AI2619 Digital Signal and Image Processing Written Assignment 1

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1. (a) Suppose

$$y(t) = \begin{cases} c, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

we have

$$\begin{aligned} \|x-y\|^2 &= \langle x-y, x-y \rangle \\ &= \int_{-\infty}^{+\infty} [x(t)-y(t)]^2 \,\mathrm{d}t \\ &= \int_{-\infty}^{+\infty} x^2(t)\,\mathrm{d}t + \int_{-\infty}^{+\infty} y^2(t)\,\mathrm{d}t - 2\int_{-\infty}^{+\infty} x(t)y(t)\,\mathrm{d}t \\ &= c^2T - 2c\int_0^T x(t)\,\mathrm{d}t + \int_{-\infty}^{+\infty} x^2(t)\,\mathrm{d}t \\ &= T\left(c - \frac{1}{T}\int_0^T x(t)\,\mathrm{d}t\right)^2 + \int_{-\infty}^{+\infty} x^2(t)\,\mathrm{d}t - \frac{1}{T}\left(\int_0^T x(t)\,\mathrm{d}t\right)^2 \end{aligned}$$

To minimize $\|x-y\|^2$, we have

$$c = \frac{1}{T}\int_0^T x(t)\,\mathrm{d}t$$

Therefore

$$y = \mathcal{P}_V(x) = \begin{cases} \dfrac{1}{T}\displaystyle\int_0^T x(t)\,\mathrm{d}t, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

(b) Let

$$e(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

Any function $f \in V$ can be expressed by $e(t)$ multiplied with a scaler, so V is the span of $e = \{e(t)\}$. Suppose $y = ke$, we have

$$\begin{aligned}\langle x - y, e\rangle &= \langle x - ke, e\rangle \\ &= \int_{-\infty}^{+\infty} [x(t) - ke(t)]\, e(t)\, \mathrm{d}t \\ &= \int_0^T [x(t) - ke(t)]\, \mathrm{d}t \\ &= \int_0^T x(t)\, \mathrm{d}t - kT = 0\end{aligned}$$

which yields

$$k = \frac{1}{T}\int_0^T x(t)\, \mathrm{d}t$$

Therefore

$$y = \mathcal{P}_V(x) = \begin{cases} \frac{1}{T}\int_0^T x(t)\, \mathrm{d}t, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

(c) Note that 1a and 1b give the same function y, which shows the two methods give the same result. From the perspective of signal processing, I prefer the latter because its definition is explicit and inner product can be calculated easily within linear time. However, the former could be better in a more complex space. Minimization problems can be efficiently solved by algorithms like gradient descent, which is increasingly popular nowadays.

2. (a) The coefficients can be determined by several specific values, suppose

$$f(t) = \lambda_{-1}\varphi_{-1}(t) + \lambda_0\varphi_0(t) + \lambda_1\varphi_1(t)$$

Let $t = -1, 0, 1$ respectively, we have

$$\begin{aligned} f(-1) &= \lambda_{-1}\cdot 1 + \lambda_0 \cdot 0 + \lambda_1 \cdot 0 \\ f(0) &= \lambda_{-1}\cdot 0 + \lambda_0 \cdot 1 + \lambda_1 \cdot 0 \\ f(1) &= \lambda_{-1}\cdot 0 + \lambda_0 \cdot 0 + \lambda_1 \cdot 1 \end{aligned}$$

so $\lambda_{-1} = f(-1), \lambda_0 = f(0), \lambda_1 = f(1)$, namely

$$f(t) = f(-1)\varphi_{-1}(t) + f(0)\varphi_0(t) + f(1)\varphi_1(t)$$

Any $f \in V$ can be specified as

$$f(t) = \begin{cases} at + c, & 0 \le t \le 1 \\ bt + c, & -1 \le t < 0 \end{cases}$$

which is equivalent to

$$f(t) = (c - b)\varphi_{-1}(t) + c\varphi_0(t) + (c + a)\varphi_1(t)$$

Therefore, any $f \in V$ can be expressed as a linear combination of $\varphi_{-1}, \varphi_0, \varphi_1$.

(b) By the definition of inner product, we have

$$\begin{aligned}
\langle \varphi_{-1}, \varphi_0 \rangle &= \int_{-1}^{1} \varphi_{-1}(t)\varphi_0(t)\,\mathrm{d}t = \int_{-1}^{1} -t(t+1)\,\mathrm{d}t = \frac{1}{6} \\
\langle \varphi_{-1}, \varphi_1 \rangle &= \int_{-1}^{1} \varphi_{-1}(t)\varphi_1(t)\,\mathrm{d}t = \int_{-1}^{1} 0\,\mathrm{d}t = 0 \\
\langle \varphi_0, \varphi_1 \rangle &= \int_{-1}^{1} \varphi_0(t)\varphi_1(t)\,\mathrm{d}t = \int_{-1}^{1} t(1-t)\,\mathrm{d}t = \frac{1}{6}
\end{aligned}$$

Therefore, only φ_{-1} and φ_1 are orthogonal.

(c) For normalization, we have

$$\begin{aligned}
\|\varphi_{-1}\|^2 &= \langle \varphi_{-1}, \varphi_{-1} \rangle = \int_{-1}^{1} \varphi_{-1}^2(t)\,\mathrm{d}t = \frac{1}{3} \\
\|\varphi_1\|^2 &= \langle \varphi_1, \varphi_1 \rangle = \int_{-1}^{1} \varphi_1^2(t)\,\mathrm{d}t = \frac{1}{3}
\end{aligned}$$

By kindergarten formula

$$\begin{aligned}
\mathcal{P}_{V_1}(\varphi_0) &= \left\langle \varphi_0, \frac{\varphi_{-1}}{\|\varphi_{-1}\|} \right\rangle \frac{\varphi_{-1}}{\|\varphi_{-1}\|} + \left\langle \varphi_0, \frac{\varphi_1}{\|\varphi_1\|} \right\rangle \frac{\varphi_1}{\|\varphi_1\|} \\
&= 3\langle \varphi_0, \varphi_{-1} \rangle \varphi_{-1} + 3\langle \varphi_0, \varphi_1 \rangle \varphi_1 \\
&= 3\varphi_{-1} \int_{-1}^{0} -t(1+t)\,\mathrm{d}t + 3\varphi_1 \int_{-1}^{0} t(1-t)\,\mathrm{d}t \\
&= \frac{1}{2}(\varphi_{-1} + \varphi_1)
\end{aligned}$$

Thus

$$\hat{\varphi}_0 = \varphi_0 - \mathcal{P}_{V_1}(\varphi_0) = \varphi_0 - \frac{1}{2}(\varphi_{-1} + \varphi_1)$$

namely

$$\hat{\varphi}_0(t) = \begin{cases} 1 - \dfrac{3}{2}t, & 0 \le t \le 1 \\ 1 + \dfrac{3}{2}t, & -1 \le t < 0 \end{cases}$$

Since $\langle \hat{\varphi}_0, \varphi_{-1} \rangle = \langle \hat{\varphi}_0, \varphi_1 \rangle = \langle \varphi_{-1}, \varphi_1 \rangle = 0$, we can claim that $\{\varphi_{-1}, \hat{\varphi}_0, \varphi_1\}$ is an orthogonal basis of V. They are plotted on the same graph as follow

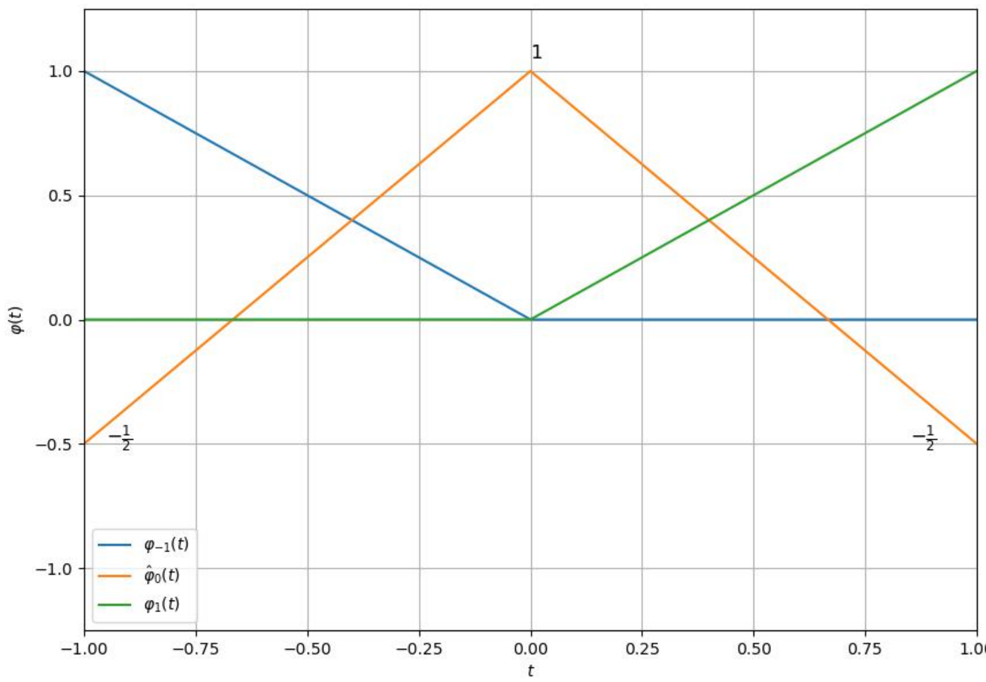

Figure 1: function family $\{\varphi_{-1}, \hat{\varphi}_0, \varphi_1\}$

(d) The best approximation of g is a function $f \in V$ that minimizes the difference between f and g, namely

$$f = \arg\min_{f\in V} \|f - g\|$$

In other words, f is the projection of g onto V. For normalization, we have

$$\|\hat{\varphi}_0\|^2 = \langle \hat{\varphi}_0, \hat{\varphi}_0 \rangle = \int_{-1}^{1} \hat{\varphi}_0^2(t)\,\mathrm{d}t = \frac{1}{2}$$

Then

$$\begin{aligned}
\mathcal{P}_V(g) &= \left\langle g, \frac{\varphi_{-1}}{\|\varphi_{-1}\|} \right\rangle \frac{\varphi_{-1}}{\|\varphi_{-1}\|} + \left\langle g, \frac{\hat{\varphi}_0}{\|\hat{\varphi}_0\|} \right\rangle \frac{\hat{\varphi}_0}{\|\hat{\varphi}_0\|} + \left\langle g, \frac{\varphi_1}{\|\varphi_1\|} \right\rangle \frac{\varphi_1}{\|\varphi_1\|} \\
&= 3\langle g, \varphi_{-1}\rangle \varphi_{-1} + 2\langle g, \hat{\varphi}_0\rangle \hat{\varphi}_0 + 3\langle g, \varphi_1\rangle \varphi_1 \\
&= 3\varphi_{-1} \int_{-1}^{0} g(t)\varphi_{-1}(t)\,\mathrm{d}t + 2\hat{\varphi}_0 \int_{-1}^{1} g(t)\hat{\varphi}_0(t)\,\mathrm{d}t + 3\varphi_1 \int_{0}^{1} g(t)\varphi_1(t)\,\mathrm{d}t \\
&= \frac{3}{2}\varphi_{-1} + \frac{7}{6}\hat{\varphi}_0 + \frac{2}{3}\varphi_1
\end{aligned}$$

Therefore

$$f(t) = \mathcal{P}_V(g) = \begin{cases} \dfrac{7}{6} - \dfrac{13}{12}t, & 0 \le t \le 1 \\ \dfrac{7}{6} + \dfrac{1}{4}t, & -1 \le t < 0 \end{cases}$$

Both $g(t)$ and $f(t)$ are plotted on the same graph as follow

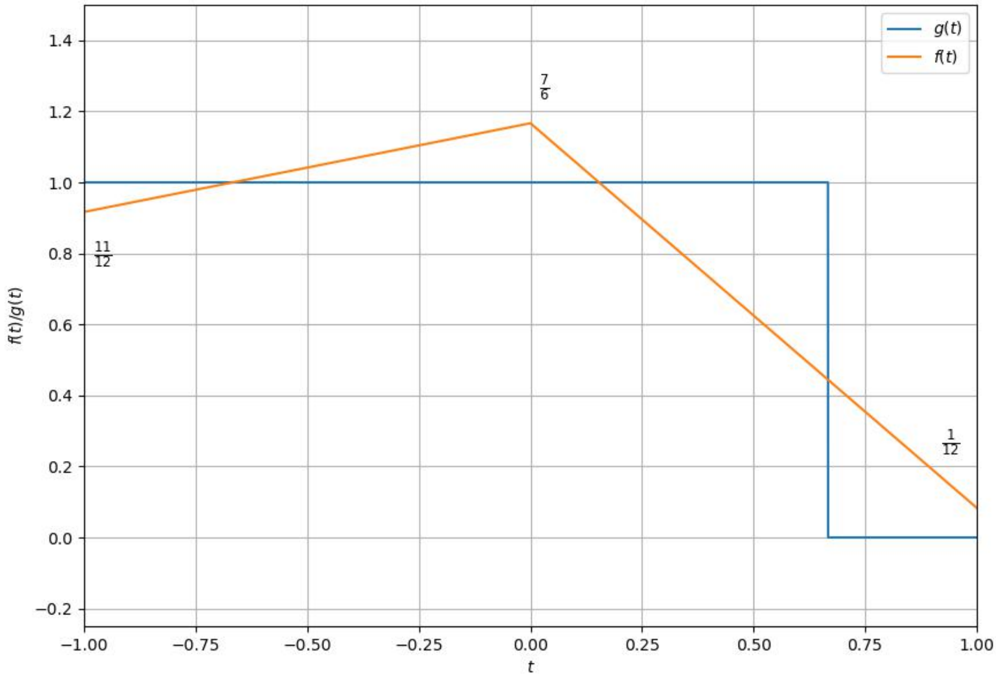

Figure 2: function $g(t)$ and approximation $f(t)$