AI2613 Stochastic Processes Homework 4

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Problem 1

(a) Since the arrival of the customers is a Poisson process with rate λ. Suppose the number of customers arriving within time $[T-s, T]$ is X, then X satisfies the Poisson distribution with mean λs, namely

$$X \sim \mathcal{P}(\lambda s)$$

The probability that there is exactly one customer arriving within time $[T-s, T]$ is

$$\mathbf{Pr}\left[X = 1\right] = \lambda s e^{-\lambda s}$$

Therefore, the probability that Joe achieves his goal is $f(\lambda, s) = \lambda s e^{-\lambda s}$.

(b) Take the derivative of $f(\lambda, s)$ with respect to s, we have

$$\frac{\partial f}{\partial s} = \lambda e^{-\lambda s}(1 - \lambda s) = 0$$

Therefore, the optimal value is $s^* = \frac{1}{\lambda}$ and the corresponding probability is $f^* = \frac{1}{e}$.

Problem 2

(a) By the definition of Poisson distribution, we have

$$\mathbf{Pr}\left[X = \lambda + k\right] = \frac{\lambda^{\lambda+k}}{(\lambda+k)!} e^{-\lambda}$$
$$\mathbf{Pr}\left[X = \lambda - k - 1\right] = \frac{\lambda^{\lambda-k-1}}{(\lambda-k-1)!} e^{-\lambda}$$

Take the ratio of the two probabilities, we have

$$\frac{\mathbf{Pr}\left[X = \lambda + k\right]}{\mathbf{Pr}\left[X = \lambda - k - 1\right]} = \frac{\lambda^{\lambda+k}}{\lambda^{\lambda-k-1}} \frac{(\lambda-k-1)!}{(\lambda+k)!} = \frac{\lambda^{2k+1}}{(\lambda+k)(\lambda+k-1)\cdots(\lambda-k)}$$
$$= \prod_{i=1}^{k} \frac{\lambda^2}{(\lambda+i)(\lambda-i)} = \prod_{i=1}^{k} \frac{\lambda^2}{\lambda^2 - i^2} \geq 1$$

Hence, it holds that

$$\mathbf{Pr}\left[X = \lambda + k\right] \geq \mathbf{Pr}\left[X = \lambda - k - 1\right]$$

Notice that

$$\mathbf{Pr}\left[X \geq \lambda\right] = \sum_{k=0}^{\infty} \mathbf{Pr}\left[X = \lambda + k\right] \geq \sum_{k=0}^{\lambda-1} \mathbf{Pr}\left[X = \lambda - k - 1\right] = \mathbf{Pr}\left[X < \lambda\right]$$

Therefore, we can conclude that

$$\mathbf{Pr}\left[X \geq \lambda\right] = \frac{\mathbf{Pr}(X \geq \lambda)}{\mathbf{Pr}\left[X \geq \lambda\right] + \mathbf{Pr}\left[X < \lambda\right]} \geq \frac{1}{2}$$

(b) Since $\mathbf{E}\left[f(X_1, X_2, \ldots, X_n)\right]$ is monotonically increasing with respect to m, we have

$$\begin{aligned}
\mathbf{E}\left[f(Y_1, Y_2, \ldots, Y_n)\right] &= \sum_{k=0}^{\infty} \mathbf{E}\left[f(Y_1, Y_2, \ldots, Y_n) \;\middle|\; \sum_{i=1}^{n} Y_i = k\right] \mathbf{Pr}\left[\sum_{i=1}^{n} Y_i = k\right] \\
&\geq \sum_{k=m}^{\infty} \mathbf{E}\left[f(Y_1, Y_2, \ldots, Y_n) \;\middle|\; \sum_{i=1}^{n} Y_i = k\right] \mathbf{Pr}\left[\sum_{i=1}^{n} Y_i = k\right] \\
&\geq \mathbf{E}\left[f(Y_1, Y_2, \ldots, Y_n) \;\middle|\; \sum_{i=1}^{n} Y_i = m\right] \sum_{k=m}^{\infty} \mathbf{Pr}\left[\sum_{i=1}^{n} Y_i = k\right] \\
&= \mathbf{E}\left[f(X_1, X_2, \ldots, X_n)\right] \sum_{k=m}^{\infty} \mathbf{Pr}\left[\sum_{i=1}^{n} Y_i = k\right]
\end{aligned}$$

Note that $\sum_{i=1}^{n} Y_i \sim \mathcal{P}(m)$. By the previous conclusion, we have

$$\sum_{k=m}^{\infty} \mathbf{Pr}\left[\sum_{i=1}^{n} Y_i = k\right] = \mathbf{Pr}\left[\sum_{i=1}^{n} Y_i \geq m\right] \geq \frac{1}{2}$$

Therefore, we can conclude that

$$\mathbf{E}\left[f(X_1, X_2, \ldots, X_n)\right] \leq 2 \cdot \mathbf{E}\left[f(Y_1, Y_2, \ldots, Y_n)\right]$$

(c) Let X_i denote the number of students born on the i-th day. Then the distribution of $(X_1, X_2, \ldots, X_m)$ is exactly the same as that of $(Y_1, Y_2, \ldots, Y_m)$ conditioned on $\sum_{i=1}^{m} Y_i = n$, where $Y_i \sim \mathcal{P}\left(\frac{n}{m}\right)$ for $i = 1, 2, \ldots, m$. The function $f : \mathbb{N}^n \to \{0, 1\}$ is defined as

$$f(X_1, X_2, \ldots, X_m) = \mathbb{1}\left[\max\left\{X_1, X_2, \ldots, X_m\right\} \geq 5\right]$$

It is obvious that $\mathbf{E}\left[f(X_1, X_2, \ldots, X_m)\right]$ is monotonically increasing with respect to n. By the previous conclusion, we have

$$\begin{aligned}
&\mathbf{Pr}\left[\max\left\{X_1, X_2, \ldots, X_m\right\} \geq 5\right] = \mathbf{E}\left[f(X_1, X_2, \ldots, X_m)\right] \leq 2 \cdot \mathbf{E}\left[f(Y_1, Y_2, \ldots, Y_m)\right] \\
&= 2 \cdot \mathbf{Pr}\left[\max\left\{Y_1, Y_2, \ldots, Y_m\right\} \geq 5\right] = 2 \cdot \left(1 - \mathbf{Pr}\left[\max\left\{Y_1, Y_2, \ldots, Y_m\right\} < 5\right]\right) \\
&= 2 \cdot \left(1 - \prod_{i=1}^{m} \mathbf{Pr}\left[Y_i < 5\right]\right) = 2 \cdot \left(1 - e^{-n}\left[\sum_{k=0}^{4} \frac{\left(\frac{n}{m}\right)^k}{k!}\right]^m\right) \approx 0.0094 < 0.01
\end{aligned}$$

which verifies that the probability is at most 1%.

Problem 3

(a) Let $B(t)$ denote $c^{-\frac{1}{2}}W(ct)$. By definition of the standard Brownian motion, we have

$$W(cs+ct)-W(cs)\sim\mathcal{N}(0,ct)$$

which yields

$$B(s+t)-B(s)=c^{-\frac{1}{2}}\left[W(cs+ct)-W(cs)\right]\sim\mathcal{N}(0,t)$$

It is natural that for any $0\le t_0\le t_1\le\cdots\le t_n$, the random variables $B(t_1)-B(t_0), B(t_2)-B(t_1),\ldots,B(t_n)-B(t_{n-1})$ are mutually independent. In addtion, it is obvious that $B(0)=0$. Therefore, we can conclude that $\{B(t)\}_{t\ge 0}$ is also a standard Brownian motion.

(b) For any $s,t>0$, it holds that

$$X(s+t)-X(s)=W(c+s+t)-W(c+s)\sim\mathcal{N}(0,t)$$

It is natural that $X(t_1)-X(t_0), X(t_2)-X(t_1),\ldots,X(t_n)-X(t_{n-1})$ are mutually independent and $X(0)=W(c+0)-W(c)=0$. For any $0\le s\le c\le c+t$, the random variables $W(s)-W(0)$ and $W(c+t)-W(c)$ are independent, namely $X(t)$ is independent of $W(s)$. Therefore, $\{X(t)\}_{t\ge 0}$ is a standard Brownian motion independent of $\{W(t)\}_{0\le t\le c}$.

(c) Expand the conditioned probability, we have

$$\mathbf{Pr}\left[W(1)>0\mid W(1/2)>0\right]=\frac{\mathbf{Pr}\left[W(1)>0\wedge W(1/2)>0\right]}{\mathbf{Pr}\left[W(1/2)>0\right]}$$

Note that $W(1)-W(1/2)\sim\mathcal{N}(0,1/2)$. By total probability theorem, we have

$$\begin{aligned}
&\mathbf{Pr}\left[W(1)>0\wedge W(1/2)>0\right]\\
=&\int_0^{+\infty}\mathbf{Pr}\left[W(1)>0\mid W(1/2)=x\right]\cdot\frac{1}{\sqrt{\pi}}\exp\left(-x^2\right)\mathrm{d}x\\
=&\int_0^{+\infty}\mathbf{Pr}\left[W(1)-W(1/2)>-x\right]\cdot\frac{1}{\sqrt{\pi}}\exp\left(-x^2\right)\mathrm{d}x\\
=&\int_0^{+\infty}\left[\int_{-x}^{+\infty}\frac{1}{\sqrt{\pi}}\exp\left(-y^2\right)\mathrm{d}y\right]\cdot\frac{1}{\sqrt{\pi}}\exp\left(-x^2\right)\mathrm{d}x\\
=&\int_0^{+\infty}\left[\frac{1}{2}+\int_0^{x}\frac{1}{\sqrt{\pi}}\exp\left(-y^2\right)\mathrm{d}y\right]\cdot\frac{1}{\sqrt{\pi}}\exp\left(-x^2\right)\mathrm{d}x\\
=&\frac{1}{2}\int_0^{+\infty}\frac{1}{\sqrt{\pi}}\exp\left(-x^2\right)\mathrm{d}x+\frac{1}{\pi}\int_0^{+\infty}\mathrm{d}x\int_0^{x}e^{-(x^2+y^2)}\mathrm{d}y\\
=&\frac{1}{4}+\frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty}e^{-(x^2+y^2)}\mathrm{d}x\,\mathrm{d}y=\frac{1}{4}+\frac{1}{2\pi}\times\frac{\pi}{4}=\frac{3}{8}
\end{aligned}$$

By symmetry, we have $\mathbf{Pr}\left[W(1/2)>0\right]=\frac{1}{2}$. Therefore, we can conclude that

$$\mathbf{Pr}\left[W(1)>0\mid W(1/2)>0\right]=\frac{3}{4}$$

Problem 4

(a) Since $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion, we have

$$W(t) \sim \mathcal{N}(0,t)$$
$$t^{-\frac{1}{2}}W(t) \sim \mathcal{N}(0,1)$$

Note that $\xi \sim \mathcal{N}(0,1)$, which yields

$$\mathbf{Pr}[X(t) \leq \delta] = \mathbf{Pr}[\mu t + \sigma W(t) \leq \delta] = \mathbf{Pr}\left[t^{-\frac{1}{2}}W(t) \leq \frac{\delta - \mu t}{\sigma\sqrt{t}}\right] = \mathbf{Pr}\left[\xi \leq \frac{\delta - \mu t}{\sigma\sqrt{t}}\right]$$

(b) Similarly, we know that $-t^{-\frac{1}{2}}W(t) \sim \mathcal{N}(0,1)$. By the Fubini's theorem, we have

$$\begin{aligned}
\mathbf{E}[T] &= \mathbf{E}\left[\int_0^{+\infty} \mathbb{1}[0 \leq X(t) \leq \delta]\,\mathrm{d}t\right] = \int_0^{+\infty} \mathbf{E}[\mathbb{1}[0 \leq X(t) \leq \delta]]\,\mathrm{d}t \\
&= \int_0^{+\infty} \mathbf{Pr}[0 \leq X(t) \leq \delta]\,\mathrm{d}t = \int_0^{+\infty} \mathbf{Pr}[0 \leq \mu t + \sigma W(t) \leq \delta]\,\mathrm{d}t \\
&= \int_0^{+\infty} \mathbf{Pr}\left[\frac{\mu t - \delta}{\sigma\sqrt{t}} \leq -\frac{W(t)}{\sqrt{t}} \leq \frac{\mu t}{\sigma\sqrt{t}}\right]\mathrm{d}t = \int_0^{+\infty} \mathbf{Pr}\left[\frac{\mu t - \delta}{\sigma\sqrt{t}} \leq \xi \leq \frac{\mu\sqrt{t}}{\sigma}\right]\mathrm{d}t
\end{aligned}$$

(c) Assume it holds that $\frac{\mu t-\delta}{\sigma\sqrt{t}} \leq \xi \leq \frac{\mu\sqrt{t}}{\sigma}$, we have

$$\begin{cases} \mu\sqrt{t} - \sigma\xi \geq 0 \\ \mu t - \sigma\xi\sqrt{t} - \delta \leq 0 \end{cases}$$

which yields

$$\begin{cases} \sqrt{t} \geq \dfrac{\sigma\xi}{\mu} \\ \sqrt{t} \leq \dfrac{\sigma\xi + \sqrt{\sigma^2\xi^2 + 4\mu\delta}}{2\mu} \end{cases}$$

Hence, the function can be specified as

$$f(\delta,\xi) = \left(\frac{\sigma\xi + \sqrt{\sigma^2\xi^2 + 4\mu\delta}}{2\mu}\right)^2$$

Notice that $f(0,\xi) = \left(\frac{\sigma\xi}{\mu}\right)^2$. Therefore, it can be verified that

$$\mathbf{Pr}\left[\frac{\mu t - \delta}{\sigma\sqrt{t}} \leq \xi \leq \frac{\mu\sqrt{t}}{\sigma}\right] = \mathbf{Pr}[f(0,\xi) \leq t \leq f(\delta,\xi)]$$

(d) According to the properties of expectation, we have

$$\mathbf{E}[f(\delta,\xi)] = \mathbf{E}\left[\left(\frac{\sigma\xi + \sqrt{\sigma^2\xi^2 + 4\mu\delta}}{2\mu}\right)^2\right] = \frac{\sigma^2}{2\mu^2}\mathbf{E}\left[\xi^2\right] + \frac{\delta}{\mu} + \frac{\sigma}{2\mu^2}\mathbf{E}\left[\xi\sqrt{\sigma^2\xi^2 + 4\mu\delta}\right]$$

Since $\xi \sim \mathcal{N}(0,1)$, we have

$$\mathbf{E}\left[\xi^2\right] = \mathbf{E}^2[\xi] + \mathbf{Var}[\xi] = 1$$

Note that $\varphi(\xi) = \xi\sqrt{\sigma^2\xi^2 + 4\mu\delta}$ is an odd function with respect to ξ, which yields

$$\mathbf{E}\left[\xi\sqrt{\sigma^2\xi^2 + 4\mu\delta}\right] = 0$$

Therefore, we can conclude that

$$\mathbf{E}[f(\delta,\xi)] = \frac{\sigma^2}{2\mu^2} + \frac{\delta}{\mu}$$

(e) By the previous results, we have

$$\begin{aligned}
\mathbf{E}[T] &= \int_0^{+\infty} \mathbf{Pr}\left[\frac{\mu t - \delta}{\sigma\sqrt{t}} \le \xi \le \frac{\mu\sqrt{t}}{\sigma}\right] \mathrm{d}t = \int_0^{+\infty} \mathbf{Pr}\left[f(0,\xi) \le t \le f(\delta,\xi)\right] \mathrm{d}t \\
&= \int_0^{+\infty} \mathbf{Pr}\left[f(\delta,\xi) \ge t\right] \mathrm{d}t - \int_0^{+\infty} \mathbf{Pr}\left[f(0,\xi) \ge t\right] \mathrm{d}t \\
&= \mathbf{E}[f(\delta,\xi)] - \mathbf{E}[f(0,\xi)] = \frac{\sigma^2}{2\mu^2} + \frac{\delta}{\mu} - \frac{\sigma^2}{2\mu^2} = \frac{\delta}{\mu}
\end{aligned}$$

which verifies the conclusion.

References

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[3] Lalley S. Brownian Motion. University of Chicago.