Homework 3

1. If $C_1 \cap C_2 = \emptyset$, it reduces to separating hyperplane theorem on slides.

Now we just consider $C_1 \cap C_2 \neq \emptyset$, since $(\text{int } C_1) \cap C_2 = \emptyset$, any intersection x_0 must fall on ∂C_1, namely $x_0 \in C_1 \cap \partial C_1$. Then we have $x_0 \in C_2 \cap \partial C_2$, otherwise, if $x_0 \in \text{int } C_2$, there exists a ball $U(x_0, \varepsilon) \subset C_2$ and $x_0' \in U(x_0, \varepsilon)$ s.t. $x_0' \in \text{int } C_1$, causing $(\text{int } C_1) \cap C_2 \neq \emptyset$.

Let $C = C_1 - C_2$ which is a nonempty convex set.

Lemma: $0 \in \partial C$. Proof: Since $x_0 \in C_1 \cap C_2$, we have $0 \in C$. If $0 \in \text{int } C$, there exists a ball $U(0, \varepsilon) \subset C$, $\forall d \in U(0, \varepsilon)$, $\exists x_1 \in C_1, x_2 \in C_2 : d = x_1 - x_2 \in \text{int } C$, namely $x_1 = x_2 + d \in C_1$, this is impossible for $x_2 \notin \text{int } C_1$.

Since $0 \in \partial C$, by supporting hyperplane theorem, there exists $w \neq 0$ s.t. $\forall x \in C$: $\langle w, x \rangle \leq 0$, namely $\forall x_1 \in C_1, \forall x_2 \in C_2 : w^T x_1 \leq w^T x_2$, take $b = \sup_{x_1 \in C_1} w^T x_1$, then $\forall x_1 \in C_1 : w^T x_1 \leq b$, $\forall x_2 \in C_2 : w^T x_2 \geq b$. Q.E.D.

2. (a) $\forall x, y \in S_\alpha$, we have $f(x) < \alpha$, $f(y) < \alpha$, $\forall \theta \in [0,1]$:

$$f(\theta x + \bar{\theta} y) \leq \theta f(x) + \bar{\theta} f(y) < (\theta + \bar{\theta})\alpha = \alpha$$

Thus $\theta x + \bar{\theta} y \in S_\alpha$ $(\alpha < +\infty)$.

When $\alpha = +\infty$, $S_\alpha = \text{dom} f$ is convex (this will be shown in (b))

Similarly, $\forall x, y \in C_\alpha$, $\forall \theta \in [0,1]$: $\theta x + \bar{\theta} y \in C_\alpha$ $(\alpha < +\infty)$

When $\alpha = +\infty$, $C_\alpha = \mathbb{R}^n$ is still convex.

Therefore, $\forall \alpha \in (-\infty, +\infty]$, S_α and C_α are convex.

(b) $\text{dom} f = S_{+\infty} = \{x : f(x) < +\infty\}$, $\forall x, y \in S_{+\infty}$, $\forall \theta \in [0,1]$:

$$f(\theta x + \bar{\theta} y) \leq \theta f(x) + \bar{\theta} f(y) < \theta \cdot +\infty + \bar{\theta} \cdot +\infty = +\infty$$

Then $\theta x + \bar{\theta} y \in S_{+\infty}$, thus $\text{dom} f = S_{+\infty}$ is convex.

(c) Let $\alpha = \inf_{x \in X} f(x)$

If $\alpha \notin f(X)$, then $M = C_\alpha = \emptyset$ is convex.

If $\alpha \in f(X)$, then $\alpha = \min_{x \in X} f(x)$, $C_\alpha = \{x : f(x) \leq \alpha\} = \{x : f(x) = \min_{x \in X} f(x)\} = M$

According to conclusion in (a), $M = C_\alpha$ is convex.

3、Suppose $\theta \in (\theta_0, 1)$. Let $\alpha = \frac{\theta - \theta_0}{\bar{\theta}_0}$, $\bar{\alpha} = \frac{\bar{\theta}}{\bar{\theta}_0}$,

Then
$$\begin{aligned} f(\theta x + \bar{\theta} y) &= f[\alpha x + \bar{\alpha}(\theta_0 x + \bar{\theta}_0 y)] \\ &\leq \alpha f(x) + \bar{\alpha} f(\theta_0 x + \bar{\theta}_0 y) \\ &< \alpha f(x) + \bar{\alpha}[\theta_0 f(x) + \bar{\theta}_0 f(y)] \\ &= (\alpha + \bar{\alpha}\theta_0) f(x) + \bar{\alpha}\bar{\theta}_0 f(y) \\ &= \theta f(x) + \bar{\theta} f(y) \end{aligned}$$

For $\theta \in (0, \theta_0)$, the inequality can be proved similarly.

Therefore, $\forall \theta \in (0,1)$: $f(\theta x + \bar{\theta} y) < \theta f(x) + \bar{\theta} f(y)$. Q.E.D.

4、Since f is differentiable and convex, by first-order condition we have:

$$f(y) \geq f(x) + \nabla f(x)^T \cdot (y-x) \qquad ①$$

$$f(x) \geq f(y) + \nabla f(y)^T \cdot (x-y) \qquad ②$$

Then ①+② is

$$f(x) + f(y) \geq f(x) + f(y) + [\nabla f(x) - \nabla f(y)]^T \cdot (y-x)$$

Namely $\forall x, y \in \mathrm{dom} f$: $\langle \nabla f(x) - \nabla f(y), x-y \rangle \geq 0$.