Homework 4

2022.10.26

1、(a) $f(x) = x^T A x$ where $A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 2 \end{pmatrix}$.

Since $|1| = 1 > 0$, $\begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4} > 0$, $\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 2 \end{vmatrix} = \frac{5}{4} > 0$.

We have $A \succ 0$, thus quadratic function f is convex.

(b) Consider the Hessian matrix $H = \begin{pmatrix} \frac{6}{x_1^4 x_2^2} & \frac{4}{x_1^3 x_2^3} \\ \frac{4}{x_1^3 x_2^3} & \frac{6}{x_1^2 x_2^4} \end{pmatrix}$,

where $D_1(H) = \frac{6}{x_1^4 x_2^2} > 0$, $D_2(H) = \frac{20}{x_1^6 x_2^6} > 0$.

Thus $H \succ 0$ and f is convex.

(c) Consider the Hessian matrix $H = \begin{pmatrix} 2x_2^3 & 6x_1x_2^2 \\ 6x_1x_2^2 & 6x_1^2x_2 \end{pmatrix}$,

where $D_2(H) = D_2(-H) = -24x_1^2x_2^4 < 0$.

Thus H is indefinite and f is neither convex nor concave.

(d) Consider the Hessian matrix $H = \begin{pmatrix} -\frac{1}{4}x_1^{-\frac{3}{2}}x_2^{-\frac{1}{2}} & -\frac{1}{4}x_1^{-\frac{1}{2}}x_2^{-\frac{3}{2}} \\ -\frac{1}{4}x_1^{-\frac{1}{2}}x_2^{-\frac{3}{2}} & \frac{3}{4}x_1^{\frac{1}{2}}x_2^{-\frac{5}{2}} \end{pmatrix}$

where $D_2(H) = D_2(-H) = -\frac{1}{4} \cdot \frac{1}{x_1 x_2^3} < 0$.

Thus H is indefinite and f is neither convex nor concave.

(e) Consider the Hessian matrix $H = \begin{pmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\ \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & -\alpha(1-\alpha)x_1^{\alpha}x_2^{-\alpha-1} \end{pmatrix}$

where $D_{11}(-H) = \alpha(1-\alpha)x_1^{\alpha-2}x_2^{1-\alpha} \geq 0$, $D_{22}(-H) = \alpha(1-\alpha)x_1^{\alpha}x_2^{-\alpha-1} \geq 0$, $D_{12}(-H) = 0$.

Thus $H \preceq 0$ and f is concave.

(f) Let $f(x) = h(g(x))$ where $g(x) = \|Ax+b\|$ is convex,

$h(x) = \begin{cases} x^5, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is convex and increasing.

(Because $h'(x) = \begin{cases} 5x^4, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is increasing.)

By scaler composition, f is convex.

2、(a) Let $f(x) = |x|$ which is convex.

Then $\mathrm{epi} f = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R}, x_2 \geq |x|\}$ is convex,

namely S is convex.

(b) Let $f(x)=x^3$, then $S=\{(x_1,x_2)\in\mathbb{R}^2 : x_1\in\mathbb{R},\ x_2\geq x^3\}=\mathrm{epi} f$.

However, f is not convex, thus $S=\mathrm{epi} f$ is not convex.

(c) Let $f(x_1,x_2)=x_1\log x_1+x_2\log x_2$, whose Hessian matrix $H=\begin{pmatrix}\frac{1}{x_1} & 0\\ 0 & \frac{1}{x_2}\end{pmatrix}\succ 0$.

then we know f is convex, and $S=\{x\in\mathbb{R}^2 : x>0,\ f(x)\leq 2\}$ is a sublevel set of f, thus S is convex.

(d) Let $S_1=\{x\in\mathbb{R}^2 : \log(1+\|Ax+b\|^3)\leq 3\}=\{x\in\mathbb{R}^2 : \|Ax+b\|^3\leq e^3-1\}$

$S_2=\{x\in\mathbb{R}^2 : x_1\geq\log(1+e^{x_1+5x_2})\}=\{x\in\mathbb{R}^2 : \log(1+e^{x_1+5x_2})-x_1\leq 0\}$

Let $f(x)=\|Ax+b\|^3$. It can be proved convex which is similar to 1(f), and S_1 is a sublevel set of f, thus S_1 is convex.

Let $g(x)=\log(1+e^{x_1+5x_2})-x_1\leq 0$. Its Hessian matrix $H=\frac{e^{x_1+5x_2}}{(1+e^{x_1+5x_2})^2}\cdot\begin{pmatrix}1 & 5\\ 5 & 25\end{pmatrix}\succeq 0$.

Thus, $g(x)$ is convex. Since S_2 is a sublevel set of g, S_2 is convex.

Therefore, $S=S_1\cap S_2$ is also convex.

3. Let $f(x)=x\log x$, then $f''(x)=\frac{1}{x}>0$ $(x>0)$, so f is convex.

Since $x,y\in\Delta_{n-1}$, we have $x_i, y_i\in(0,1]$, $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i=1$.

Thus,
$$
\begin{aligned}
KL(x\|y) &= \sum_{i=1}^n x_i\log\frac{x_i}{y_i}\\
&= \sum_{i=1}^n y_i\cdot\frac{x_i}{y_i}\log\frac{x_i}{y_i}\\
&= \sum_{i=1}^n y_i f\left(\frac{x_i}{y_i}\right)\\
&\geq f\left(\sum_{i=1}^n y_i\cdot\frac{x_i}{y_i}\right) \qquad \text{(Jensen's inequality)}\\
&= f\left(\sum_{i=1}^n x_i\right)\\
&= f(1)=0.
\end{aligned}
$$

4. (a) Since $x_1^2-3x_2$ is not affine, it is not a convex optimization problem.

(b) For $f(x)=x_1^2+x_2^4$, its Hessian matrix $\begin{pmatrix}2 & 0\\ 0 & 12x_2^2\end{pmatrix}\succeq 0$, thus f is convex.

For $g_1(x)=(x_1-x_2)^2+4x_1x_2+e^{x_1+x_2}$, its Hessian matrix $(2+e^{x_1+x_2})\cdot\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\succeq 0$, thus g_1 is convex.

For $g_2(x)=x_1^2-2x_1x_2+x_2^2+x_1+x_2$, its Hessian matrix $\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix}\succeq 0$, thus g_2 is convex.

And $6x_1-7x_2=0$ is affine.

Therefore, it is a convex optimization problem.