1,j}-t_{i,j}}{\Delta x}\right)$$

$$-l\Delta x\left(\frac{t_{i,j}-t_{i,j-1}}{\Delta y}\right)+l\Delta x\left(\frac{t_{i,j+1}-t_{i,j}}{\Delta y}\right)=0$$

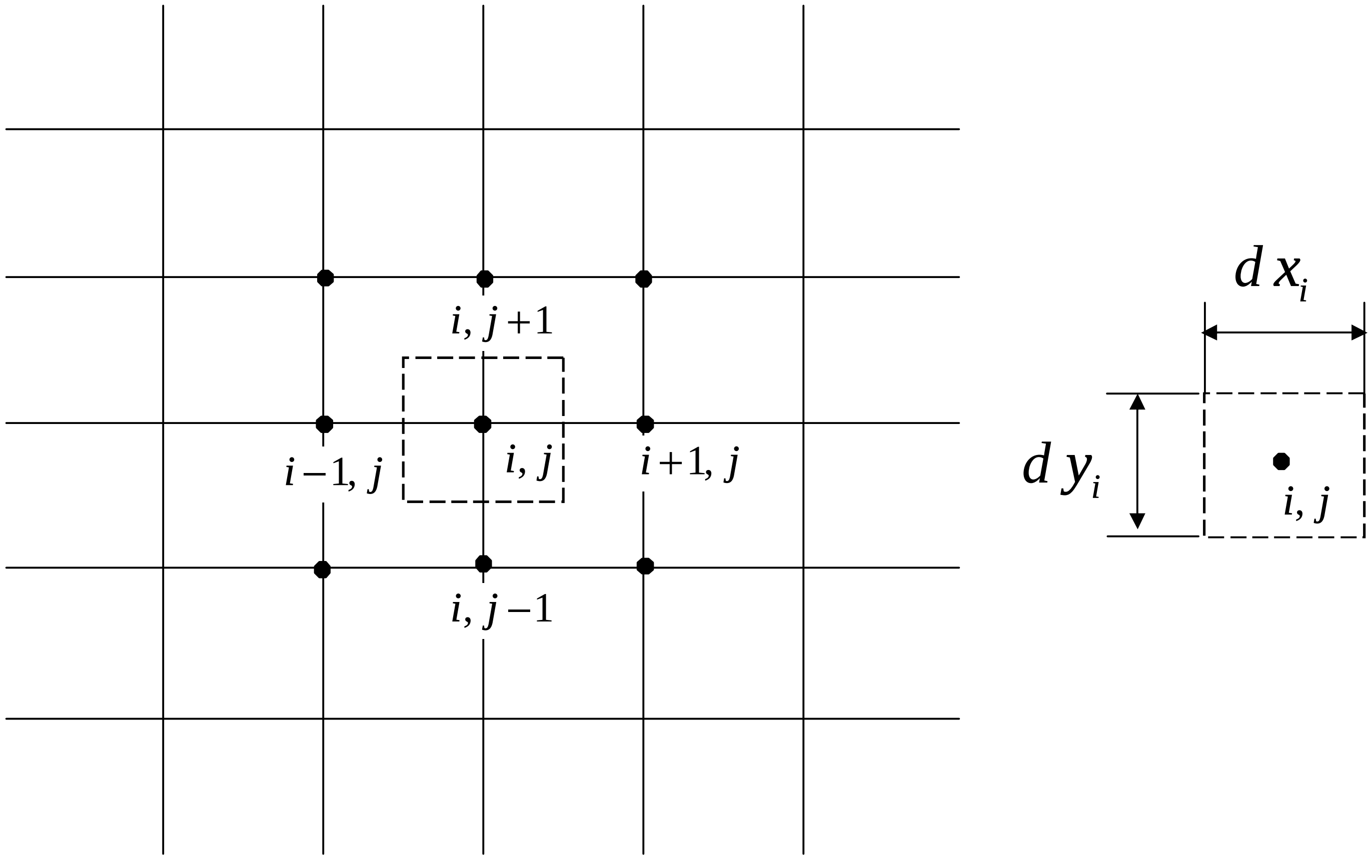
若 $\Delta x = \Delta y$

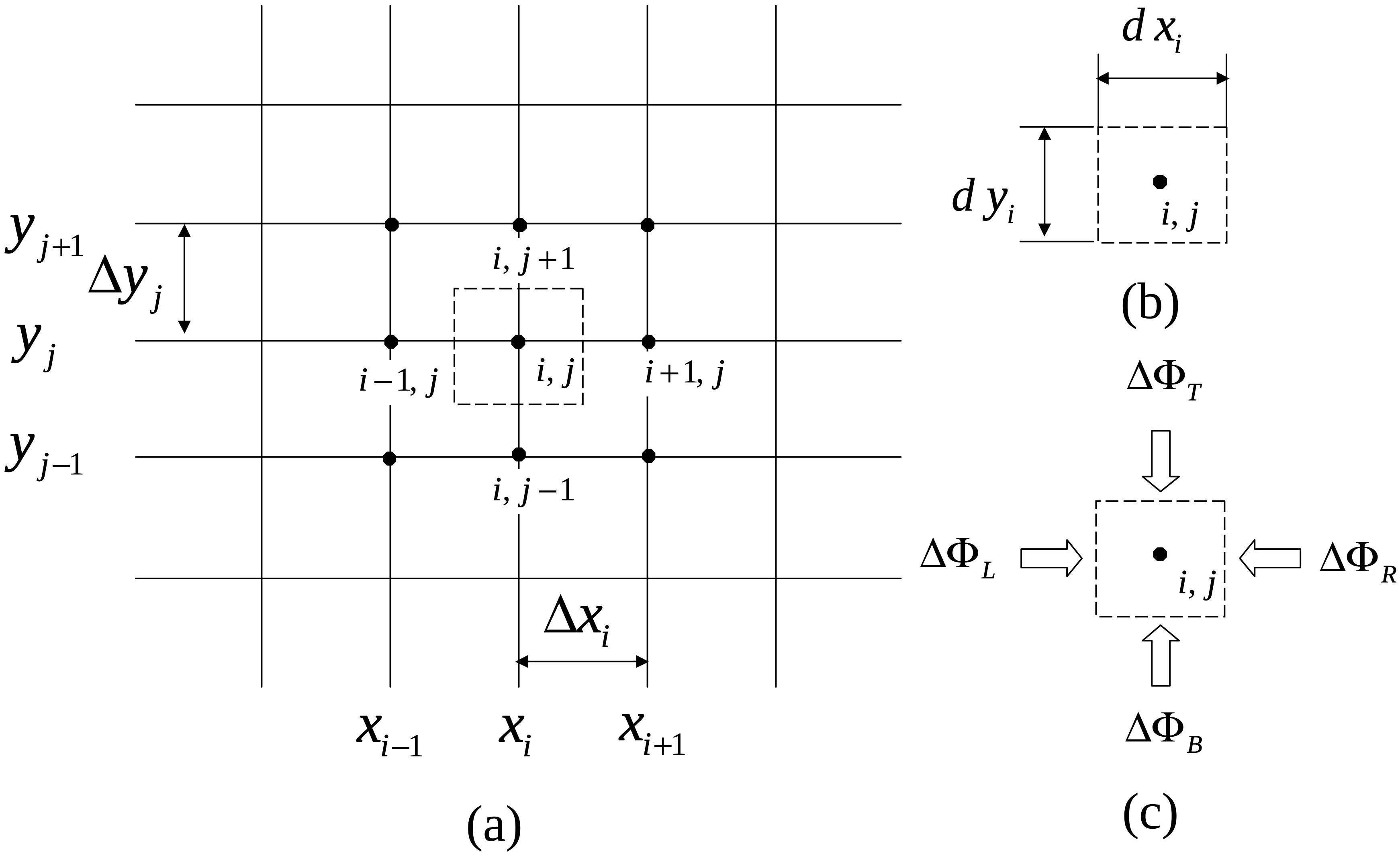
$$-4t_{i,j}+t_{i-1,j}+t_{i+1,j}+t_{i,j-1}+t_{i,j+1}=0$$

4-1离散化方法中的常用术语

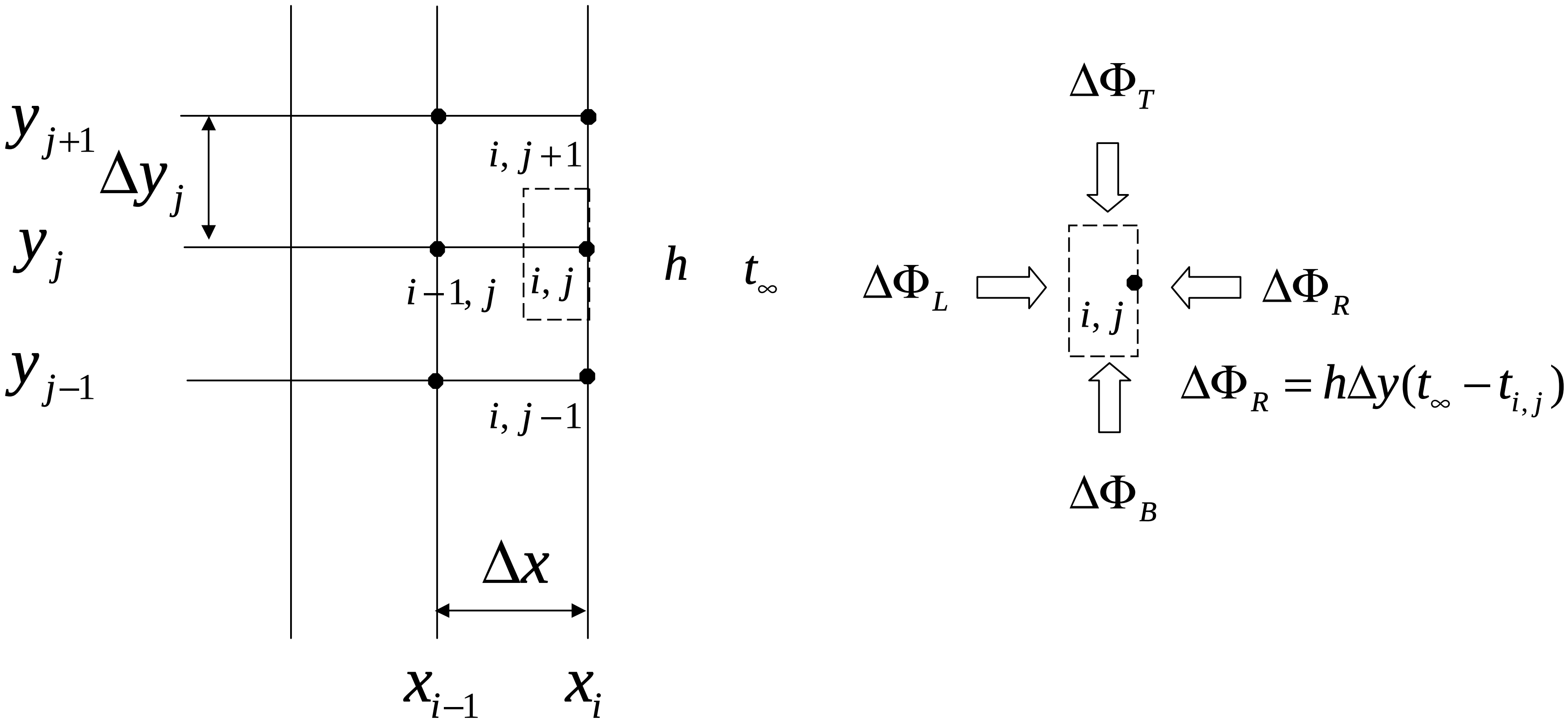
- 离散化（Discretization)——把求解的区域划分称若干互不重叠的子区域;
- 节点(Nodes)——用来表示子区域状态的特征点，如果节点处于区域的边界则成为边界节点，否则成为内部节点;
- 网格（Grids)——节点间按照一定的规则分割所形成的几何图形;
- 控制容积（微元体）——包围一个节点的子区域

网格、节点、控制容积示意图





y_{j+1}
Δy_j
y_j
y_{j-1}
$i, j+1$
$i-1, j$
i, j
$i+1, j$
$i, j-1$
Δx_i
x_{i-1}
x_i
x_{i+1}
(a)
$d\,x_i$
$d\,y_i$
i, j
(b)
$\Delta\Phi_T$
$\Delta\Phi_L$
i, j
$\Delta\Phi_R$
$\Delta\Phi_B$
(c)


y_{j+1}
Δy_j
y_j
y_{j-1}
$i, j+1$
$i-1, j$
i, j
$i, j-1$
h
t_∞
Δx
x_{i-1}
x_i
$\Delta\Phi_T$
$\Delta\Phi_L$
$\Delta\Phi_R$
$\Delta\Phi_B$
$\Delta\Phi_R = h\Delta y(t_\infty - t_{i,j})$

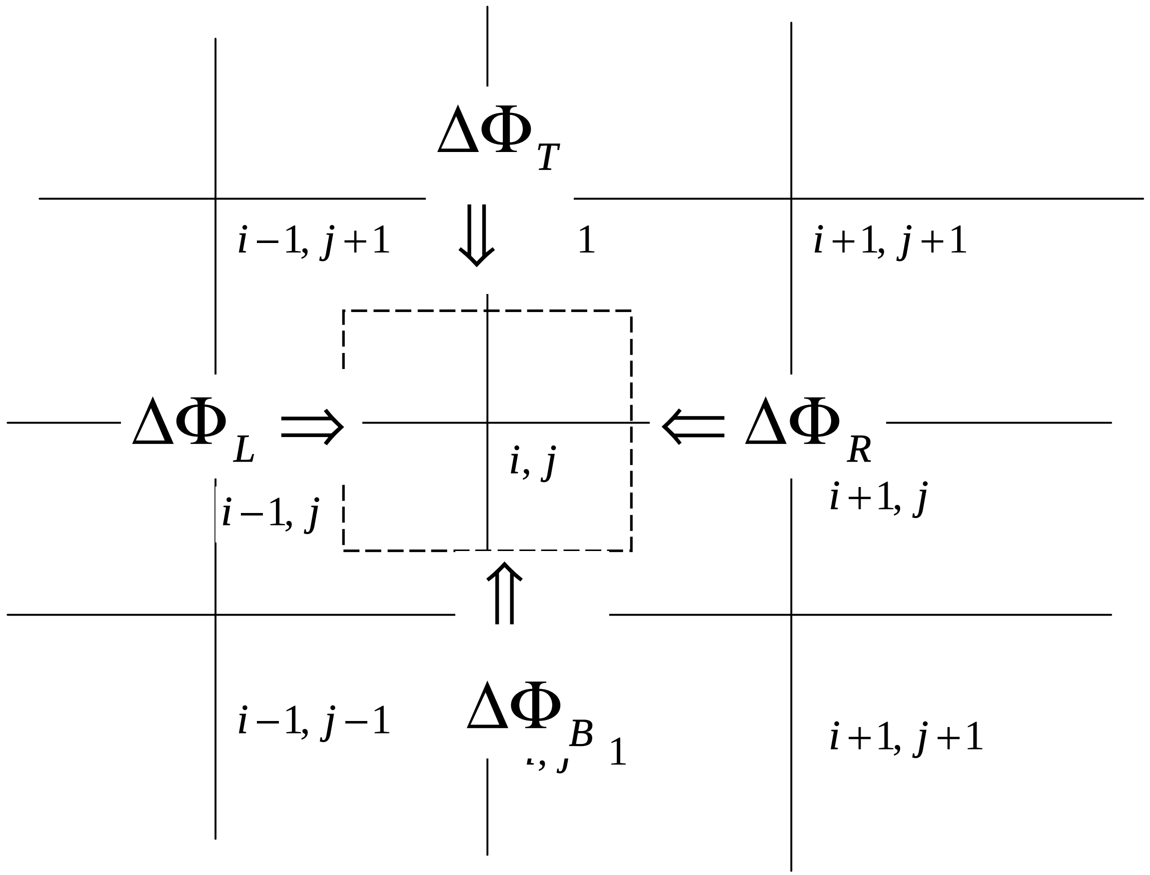
(a) (b)

4.2 二维稳态导热问题的计算

- 本课程采用基于控制容积能量守恒的有限差分法
- 首先将求解区域离散化，确定节点和网格
- 围绕节点划分控制容积
- 建立控制容积的能量守恒方程
- 用傅里叶定律表示导入控制容积的热流量
- 用差商近似表示导数
- 形成关于节点温度的离散化代数方程
- 建立每个待求温度的离散化代数方程，形成代数方程组
- 求解代数方程组，获得离散节点上的温度分布

节点的代数方程的建立

稳态无内热源条件下 $$\Delta\Phi_L + \Delta\Phi_R + \Delta\Phi_B + \Delta\Phi_T = 0$$



节点代数方程表达式

$$\begin{cases} \Delta\Phi_L = -l\Delta y\left(\dfrac{t_{i,j} - t_{i-1,j}}{\Delta x}\right), & \Delta\Phi_R = l\Delta y\left(\dfrac{t_{i+1,j} - t_{i,j}}{\Delta x}\right) \\ \Delta\Phi_B = -l\Delta x\left(\dfrac{t_{i,j} - t_{i,j-1}}{\Delta y}\right), & \Delta\Phi_T = l\Delta x\left(\dfrac{t_{i,j+1} - t_{i,j}}{\Delta y}\right) \end{cases}$$

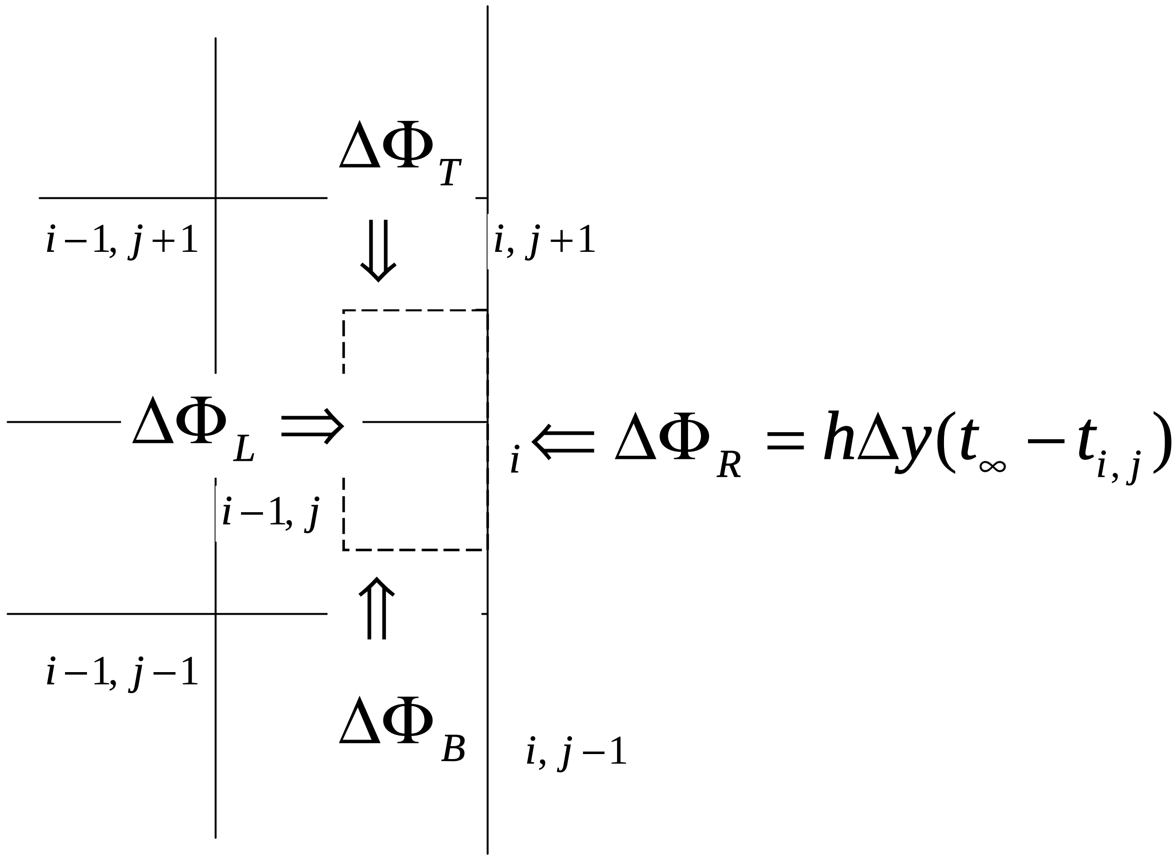
$$-l\Delta y\left(\frac{t_{i,j} - t_{i-1,j}}{\Delta x}\right) + l\Delta y\left(\frac{t_{i+1,j} - t_{i,j}}{\Delta x}\right)$$
$$-l\Delta x\left(\frac{t_{i,j} - t_{i,j-1}}{\Delta y}\right) + l\Delta x\left(\frac{t_{i,j+1} - t_{i,j}}{\Delta y}\right) = 0$$

若 $\Delta x = \Delta y$

$$-4t_{i,j} + t_{i-1,j} + t_{i+1,j} + t_{i,j-1} + t_{i,j+1} = 0$$

边界节点的代数方程

稳态无内热源条件下 $\Delta\Phi_L + \Delta\Phi_R + \Delta\Phi_B + \Delta\Phi_T = 0$



能量平衡关系及节点方程

$$\begin{cases} \Delta\Phi_L = -l\Delta y\left(\dfrac{t_{i,j} - t_{i-1,j}}{\Delta x}\right), \quad \Delta\Phi_R = -h\Delta y(t_{i,j} - t_\infty) \\ \Delta\Phi_B = -l\dfrac{\Delta x}{2}\left(\dfrac{t_{i,j} - t_{i,j-1}}{\Delta y}\right), \quad \Delta\Phi_T = l\dfrac{\Delta x}{2}\left(\dfrac{t_{i,j+1} - t_{i,j}}{\Delta y}\right) \end{cases}$$

$$-l\Delta y\left(\frac{t_{i,j} - t_{i-1,j}}{\Delta x}\right) + h\Delta y(t_\infty - t_{i,j})$$

$$-l\frac{\Delta x}{2}\left(\frac{t_{i,j} - t_{i,j-1}}{\Delta y}\right) + l\frac{\Delta x}{2}\left(\frac{t_{i,j+1} - t_{i,j}}{\Delta y}\right) = 0$$

若 $\Delta x = \Delta y$

$$-\left(4 + \frac{2h\Delta x}{l}\right)t_{i,j} + 2t_{i-1,j} + t_{i,j-1} + t_{i,j+1} + \frac{2h\Delta x}{l}t_\infty = 0$$

代数方程组的求解方法

直接解法

- 高斯消元法
- 主元消去法

迭代解法

- 高斯(Guass)—赛德尔(Seidel)迭代法
- 雅可比（Jacobi)迭代法

直接解法

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

迭代解法的原理

内部节点

$$-4t_{i,j} + t_{i-1,j} + t_{i+1,j} + t_{i,j-1} + t_{i,j+1} = 0$$

$$t_{i,j} = \frac{1}{4}\left(t_{i-1,j} + t_{i+1,j} + t_{i,j-1} + t_{i,j+1}\right)$$

边界节点

$$-\left(4 + \frac{2h\Delta x}{l}\right)t_{i,j} + 2t_{i-1,j} + t_{i,j-1} + t_{i,j+1} + \frac{2h\Delta x}{l}t_{\infty} = 0$$

$$t_{i,j} = \frac{2t_{i-1,j} + t_{i,j-1} + t_{i,j+1} + \frac{2h\Delta x}{l}t_{\infty}}{4 + \frac{2h\Delta x}{l}}$$

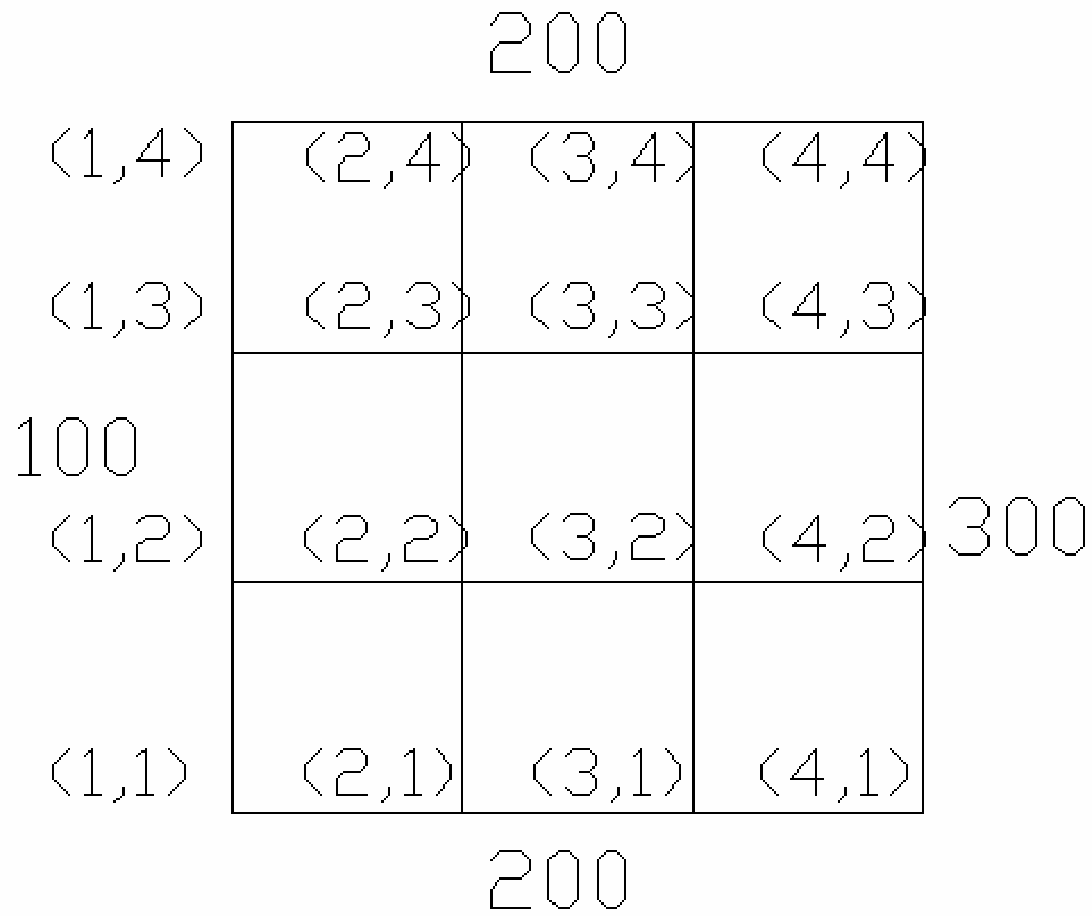
迭代公式

$$t_{i,j}^{(k+1)} = \frac{1}{4}\left(t_{i-1,j}^{(k)} + t_{i+1,j}^{(k)} + t_{i,j-1}^{(k)} + t_{i,j+1}^{(k)}\right)$$

$$t_{i,j}^{(k+1)} = \frac{2t_{i-1,j}^{(k)} + t_{i,j-1}^{(k)} + t_{i,j+1}^{(k)} + \frac{2h\Delta x}{l}t_{\infty}}{4 + \frac{2h\Delta x}{l}}$$

迭代步骤

- 给定各节点初始温度；
- 代入迭代公式，计算各点新的温度；
- 判断两次得到的温度是否非常接近，如偏差小于预定值，停止计算输出结果；
- 将得到的节点温度作为初值再代入迭代公式中计算。
- 说明： 计算中总会用到新的值，这种迭代方法称为G-S方法。


200
(1,4)
(2,4)
(3,4)
(4,4)
(1,3)
(2,3)
(3,3)
(4,3)
100
(1,2)
(2,2)
(3,2)
(4,2)
300
(1,1)
(2,1)
(3,1)
(4,1)
200

迭代公式

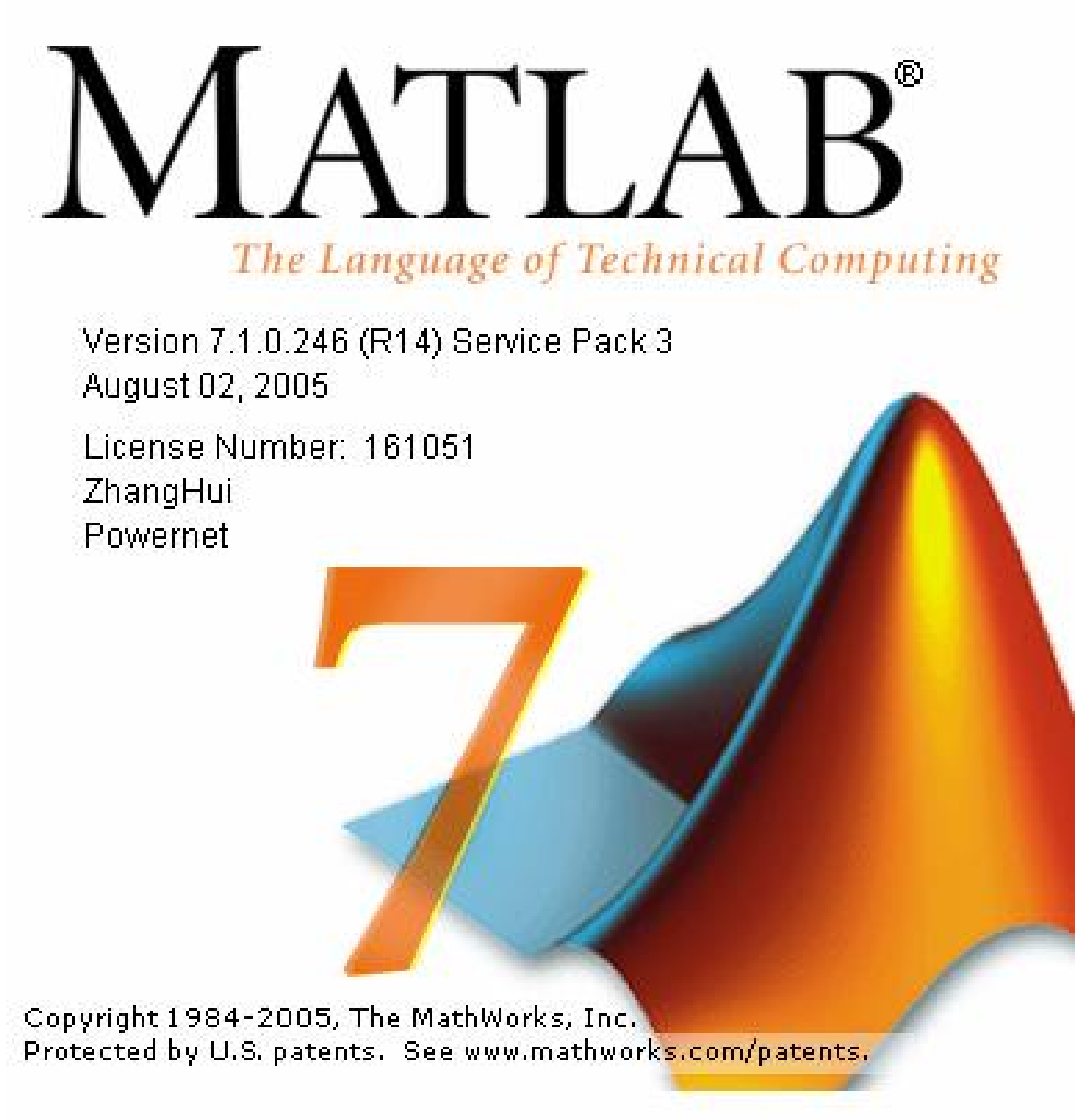
$$t_{2,2}^{(n+1)} = \frac{1}{4}(t_{3,2}^{(n)} + t_{1,2}^{(n)} + t_{2,3}^{(n)} + t_{2,1}^{(n)})$$

$$t_{3,2}^{(n+1)} = \frac{1}{4}(t_{4,2}^{(n)} + t_{2,2}^{(n+1)} + t_{3,3}^{(n)} + t_{3,1}^{(n)})$$

$$t_{2,3}^{(n+1)} = \frac{1}{4}(t_{3,3}^{(n)} + t_{1,3}^{(n)} + t_{2,4}^{(n)} + t_{2,2}^{(n+1)})$$

$$t_{3,3}^{(n+1)} = \frac{1}{4}(t_{4,3}^{(n)} + t_{2,3}^{(n+1)} + t_{3,4}^{(n)} + t_{3,2}^{(n+1)})$$

迭代演示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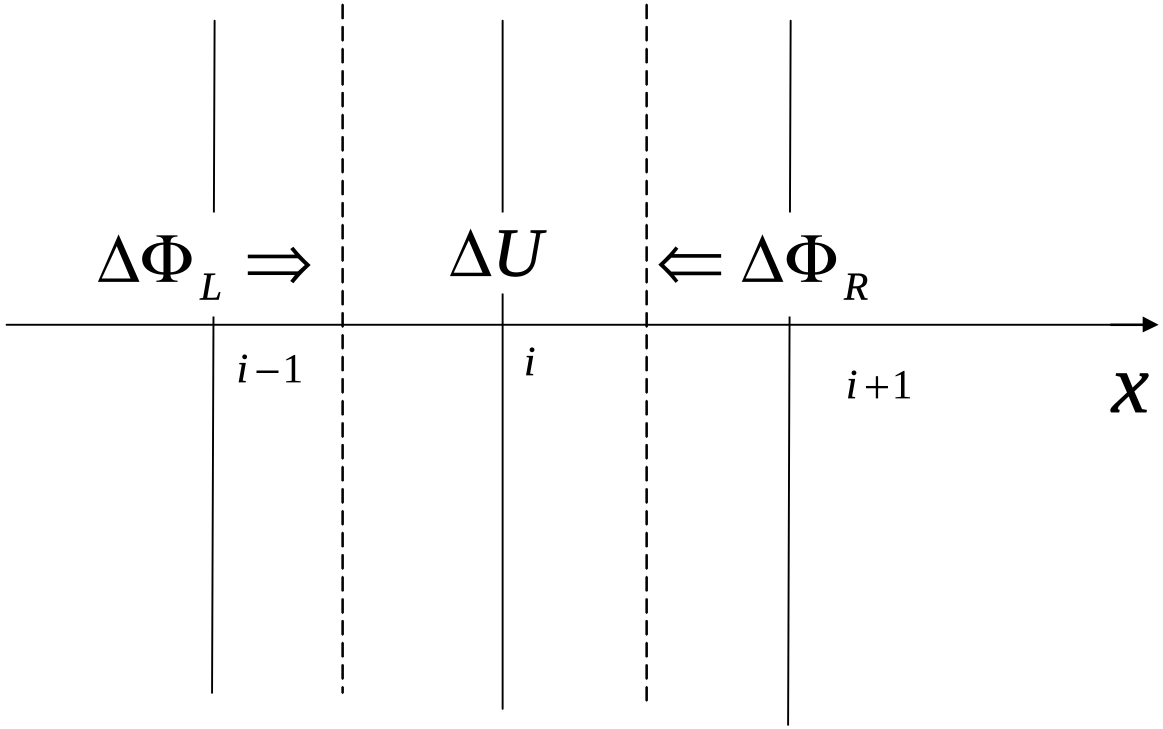
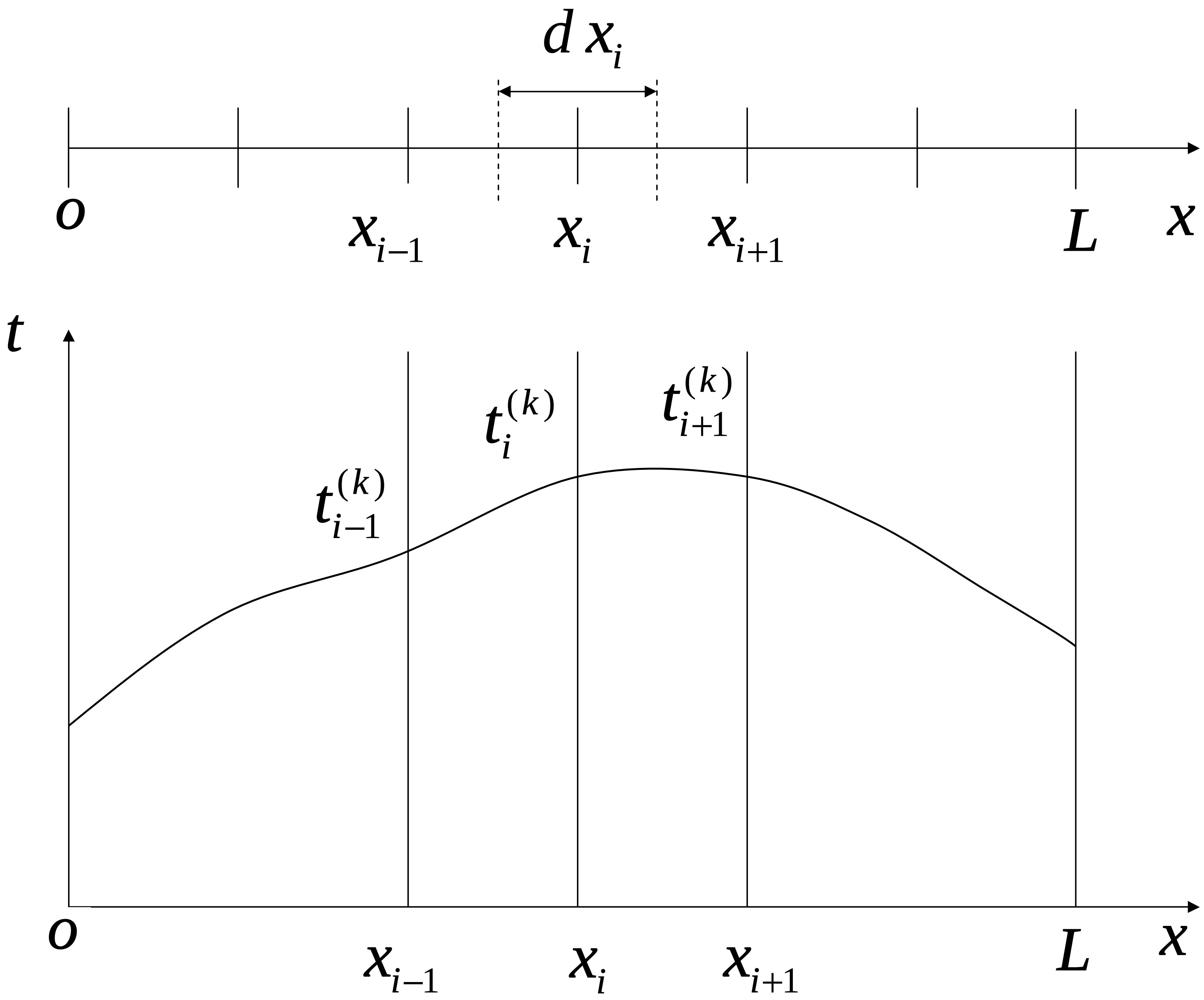
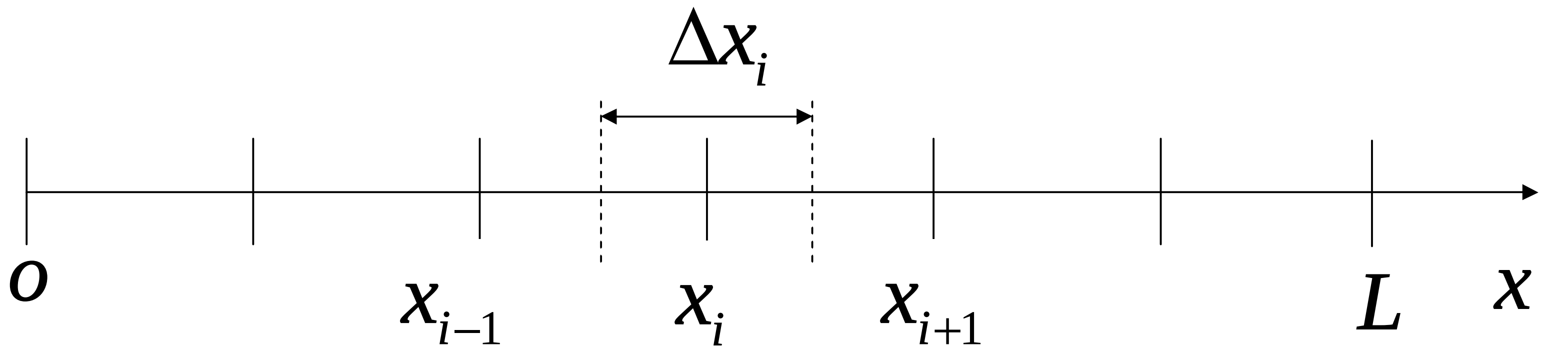
4.3 一维非稳态导热问题的计算

$$\Delta U = \Delta\Phi_L + \Delta\Phi_R$$

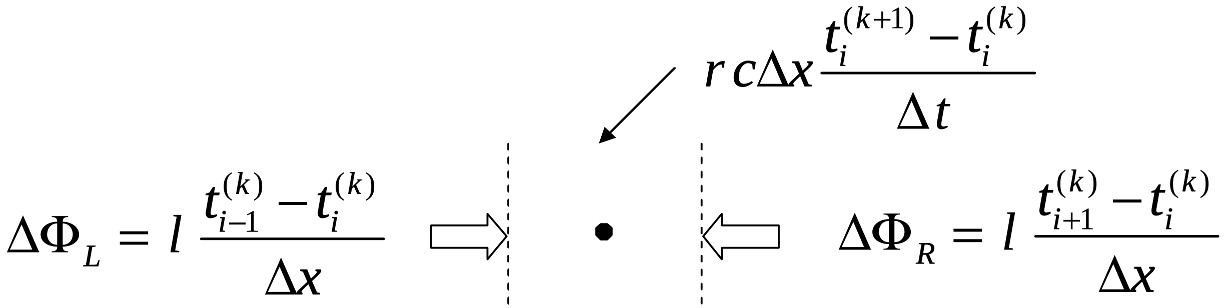




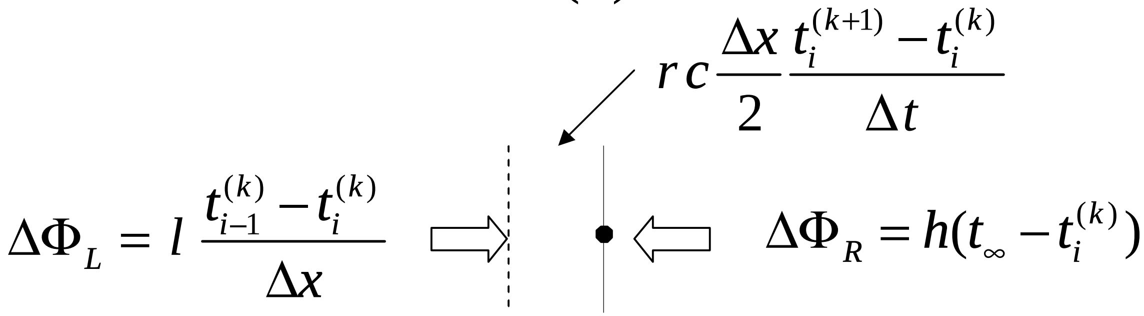
dx_i
o
x_{i-1}
x_i
x_{i+1}
L
x
t
$t_{i-1}^{(k)}$
$t_i^{(k)}$
$t_{i+1}^{(k)}$
o
x_{i-1}
x_i
x_{i+1}
L
x


Δx_i
o
x_{i-1}
x_i
x_{i+1}
L
x

(a)


$r\,c\Delta x \dfrac{t_i^{(k+1)} - t_i^{(k)}}{\Delta t}$
$\Delta\Phi_L = l\,\dfrac{t_{i-1}^{(k)} - t_i^{(k)}}{\Delta x}$
$\Delta\Phi_R = l\,\dfrac{t_{i+1}^{(k)} - t_i^{(k)}}{\Delta x}$

(b)


$r\,c\dfrac{\Delta x}{2}\dfrac{t_i^{(k+1)} - t_i^{(k)}}{\Delta t}$
$\Delta\Phi_L = l\,\dfrac{t_{i-1}^{(k)} - t_i^{(k)}}{\Delta x}$
$\Delta\Phi_R = h(t_\infty - t_i^{(k)})$

(c)

显式格式和隐式格式

$$r c \frac{t_i^{(k+1)} - t_i^{(k)}}{\Delta t} \Delta x = -l \left(\frac{t_i^{(k)} - t_{i-1}^{(k)}}{\Delta x} \right) + l \left(\frac{t_{i+1}^{(k)} - t_i^{(k)}}{\Delta x} \right)$$

$$r c \frac{t_i^{(k+1)} - t_i^{(k)}}{\Delta t} \Delta x = -l \left(\frac{t_i^{(k+1)} - t_{i-1}^{(k+1)}}{\Delta x} \right) + l \left(\frac{t_{i+1}^{(k+1)} - t_i^{(k+1)}}{\Delta x} \right)$$

内部节点的离散化方程（显式）

$$r\,c\frac{t_i^{(k+1)}-t_i^{(k)}}{\Delta t}\Delta x=-l\left(\frac{t_i^{(k)}-t_{i-1}^{(k)}}{\Delta x}\right)+l\left(\frac{t_{i+1}^{(k)}-t_i^{(k)}}{\Delta x}\right)$$

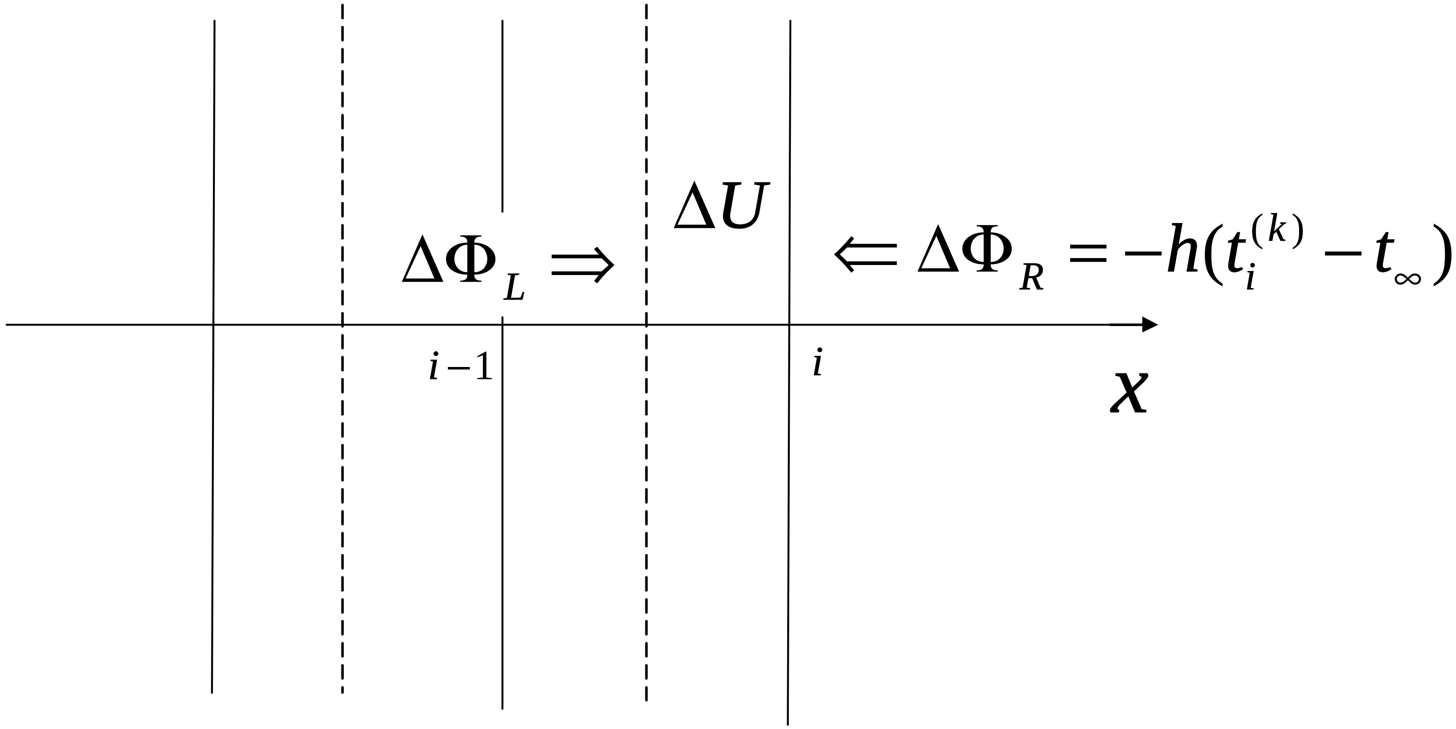
$$t_i^{(k+1)}=t_i^{(k)}+\frac{l\,\Delta t}{r\,c\left(\Delta x\right)^2}\left(t_{i-1}^{(k)}+t_{i+1}^{(k)}-2t_i^{(k)}\right)$$

$$t_i^{(k+1)}=\left[1-\frac{2a\Delta t}{\left(\Delta x\right)^2}\right]t_i^{(k)}+\frac{a\Delta t}{\left(\Delta x\right)^2}\left(t_{i-1}^{(k)}+t_{i+1}^{(k)}\right)$$

$$t_i^{(k+1)}=\left(1-2Fo\right)t_i^{(k)}+Fo\left(t_{i-1}^{(k)}+t_{i+1}^{(k)}\right)$$

边界节点

$$\Delta U = \Delta\Phi_L + \Delta\Phi_R$$



边界节点的离散化方程（显式）

$$r\,c\,\frac{t_i^{(k+1)}-t_i^{(k)}}{\Delta t}\,\frac{\Delta x}{2}=-l\left(\frac{t_i^{(k)}-t_{i-1}^{(k)}}{\Delta x}\right)+h(t_\infty-t_i^{(k)})$$

$$t_i^{(k+1)}=t_i^{(k)}-\frac{2\,l\,\Delta t}{r\,c\Delta x}\left(\frac{t_i^{(k)}-t_{i-1}^{(k)}}{\Delta x}\right)+\frac{2h\Delta t}{r\,c\Delta x}(t_\infty-t_i^{(k)})$$

$$Bi=\frac{h\Delta x}{l},\quad Fo=\frac{a\Delta t}{(\Delta x)^2},\quad a=\frac{l}{r\,c}$$

$$t_i^{(k+1)}=t_i^{(k)}-2Fo\left(t_i^{(k)}-t_{i-1}^{(k)}\right)+2Bi\;Fo(t_\infty-t_i^{(k)})$$

$$t_i^{(k+1)}=\left[1-2Fo(1+Bi)\right]t_i^{(k)}+2Fo\;t_{i-1}^{(k)}+2Bi\;Fot_\infty$$

内部节点的离散化方程（隐式）

$$rc\frac{t_i^{(k+1)}-t_i^{(k)}}{\Delta t}\Delta x=-l\left(\frac{t_i^{(k+1)}-t_{i-1}^{(k+1)}}{\Delta x}\right)+l\left(\frac{t_{i+1}^{(k+1)}-t_i^{(k+1)}}{\Delta x}\right)$$

$$t_i^{(k+1)}=t_i^{(k)}+\frac{l\Delta t}{rc(\Delta x)^2}\left(t_{i-1}^{(k+1)}+t_{i+1}^{(k+1)}-2t_i^{(k+1)}\right)$$

$$t_i^{(k+1)}=t_i^{(k)}-\frac{2a\Delta t}{(\Delta x)^2}t_i^{(k+1)}+\frac{a\Delta t}{(\Delta x)^2}\left(t_{i-1}^{(k+1)}+t_{i+1}^{(k+1)}\right)$$

$$t_i^{(k+1)}(1+2Fo)=t_i^{(k)}+Fo\left(t_{i-1}^{(k+1)}+t_{i+1}^{(k+1)}\right)$$

$$t_i^{(k+1)}=\frac{1}{(1+2Fo)}t_i^{(k)}+\frac{Fo}{(1+2Fo)}\left(t_{i-1}^{(k+1)}+t_{i+1}^{(k+1)}\right)$$

边界节点的离散化方程（隐式）

$$\rho c \frac{t_i^{(k+1)} - t_i^{(k)}}{\Delta t} \frac{\Delta x}{2} = -\lambda \left(\frac{t_i^{(k+1)} - t_{i-1}^{(k+1)}}{\Delta x} \right) + h(t_\infty - t_i^{(k+1)})$$

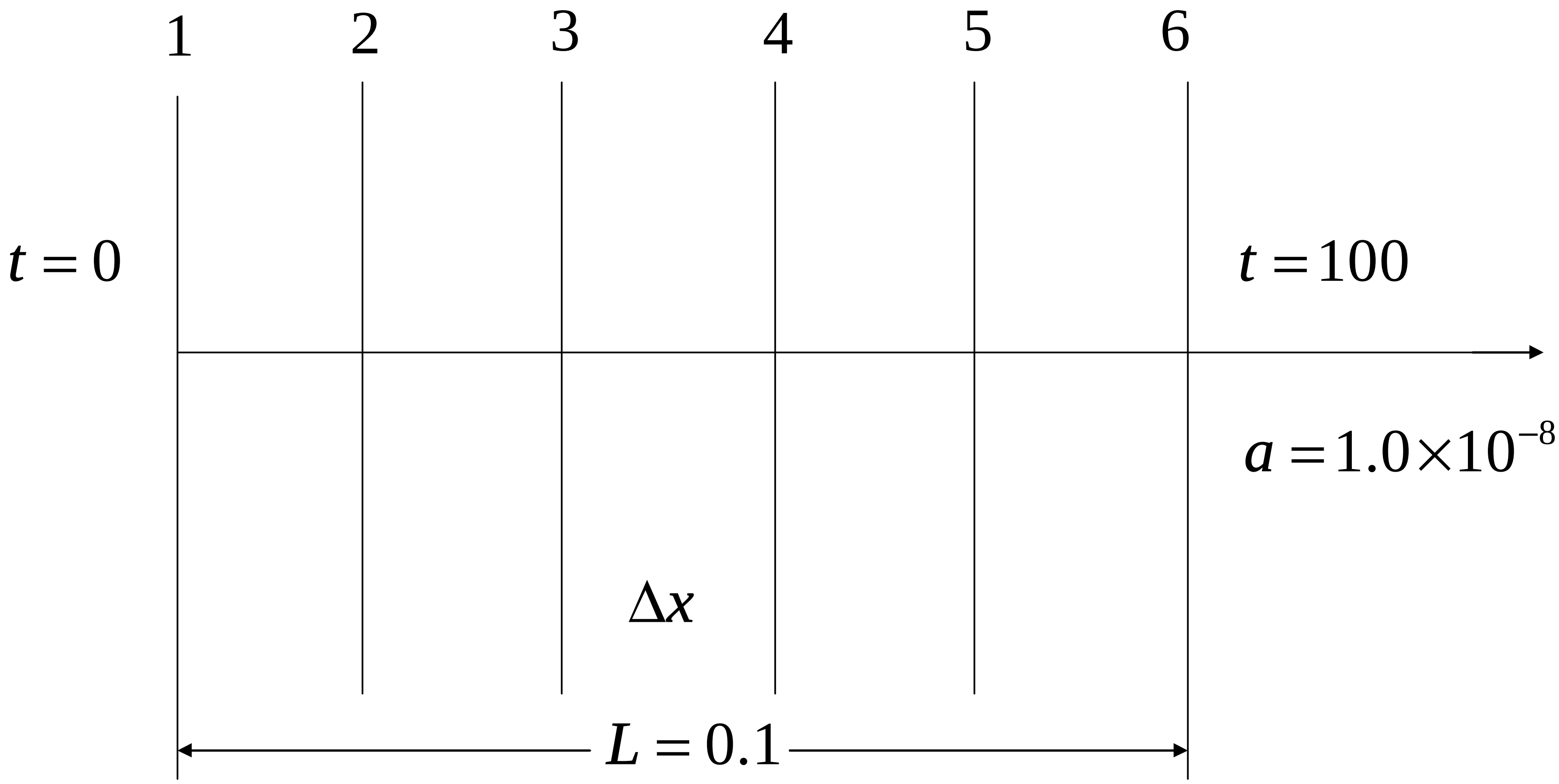
$$t_i^{(k+1)} = t_i^{(k)} - \frac{2\lambda\Delta t}{\rho c\Delta x}\left(\frac{t_i^{(k+1)} - t_{i-1}^{(k+1)}}{\Delta x} \right) + \frac{2h\Delta t}{\rho c\Delta x}(t_\infty - t_i^{(k+1)})$$

$$t_i^{(k+1)} = t_i^{(k)} - 2Fo\left(t_i^{(k+1)} - t_{i-1}^{(k+1)}\right) + 2Bi\,Fo(t_\infty - t_i^{(k+1)})$$

$$t_i^{(k+1)}\left[1 + 2Fo(1 + Bi)\right] = t_i^{(k)} + 2Fo\,t_{i-1}^{(k+1)} + 2Bi\,Fot_\infty$$

$$t_i^{(k+1)} = \frac{1}{\left[1 + 2Fo(1 + Bi)\right]} t_i^{(k)} + \frac{1}{\left[1 + 2Fo(1 + Bi)\right]}\left(2Fo\,t_{i-1}^{(k+1)} + 2Bi\,Fot_\infty\right)$$

计算实例



显式格式的不稳定性

$$t_i^{(k+1)} = \left(1 - 2Fo\right) t_i^{(k)} + Fo\left(t_{i-1}^{(k)} + t_{i+1}^{(k)}\right)$$

$$t_i^{(k+1)} = \left[1 - 2Fo(1 + Bi)\right] t_i^{(k)} + 2Fo\, t_{i-1}^{(k)} + 2Bi\; Fot_{\infty}$$

稳定性准则

$$1 - 2Fo \geq 0$$

$$1 - 2Fo(1 + Bi) \geq 0$$

导致不稳定的原因

对于内部节点，如果

$$1-2Fo<0$$

将导致前某点前一时刻的温度越高，后一个时刻的温度越低，这与实际情况是不相符的。出现这个问题的本质是，时间步长与空间步长不协调，导致数值精度下降而引起了计算结果的振荡。

隐式格式绝对稳定

$$t_i^{(k+1)}=\frac{1}{(1+2Fo)}t_i^{(k)}+\frac{Fo}{(1+2Fo)}\left(t_{i-1}^{(k+1)}+t_{i+1}^{(k+1)}\right)$$

$$t_i^{(k+1)}=\frac{1}{[1+2Fo(1+Bi)]}t_i^{(k)}+\frac{1}{[1+2Fo(1+Bi)]}\left(2Fo\,t_{i-1}^{(k+1)}+2Bi\,Fot_\infty\right)$$

显式格式可直接计算

隐式格式需要迭代计算

本章要点

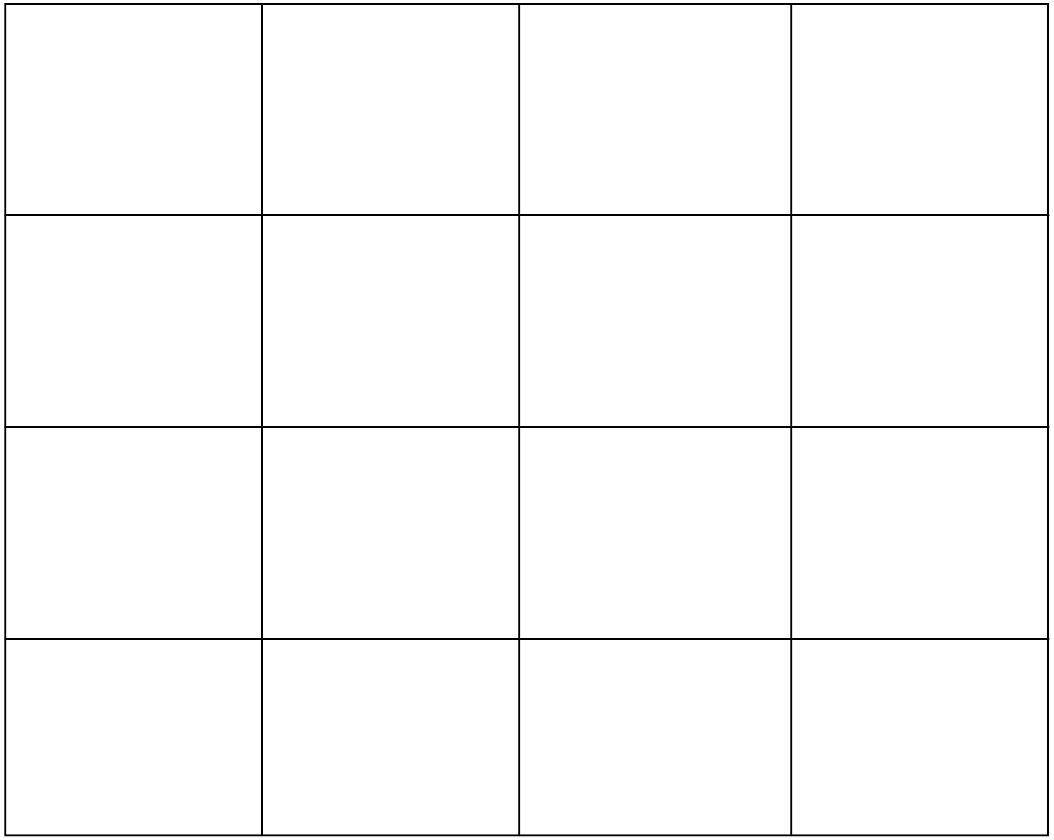
- 数值方法的基本概念和过程
- 二维稳态问题的有限差分方法
- 内部节点和边界节点方程的推导
- 代数方程组的迭代方法
- 一维非稳态问题的内部和边界节点方程
- 显示格式和隐式格式
- 代数方程（组）的求解方法
- 显式格式的不稳定性

二维稳态计算练习

150℃

T_f=10℃，λ=15W/(m·℃)

h=15W/（m^2·℃）



100℃

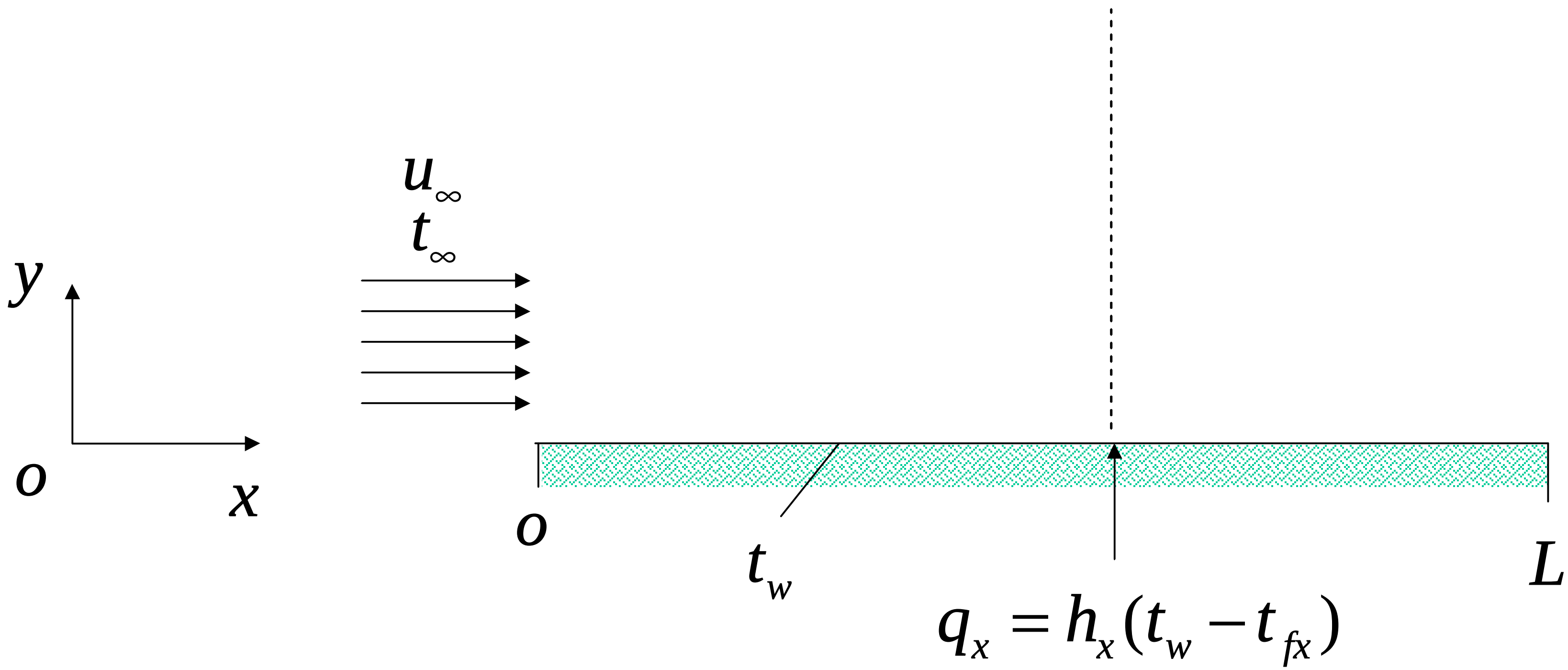
绝热

计算要求

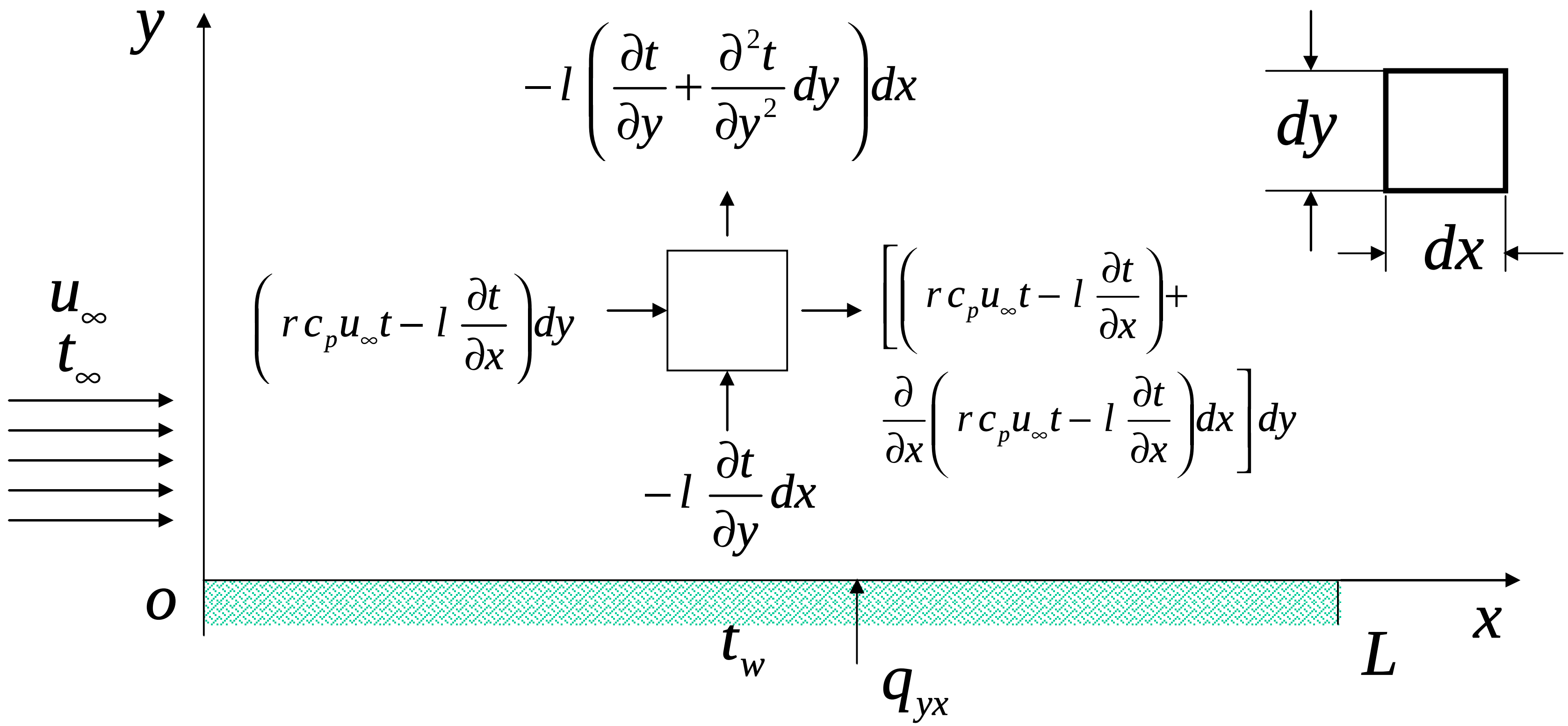
1. 写出各未知温度节点的代数方程
2. 分别给出G-S迭代和Jacobi迭代程序
3. 程序中给出两种自动判定收敛的方法
4. 考察三种不同初值时的收敛快慢
5. 上、右边界的热流量（λ=15W/(m℃））
6. 绘出最终结果的等值线

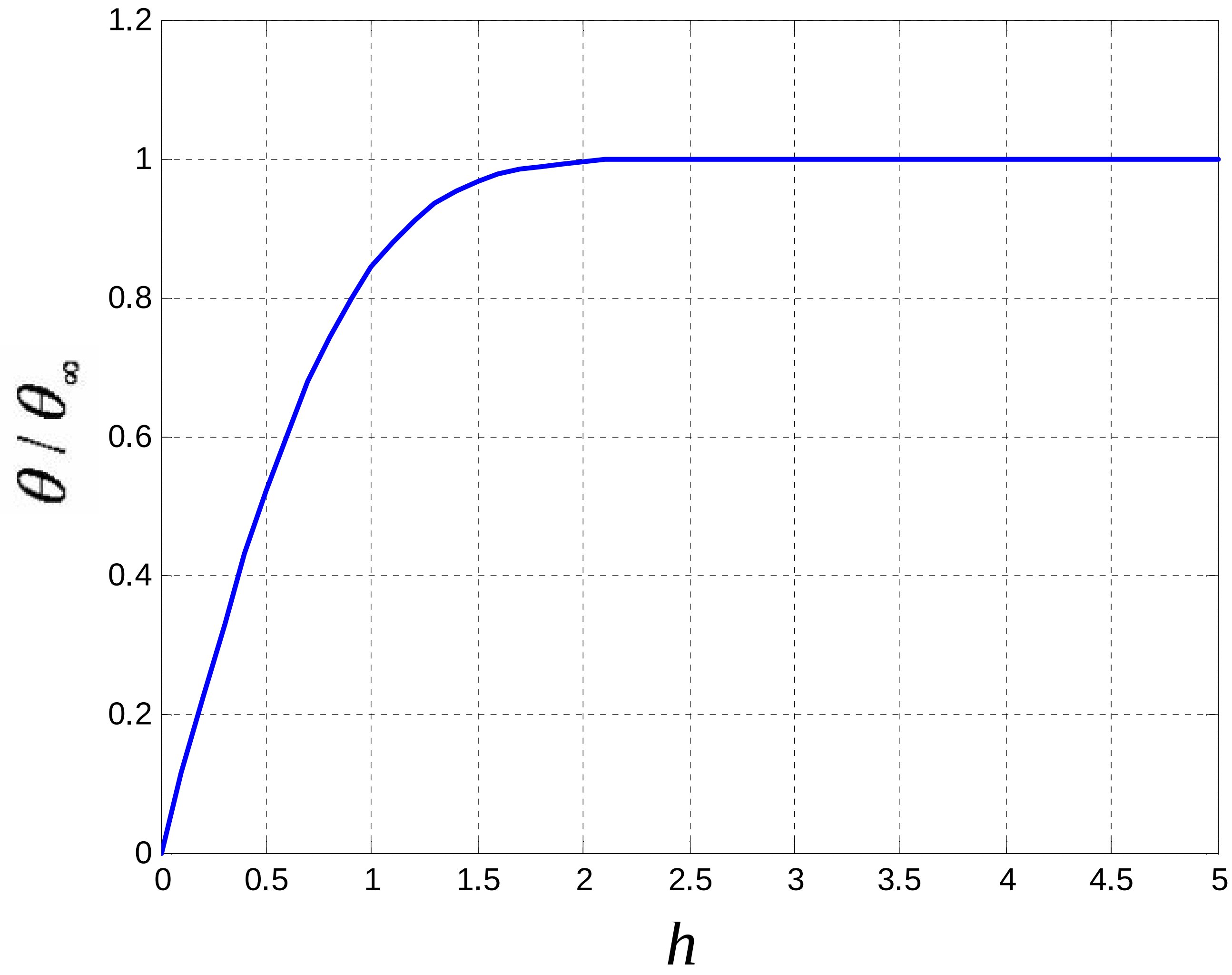
报告要求

1. 原始题目及要求
2. 各节点的离散化的代数方程
3. 源程序
4. 不同初值时的收敛快慢
5. 上右边界的热流量（λ=15W/(m℃））
6. 计算结果的等温线图
7. 计算小结



u_∞
t_∞
y
o
x
o
t_w
L
$q_x = h_x(t_w - t_{fx})$


y
$-l\left(\frac{\partial t}{\partial y}+\frac{\partial^2 t}{\partial y^2}dy\right)dx$
dy
dx
u_∞
t_∞
$\left(r c_p u_\infty t-l\frac{\partial t}{\partial x}\right)dy$
$\left[\left(r c_p u_\infty t-l\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial x}\left(r c_p u_\infty t-l\frac{\partial t}{\partial x}\right)dx\right]dy$
$-l\frac{\partial t}{\partial y}dx$
o
x
t_w
L
q_{yx}


1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
θ / θ∞
h