AI2619 Digital Signal and Image Processing Written Assignment 5

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1. The cumulative distribution functions are specified as

$$F_r(r) = 2r - r^2, \quad 0 \le r \le 1$$

$$F_z(z) = \begin{cases} 2z^2, & 0 \le z \le \dfrac{1}{2} \\ -2z^2 + 4z - 1, & \dfrac{1}{2} < z \le 1 \end{cases}$$

When $0 \le z \le \frac{1}{2}$, we have

$$2r - r^2 = 2z^2$$

which yields

$$z = \sqrt{r - \frac{1}{2}r^2}$$

When $\frac{1}{2} < z \le 1$, we have

$$2r - r^2 = -2z^2 + 4z - 1$$

which yields

$$z = \frac{\sqrt{2}}{2}r + 1 - \frac{\sqrt{2}}{2}$$

Therefore, the transformation is specified as

$$z = \begin{cases} \sqrt{r - \dfrac{1}{2}r^2}, & 0 \le r \le 1 - \dfrac{\sqrt{2}}{2} \\ \dfrac{\sqrt{2}}{2}r + 1 - \dfrac{\sqrt{2}}{2}, & 1 - \dfrac{\sqrt{2}}{2} < r \le 1 \end{cases}$$

2. (a) Let $c \times d$ denote the size of the zero-padded image. Let g denote the filtered image. The summation of all the elements in g is specified as

$$\sum_{x=1}^{c}\sum_{y=1}^{d} g(x,y) = \sum_{x=1}^{c}\sum_{y=1}^{d} w(x,y) * f(x,y) = \sum_{x=1}^{c}\sum_{y=1}^{d}\left[\sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)f(x-s,y-t)\right]$$

Exchange the order of summation, we have

$$\sum_{x=1}^{c}\sum_{y=1}^{d}\left[\sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)f(x-s,y-t)\right] = \sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)\left[\sum_{x=1}^{c}\sum_{y=1}^{d} f(x-s,y-t)\right]$$

Note that shifting f does not change the summation of all the elements in f, namely

$$\sum_{x=1}^{c}\sum_{y=1}^{d} f(x-s,y-t)=\sum_{x=1}^{c}\sum_{y=1}^{d} f(x,y)$$

which yields

$$\sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)\left[\sum_{x=1}^{c}\sum_{y=1}^{d} f(x-s,y-t)\right]=\left[\sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)\right]\left[\sum_{x=1}^{c}\sum_{y=1}^{d} f(x,y)\right]=0$$

Therefore, we can conclude that

$$\sum_{x=1}^{c}\sum_{y=1}^{d} g(x,y)=\sum_{x=1}^{c}\sum_{y=1}^{d} w(x,y)*f(x,y)=0$$

(b) Notice that correlation and convolution are essentially the same. The kernel for correlation can be obtained by flipping the kernel for convolution both horizontally and vertically, which still has the property that

$$\sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)=0$$

Therefore, we can claim that the summation of all the elements in g is zero.

3. According to the definition and shift property, we have

$$\begin{aligned}
&\mathscr{F}\left[f(x,y)*h(x,y)\right]\\
&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(s,t)h(x-s,y-t)\,\mathrm{d}s\,\mathrm{d}t\right]e^{-j2\pi(ux+vy)}\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(s,t)\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x-s,y-t)e^{-j2\pi(ux+vy)}\,\mathrm{d}x\,\mathrm{d}y\right]\mathrm{d}s\,\mathrm{d}t\\
&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(s,t)H(u,v)e^{-j2\pi(us+vt)}\,\mathrm{d}s\,\mathrm{d}t\\
&=H(u,v)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(s,t)e^{-j2\pi(us+vt)}\,\mathrm{d}s\,\mathrm{d}t\\
&=F(u,v)H(u,v)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathscr{F}^{-1}\left[F(u,v)*H(u,v)\right]\\
&=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(s,t)H(u-s,v-t)\,\mathrm{d}s\,\mathrm{d}t\right]e^{j2\pi(xu+yv)}\,\mathrm{d}u\,\mathrm{d}v\\
&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(s,t)\left[\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H(u-s,v-t)e^{j2\pi(xu+yv)}\,\mathrm{d}u\,\mathrm{d}v\right]\mathrm{d}s\,\mathrm{d}t\\
&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(s,t)h(x,y)e^{j2\pi(xs+yt)}\,\mathrm{d}s\,\mathrm{d}t\\
&=4\pi^2 h(x,y)\cdot\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(s,t)e^{j2\pi(xs+yt)}\,\mathrm{d}s\,\mathrm{d}t\\
&=f(x,y)h(x,y)
\end{aligned}$$

which proves the 2-D convolution theorem.

4. (a) The result is shown in the following table.

0	$\frac{1}{4}$	0	0	0	0	0
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0	0	0	0
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	0
0	0	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0	0	0	0
0	$\frac{1}{4}$	0	0	0	0	0

(b) In the spatial domain, the filter can be specified as

$$h(x,y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

According to the definition of 2-D discrete Fourier transform

$$\begin{aligned} H(u,v) &= \sum_{x=-1}^{1}\sum_{y=-1}^{1} h(x,y)e^{-j\frac{2\pi}{3}(ux+vy)} \\ &= \frac{1}{4}\left(e^{j\frac{2\pi}{3}u} + e^{-j\frac{2\pi}{3}u} + e^{j\frac{2\pi}{3}v} + e^{-j\frac{2\pi}{3}v}\right) \\ &= \frac{1}{2}\left(\cos\frac{2\pi u}{3} + \cos\frac{2\pi v}{3}\right) \end{aligned}$$

where $-4 \leq u \leq 4$ and $-3 \leq v \leq 3$.