CS2601 Linear and Convex Optimization
Homework 9

Due: 2022.12.16

For this assignment, you should submit a **single** pdf file as well as your source code (.py or .ipynb files). The pdf file should include all necessary figures, the outputs of your Python code, and your answers to the questions. Do NOT submit your figures in separate files. Your answers in any of the .py or .ipynb files will NOT be graded.

1. Consider the following problem,

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^2} \quad & f(\boldsymbol{x}) = x_1^2 + (x_2-1)^2 \\ \text{s.t.} \quad & g_1(\boldsymbol{x}) = x_1 - x_2 - 1 \le 0 \\ & g_2(\boldsymbol{x}) = (x_1-1)^2 + x_2^2 - 1 \le 0 \end{aligned}$$

Write down the KKT conditions and find the optimal point $\boldsymbol{x}^*$ and the corresponding Lagrange multipliers **Consider all four cases as the example in lecture slides**.

2. Consider the following problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & -(x_1-1)^2 - (x_2-1)^2 + 1 \ge 0 \\ & -(x_1-1)^2 - x_2^2 + 1 \ge 0 \end{aligned}$$

For each of the following points, determine whether it is an optimal solution to the above problem and show your arguments,

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{x}^{(3)} = \begin{bmatrix} 1-\frac{\sqrt{2}}{2} \\ 1-\frac{\sqrt{2}}{2} \end{bmatrix}$$

Hint: Try if you can find Lagrange multipliers satisfying the KKT conditions. Note you can easily check which constraints are active.

3. Lasso. Implement the projection onto ℓ_1 ball. Use projected gradient descent to solve the Lasso problem on slide 12 of §11, you should complete the implementation of projected gradient descent method in `proj_gd.py`.

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \frac{1}{2}\|\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}\|_2^2 \\ \text{s.t.} \quad & \|\boldsymbol{w}\|_1 \le t \end{aligned} \qquad \boldsymbol{X} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad t = 1$$

Use the initial point $\boldsymbol{w}_0 = (-1, 0.5)^T$ and 0.1 step size. Report the solution and the number of iterations. Plot the trajectory of $\boldsymbol{w}_k$ and the gap $f(\boldsymbol{w}_k) - f(\boldsymbol{w}^*)$.

4. Let $\boldsymbol{x} \in \mathbb{R}^3$. Consider

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) = e^{2x_1} + e^{x_2} + e^{x_3} \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1 \end{aligned} \tag{1}$$

(a). Solve problem (1) by the Lagrange multiplier method. Show the optimal solution $\boldsymbol{x}^*$, the Lagrange multiplier λ^* and the optimal value f^*.

(b). Find the closed-form expression for the Newton direction at a feasible $\boldsymbol{x}$ by solving the KKT system.

(c). Implement the constrained Newton's method on slide 13 of §12 in the `newton_eq` function of `newton.py`. The functions `numpy.block` and `numpy.linalg.solve` might be useful. Use your implementation to solve (1) with the initial point $\boldsymbol{x}_0 = (0, 1, 0)^T$. Show the output.