Homework 2

2022.10.1

1、(a) $f(x)=x^TQx+b^Tx$ where $Q=\begin{pmatrix}2&1&0\\1&\frac{5}{2}&-1\\0&-1&3\end{pmatrix}$, $b=\begin{pmatrix}2\\-3\\2\end{pmatrix}$.

$\nabla f=2Qx+b=\begin{pmatrix}4x_1+2x_2+2\\2x_1+5x_2-2x_3-3\\-2x_2+6x_3+2\end{pmatrix}=0$, then stationary point $x_0=\begin{pmatrix}-1\\1\\0\end{pmatrix}$

Hessian matrix $\nabla^2 f=\begin{pmatrix}4&2&0\\2&5&-2\\0&-2&6\end{pmatrix}$, where $|4|>0$, $\begin{vmatrix}4&2\\2&5\end{vmatrix}=16>0$, $\begin{vmatrix}4&2&0\\2&5&-2\\0&-2&6\end{vmatrix}=80>0$

It is positive definite, thus x_0 is a local minima.

(b) $f(x)=x^TQx+b^Tx$, where $Q=\begin{pmatrix}\frac{1}{2}&1&0\\1&1&-\frac{1}{2}\\0&-\frac{1}{2}&-\frac{3}{2}\end{pmatrix}$, $b=\begin{pmatrix}0\\1\\-3\end{pmatrix}$

$\nabla f=2Qx+b=\begin{pmatrix}x_1+2x_2\\2x_1+2x_2-x_3+1\\-x_2-3x_3-3\end{pmatrix}=0$, then stationary point $x_0=\begin{pmatrix}-\frac{12}{5}\\\frac{6}{5}\\-\frac{7}{5}\end{pmatrix}$

Hessian matrix $\nabla^2 f=\begin{pmatrix}1&2&0\\2&2&-1\\0&-1&-3\end{pmatrix}$, where $|1|>0$, $\begin{vmatrix}1&2\\2&2\end{vmatrix}=-2<0$

It is indefinite, thus x_0 is a saddle point, neither local minima nor maxima.

2、$A\succeq 0$, then $|3|\geq 0$, $|1|\geq 0$, $|2|\geq 0$,

$\begin{vmatrix}3&-1\\-1&1\end{vmatrix}=3\geq 0$, $\begin{vmatrix}3&2\\2&2\end{vmatrix}=2\geq 0$, $\begin{vmatrix}1&\alpha\\\alpha&2\end{vmatrix}=2-\alpha^2\geq 0$

$\begin{vmatrix}3&-1&2\\-1&1&\alpha\\2&\alpha&2\end{vmatrix}=-3\alpha^2-4\alpha\geq 0$

Therefore, value of $\alpha\in[-\frac{4}{3},0]$.

3、For any $x_1,x_2\in f^{-1}(C)$, we have $f(x_1), f(x_2)\in C$.

Since C is convex, for any $\theta\in[0,1]$, we have

$$\begin{aligned}f(\theta x_1+\bar{\theta}x_2)&=A(\theta x_1+\bar{\theta}x_2)+b\\&=\theta(Ax_1+b)+\bar{\theta}(Ax_2+b)\\&=\theta f(x_1)+\bar{\theta}f(x_2)\in C\end{aligned}$$

Therefore $\theta x_1+\bar{\theta}x_2\in f^{-1}(C)$, indicating that $f^{-1}(C)$ is convex.

4. For any $x_1, x_2 \in C$ and $\theta \in [0,1]$, there exists $u_1, v_1 \in C_1$, $u_2, v_2 \in C_2$ s.t.

$$x_1 = u_1 + v_1, \quad x_2 = u_2 + v_2$$

Then $\theta x_1 + \bar{\theta} x_2 = \theta(u_1 + v_1) + \bar{\theta}(u_2 + v_2)$

$$= (\theta u_1 + \bar{\theta} u_2) + (\theta v_1 + \bar{\theta} v_2)$$

Since C_1, C_2 are convex, $\theta u_1 + \bar{\theta} u_2 \in C_1$, $\theta v_1 + \bar{\theta} v_2 \in C_2$

Therefore $\theta x_1 + \bar{\theta} x_2 \in C$, indicating that C is convex.

5. (a) For any $x_1, x_2 \in \operatorname{int} C$ and $\theta \in [0,1]$, there exists $r_1, r_2 > 0$ s.t.

$$B(x_1, r_1),\ B(x_2, r_2) \subset C$$

Since C is convex, for any $x_1' \in B(x_1, r_1)$, $x_2' \in B(x_2, r_2)$ and $\theta' \in [0,1]$:

$$\theta' x_1' + \bar{\theta}' x_2' \in C$$

Assume $x = \theta x_1 + \bar{\theta} x_2 \notin \operatorname{int} C$, we have $x \in \partial C$, namely $\forall \varepsilon > 0$, $\exists x' \in B(x, \varepsilon)$ s.t.

$$x' \notin C$$

However, when $\varepsilon \to 0$, x' falls into the region $\{\theta' x_1' + \bar{\theta}' x_2'\} \subset C$, then $x' \in C$, which is contradictory with $x' \notin C$.

Therefore, $x = \theta x_1 + \bar{\theta} x_2 \in \operatorname{int} C$, indicating that $\operatorname{int} C$ is convex.

(b) For any $x, y \in \bar{C}$, there exists $\{x_n\} \subset C$, $\{y_n\} \subset C$ s.t. $x_n \to x$, $y_n \to y$

Since C is convex, for any $\theta \in [0,1]$, $\{\theta x_n + \bar{\theta} y_n\} \subset C$ and $\theta x_n + \bar{\theta} y_n \to \theta x + \bar{\theta} y$

Therefore $\theta x + \bar{\theta} y \in \bar{C}$, indicating that $\bar{C}$ is convex.