[Homework 4] Poisson & Brownian Motion

Problem 1

Customers arrive according to a Poisson process of rate λ per hour. Joe does not want to stay until the store closes at $T = 10$ p.m., so he decides to close up when the first customer after time $T - s$ arrives. He wants to leave early but he does not want to lose any business so he is happy if he leaves before T and no one arrives after.

- What is the probability he achieves his goal?
- What is the optimal value of s and the corresponding success probability? (That is, the value s maximizing the success probability)

Problem 2

- Assume $X \sim \mathtt{Poisson}(\lambda)$ for some integer $\lambda \geq 1$. Prove that for any $k = 0, 1, \dots, \lambda - 1$, it holds that $\mathbf{Pr}\left[X = \lambda + k\right] \geq \mathbf{Pr}\left[X = \lambda - k - 1\right]$. Use this to conclude that $\mathbf{Pr}\left[X \geq \lambda\right] \geq \frac{1}{2}$.
- Recall the setting of Corollary 4 in Lecture 10. Prove that if $\mathbf{E}\left[f(X_1, \dots, X_n)\right]$ is monotonically increasing in m, then

$$\mathbf{E}\left[f(X_1, \dots, X_n)\right] \leq 2 \cdot \mathbf{E}\left[f(Y_1, \dots, Y_n)\right].$$

- Recall the birthday problem in Lecture 2 and assume notations there. Now suppose we would like to estimate the probability of the event "*there exists five students who share the same birthday*". Assume there are 105 students in the class ($n = 105$ and $m = 365$). Use Poisson approximation to show that the probabilty is at most 1%.

In problems below, we will use $W(t)$ to denote a standard Brownian motion

Problem 3

(a) Let $c > 0$ be a constant. Prove that $c^{-\frac{1}{2}} W(ct)$ is also a standard Brownian motion.

(b) Let $c > 0$ be a constant. Define $X(t) = W(c + t) - W(c)$. Prove that $\{X(t) : t \geq 0\}$ is a standard Brownian motion independent of $\{W(t) : 0 \leq t \leq c\}$.

(c) Compute $\mathbf{Pr}\left[W(1) > 0 \mid W(1/2) > 0\right]$.

Problem 4

In this problem, we let $X(t) = \mu t + \sigma W(t)$ be a (μ, σ^2) Brownian motion where $\mu > 0$. We also assume $\xi \sim \mathcal{N}(0, 1)$ is a standard Gaussian.

(a) Let $\delta \in \mathbb{R}$ be a number. Prove that

$$\mathbf{Pr}\left[X(t) \le \delta\right] = \mathbf{Pr}\left[\xi \le \frac{\delta - \mu \cdot t}{\sigma\sqrt{t}}\right].$$

(b) Let $T = \int_0^\infty \mathbf{1}[X(t) \in [0, \delta]] \, \mathrm{d}t$ be the time that $X(t)$ stays in the inteveral $[0, \delta]$. Prove that

$$\mathbf{E}\left[T\right] = \int_0^\infty \mathbf{Pr}\left[\frac{\mu t - \delta}{\sigma\sqrt{t}} \le \xi \le \frac{\mu\sqrt{t}}{\sigma}\right] \mathrm{d}t.$$

(c) Determine the function $f(\delta, x)$ so that for every $t \ge 0$, it holds that

$$\mathbf{Pr}\left[\frac{\mu t - \delta}{\sigma\sqrt{t}} \le \xi \le \frac{\mu\sqrt{t}}{\sigma}\right] = \mathbf{Pr}\left[f(0, \xi) \le t \le f(\delta, \xi)\right].$$

(d) Let $\delta > 0$ be a fixed constant. Compute $\mathbf{E}\left[f(\delta, \xi)\right]$ for the function f in (c).

(e) Prove that $\mathbf{E}\left[T\right] = \frac{\delta}{\mu}$.