Algorithm Design and Analysis

Assignment 2

Deadline: Nov 3, 2022

1. (20 points) Sometimes, we may have more than one shortest path in the graph. We want to determine whether there is more than one different shortest path. Given a graph $G = (V, E)$ with weight $w(e)$, and a start vertex s. Design an efficient algorithm to determine whether there exists more than one shortest path from s to every $u \in V$.

2. (20 points) Given a directed graph $G = (V, E)$, and a reward r_v for all $v \in V$. The revenue of a vertex u is defined as the maximum reward among all vertices reachable from u (including u itself). Design a linear time algorithm to output every vertex's revenue.

3. (25 points) Given a directed graph $G = (V, E)$, a source vertex $s \in V$ and a destination vertex $t \in V$. Design an efficient algorithm to determine whether there is a path from s to t containing every vertex in V? (Vertices and edges can appear in the path more than once.)

4. (35 points) Let $G = (V, E)$ be a directed acyclic graph (DAG). Suppose that G is not strongly connected, and you are allowed to add extra edges into G. What is the minimum number of extra edges to make G strongly connected? In the following questions, we let $H(G)$ and $T(G)$ denote the set of head vertices (no incoming edges) and tail vertices (no outgoing edges). Notice that an isolated vertex is both a head and a tail. Moreover, if all vertices in $T(G)$ can be reached by every vertex in $H(G)$, then we call the graph G is *fully reachable*.

   (a) (5 points) Discuss the minimum number of extra edges we need to make a *fully reachable graph* strongly connected, and prove your claim.

   (b) (10 points) If G is not *fully reachable*, prove that we can always find an new edge e, such that after adding e, $G' = (V, E \cup \{e\})$ is still a DAG, and we have $|H(G')| = |H(G)| - 1$, $|T(G')| = |T(G)| - 1$.

   (c) (10 points) Design an algorithm to find the minimum number of extra edges to make G strongly connected.

   (d) (10 points) If the given graph G is not required to be a DAG, design an algorithm to find the minimum number of extra edges to make G strongly connected.

5. How long does it take you to finish the assignment (including thinking and discussing)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Write down their names here.