[Homework 1] Review of Probability Theory

- 如果你使用mac系统和chrome浏览器，遇到了数学字体渲染问题，请换用safari打开。

作业要求

1. 本次作业允许讨论，但必须自己独立完成所有证明的写作（即不允许直接复制粘贴，或者直接照抄同学或者参考资料的现成证明）。
2. 如果有与同学讨论，必须注明所有参与讨论同学的名字。
3. 如果有参考其他资料完成作业，必须注明参考来源。

Probability Space of Tossing Coins

Let us construct the probability space of tossing an infinite sequence of independent fair coins. Let $\Omega = \{0,1\}^{\mathbb{N}}$. We can write each $\omega \in \Omega$ as an infinite sequence $\omega = (\omega_1, \omega_2, \dots)$ where $\omega_i \in \{0,1\}$.

1. Let $n \in \mathbb{N}$. For every $s = (s_1, \dots, s_n) \in \{0,1\}^n$, let

$$C_s = \{\omega \in \Omega \mid \omega_1 = s_1, \dots, \omega_n = s_n\}.$$

Prove that for every $n \in \mathbb{N}$, the collection $\{C_s\}_{s \in \{0,1\}^n}$ forms a partition of Ω.

1. Let $\mathcal{F}_n$ be the σ-algebra generated by $\{C_s\}_{s \in \{0,1\}^n}$ (that is, the minimal σ-algebra containing sets in $\{C_s\}_{s \in \{0,1\}^n}$). Note that $\mathcal{F}_n$ is called the σ-algebra of *tossing n coins*. Prove that there exists a bijection between $\mathcal{F}_n$ and $2^{\{0,1\}^n}$.
2. Prove that $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \dots$ is increasing. The collection $\{\mathcal{F}_n\}_{n \geq 1}$ is called a *filtration*.
3. Let $\mathcal{F}_\infty = \bigcup_{n \geq 1} \mathcal{F}_n$ [1]. Prove that $\mathcal{F}_\infty$ is an algebra [2] (not necessarily a σ-algebra) and $\mathcal{F}_\infty \neq 2^\Omega$.
4. Let $\mathcal{B}(\Omega) \triangleq \sigma(\mathcal{F}_\infty)$ be the minimal σ-algebra containing $\mathcal{F}_\infty$. Prove that for any $\omega \in \Omega$, it holds that $\{\omega\} \in \mathcal{B}(\Omega) \setminus \mathcal{F}_\infty$.
5. Prove that for every $A \in \mathcal{F}_\infty$, there exist some $n \in \mathbb{N}$ and $s_1, \dots, s_k \in \{0,1\}^n$ such that $A = C_{s_1} \cup \dots \cup C_{s_k}$. Although the choice of n might not be unique, prove that the value $\frac{k}{2^n}$ only depends on A.
6. Prove that there exists a unique probability measure $P : \mathcal{B}(\Omega) \to [0,1]$ satisfying for every $A \in \mathcal{F}_\infty$, $P(A) = \frac{k}{2^n}$ where k and n are defined in the last question. (You can use the Carathéodory's extension theorem)

Then $(\Omega, \mathcal{B}(\Omega), P)$ is our probability space for tossing coins, and it is isomorphic to the Lebesgue measure on $[0,1]$.

8. Formalize $X \sim \texttt{Geom}(1/2)$ in this probability space.

Conditional Expectation

(In this problem, all random variables take *discrete* value)

1. Let X be a random variable and $f : \mathbb{R} \to \mathbb{R}$ be a measurable function (that is, for every borel set $A \in \mathscr{R}$, $f^{-1}(A) \in \mathscr{R}$). We usually use $f(X)$ to denote the random variable: $\omega \in \Omega \mapsto f(X(\omega)) \in \mathbb{R}$. Prove that $f(X)$ is $\sigma(X)$-measurable.
2. Let Y, Y' be two random variables such that $\sigma(Y) = \sigma(Y')$. Prove that $\mathbf{E}[X \mid Y] = \mathbf{E}[X \mid Y']$.

(*The fact you just proved should convince you that the conditional expectation* $\mathbf{E}[X \mid Y]$ *only depends on the* σ*-algebra* $\sigma(Y)$ *(but not the value of* Y*). Let* Ω *be the set of outcomes and* $X : \Omega \to \mathbb{R}$ *be a random variable. Let* $\mathcal{F} = \sigma(Y)$ *be a* σ*-algebra for some random varialbe* Y*. Then we define* $\mathbf{E}[X \mid \mathcal{F}] \triangleq \mathbf{E}[X \mid Y]$*. This notation will be useful in the future class*)

3. (The coarser always wins) Let $\mathcal{F}_1, \mathcal{F}_2$ be two σ-algebra where $\mathcal{F}_1 = \sigma(Y_1)$ and $\mathcal{F}_2 = \sigma(Y_2)$. Assume that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and let $X : \Omega \to \mathbb{R}$ be a random variable. Prove that $\mathbf{E}[\mathbf{E}[X \mid \mathcal{F}_1] \mid \mathcal{F}_2] = \mathbf{E}[\mathbf{E}[X \mid \mathcal{F}_2] \mid \mathcal{F}_1] = \mathbf{E}[X \mid \mathcal{F}_1]$.
4. (bonus) The definition of the conditional expectation introduced in the class only applies to discrete-valued random variables. Search on the internet or in the library to find the definition of the conditional expectation $\mathbf{E}[X \mid Y]$ for *general* X and Y.

(第四题自己看就行了，不用写任何东西)

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1. $x \in \bigcup_{n \geq 1} \mathcal{F}_n \iff \exists n \geq 1 : x \in \mathcal{F}_n$. ↩
2. A set $\mathcal{F}$ is an algebra if for every $A, B \in \mathcal{F}$, it holds $A^c \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$. ↩