AI2619 Digital Signal and Image Processing
Written Assignment 2

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1. (a) By properties of unit roots, we can simplify $\tilde{X}_3[k]$ as follow

$$\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{3N-1} x[n] W_{3N}^{kn} \\
&= \sum_{n=0}^{3N-1} \left(\frac{1}{N} \sum_{p=0}^{N-1} \tilde{X}[p] W_N^{-pn} \right) W_{3N}^{kn} \\
&= \frac{1}{N} \sum_{n=0}^{3N-1} \sum_{p=0}^{N-1} \tilde{X}[p] W_N^{-pn} W_{3N}^{kn} \\
&= \frac{1}{N} \sum_{p=0}^{N-1} \tilde{X}[p] \sum_{n=0}^{3N-1} W_{3N}^{(k-3p)n} \\
&= \frac{1}{N} \sum_{p=0}^{N-1} \tilde{X}[p] \cdot 3N\delta[k-3p] \\
&= 3 \sum_{p=0}^{N-1} \tilde{X}[p] \delta[k-3p]
\end{aligned}$$

Therefore, the expression required is

$$\tilde{X}_3[k] = \begin{cases} 3\tilde{X}\left[\dfrac{k}{3}\right], & k \equiv 0 \pmod 3 \\ 0, & \text{otherwise} \end{cases}$$

(b) The DFS coefficients of $\tilde{x}[n]$ are

$$\begin{aligned}
\tilde{X}[k] &= \sum_{k=0}^{N-1} \tilde{x}[n] W_N^{kn} \\
&= \tilde{x}[0] + \tilde{x}[1] e^{-j\pi k} \\
&= 1 + 2e^{-j\pi k} \\
&= \begin{cases} 3, & k \equiv 0 \pmod 2 \\ -1, & k \equiv 1 \pmod 2 \end{cases}
\end{aligned}$$

The DFS coefficients of $\tilde{x}_3[n]$ are

$$\begin{aligned}
\tilde{X}_3[k] &= \sum_{k=0}^{3N-1} \tilde{x}_3[n] W_{3N}^{kn} \\
&= \tilde{x}[0] + \tilde{x}[1]e^{-j\frac{\pi}{3}k} + \tilde{x}[2]e^{-j\frac{2\pi}{3}k} + \tilde{x}[3]e^{-j\pi k} + \tilde{x}[4]e^{-j\frac{4\pi}{3}k} + \tilde{x}[5]e^{-j\frac{5\pi}{3}k} \\
&= 1 + 2e^{-j\frac{\pi}{3}k} + e^{-j\frac{2\pi}{3}k} + 2e^{-j\pi k} + e^{-j\frac{4\pi}{3}k} + 2e^{-j\frac{5\pi}{3}k} \\
&= \begin{cases} 9, & k \equiv 0 \pmod 6 \\ -3, & k \equiv 3 \pmod 6 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

We can see that $\tilde{X}[k]$ and $\tilde{X}_3[k]$ satisfy the relationship in 1a.

2. (a) The DTFT of $x[n]$ is

$$\begin{aligned}
X\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{+\infty} \alpha^n u[n] e^{-j\omega n} \\
&= \sum_{n=0}^{+\infty} \left(\alpha e^{-j\omega}\right)^n \\
&= \frac{1}{1-\alpha e^{-j\omega}}
\end{aligned}$$

Note that $X\left(e^{j\omega}\right)$ exists only if $|\alpha| < 1$.

(b) The DFS of $\tilde{x}[n]$ is

$$\begin{aligned}
\tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \\
&= \sum_{n=0}^{N-1} \left(\sum_{r=-\infty}^{+\infty} \alpha^{n+rN} u[n+rN] \right) W_N^{kn} \\
&= \sum_{r=-\infty}^{+\infty} \sum_{n=0}^{N-1} \alpha^{n+rN} u[n+rN] W_N^{kn} \\
&= \sum_{r=0}^{+\infty} \sum_{n=0}^{N-1} \alpha^{n+rN} W_N^{kn} \\
&= \sum_{r=0}^{+\infty} \left[\alpha^{rN} \sum_{n=0}^{N-1} \left(\alpha W_N^k\right)^n \right] \\
&= \frac{1-\alpha^N}{1-\alpha W_N^k} \sum_{r=0}^{+\infty} \alpha^{rN} \\
&= \frac{1}{1-\alpha e^{-j\frac{2\pi}{N}k}}
\end{aligned}$$

Note that $\tilde{X}[k]$ exists only if $|\alpha| < 1$.

(c) Observe the results in 2a and 2b, we can conclude that

$$\tilde{X}[k] = X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}}$$