[Homework 2] Finite Markov Chains, Coupling

作业要求

1. 本次作业允许讨论，但必须自己独立完成所有证明的写作（即不允许直接复制粘贴，或者直接照抄同学或者参考资料的现成证明）。
2. 如果有与同学讨论，必须注明所有参与讨论同学的名字。
3. 如果有参考其他资料完成作业，必须注明参考来源。
4. 如果助教发现违反上述原则（比如和已有证明或同学证明雷同），将直接打低分。同学如果不服可以来challenge，我们会就作业问题额外进行口试。

Problem 1 (Optimal Coupling)

Let Ω be a finite state space and μ, ν be two distributions over Ω. Prove that there exists a coupling ω of μ and ν such that

$$\mathbf{Pr}_{(X,Y)\sim\omega}\left[X \neq Y\right] = D_{\mathrm{TV}}(\mu, \nu).$$

You need to explicitly describe how ω is constructed.

Problem 2 （Total Variation Distance is Non-Increasing)

Let P be the transition matrix of an irreducible and aperiodic Markov chain with state space Ω. Let π be its stationary distribution. Let μ_0 be an arbitrary distribution on Ω and $\mu_t^{\mathtt{T}} = \mu_0^{\mathtt{T}} P^t$ for every $t \geq 0$. For every $t \geq 0$, let $\Delta(t) = D_{\mathtt{TV}}(\mu_t, \pi)$ be the total variation distance between μ_t and π. Using *coupling* to prove that $\Delta(t+1) \leq \Delta(t)$ for every $t \geq 0$.

Problem 3

You are tossing a coin repeatedly (In each round, the coin gives H or T with same probability). Which pattern would you expect to see faster: HH or HT? Formally, let T_1 be the first time the pattern HH appears and T_2 be the first time the pattern HT appears. What are $\mathbf{E}\left[T_1\right]$ and $\mathbf{E}\left[T_2\right]$? Can you generalize your results? (Hint: define a Markov chain with four states HH, HT, TH, TT)

Problem 4 (Path Coupling)

In this problem, we develop the *path coupling* technique to improve our analysis for sampling proper colorings. In the coupling analysis I showed to you in the class, we have to design a coupling for every pair $x, y \in \Omega$. Recall that for two colorings $x, y \in \Omega$, we use $d(x, y)$ to denote their Hamming distance. The path coupling technique allows us to design coupling only for those x, y with $d(x, y) = 1$.

Suppose now we already have a coupling satisfying $\mathbf{E}\left[d(X_{t+1}, Y_{t+1}) \mid (X_t, Y_t)\right] \le (1-\alpha)\cdot d(X_t, Y_t)$ for those (X_t, Y_t) with $d(X_t, Y_t) = 1$, I will tell you how to extend it to a coupling for arbitrary (X_t, Y_t).

Assuming $d(X, Y) = k$, we can transform the coloring X to the coloring Y by changing their disagreeing vertices one by one. This operation results in $k+1$ intermediate colorings denoted by $X = Z_0, Z_1, Z_2, \ldots, Z_k = Y$ where $d(Z_i, Z_{i+1}) = 1$ for all $i = 0, 1, \ldots, k-1$. Note that some of Z_i might not be proper colorings, and we can slightly extend our state space by adding those improper Z_i. The same transition rule applies to the new states. The new state space might not be irreducible, but it contains an irreducible component containing all proper colorings and the stationary distribution is still the uniform distribution on all proper colorings (verify this!). As before, we assume a coupling satisfying $\mathbf{E}\left[d(X_{t+1}, Y_{t+1}) \mid (X_t, Y_t)\right] \le (1-\alpha)\cdot d(X_t, Y_t)$ for those (X_t, Y_t) with $d(X_t, Y_t) = 1$. In particular, we know how to couple (Z_i, Z_{i+1}) for $i = 0, 1, \ldots, k-1$. Use (Z'_i, Z'_{i+1}) to denote the result of this coupling, namely $(Z'_i, Z'_{i+1}) = (X_{t+1}, Y_{t+1})$ where (X_{t+1}, Y_{t+1}) is sampled from the promised coupling when $(X_t, Y_t) = (Z_i, Z_{i+1})$. Consider the following random process:

- Couple (Z_0, Z_1) and obtain $(Z'_0, Z'_1) = (z_0, z_1)$;
- For $i = 1, \ldots, k-1$
- Couple (Z_i, Z_{i+1}) to obtain (Z'_i, Z'_{i+1}) and let z_{i+1} be Z'_{i+1} *conditioned on* $Z'_i = z_i$
- Output (z_0, z_k)

(1) Prove that $(X_{t+1}, Y_{t+1}) = (z_0, z_k)$ is a legal coupling of (X_t, Y_t).
(2) Prove that in the coupling above, $\mathbf{E}\left[d(z_0, z_k)\right] \le (1-\alpha)\cdot k$.
(3) Use the path coupling technique to show that $q > 2\Delta$ can guarantee our Markov chain to mix in $O(n \log n)$ steps.