AI2619 Digital Signal and Image Processing Written Assignment 3

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1. (a) $x[n] = \cos\left(\frac{\pi}{2}n\right)$ can be written as $x[n] = \{1, 0, -1, 0\}$, whose 4-point DFT is

$$X[k] = \sum_{n=0}^{3} x[n] W_4^{kn} = 1 - e^{-j\pi k}$$

which can be written as

$$X[k] = \{0, 2, 0, 2\}$$

where $k = 0, 1, 2, 3$.

(b) $h[n] = 2^n$ can be written as $h[n] = \{1, 2, 4, 8\}$, whose 4-point DFT is

$$H[k] = \sum_{n=0}^{3} h[n] W_4^{kn} = \frac{-15}{1 - 2e^{-j\frac{\pi}{2}k}}$$

which can be written as

$$H[k] = \{15, -3 + j6, -5, -3 - j6\}$$

where $k = 0, 1, 2, 3$.

(c) The circular convolution of $x[n]$ and $h[n]$ is

$$y[n] = x[n] \textcircled{4} h[n] = \sum_{k=0}^{3} x[k] h\left[(n-k)_4\right]$$

which yields

$$\begin{aligned} y[0] &= x[0]h[0] + x[1]h[3] + x[2]h[2] + x[3]h[1] = -3 \\ y[1] &= x[0]h[1] + x[1]h[0] + x[2]h[3] + x[3]h[2] = -6 \\ y[2] &= x[0]h[2] + x[1]h[1] + x[2]h[0] + x[3]h[3] = 3 \\ y[3] &= x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0] = 6 \end{aligned}$$

which can be written as

$$y[n] = \{-3, -6, 3, 6\}$$

where $n = 0, 1, 2, 3$.

(d) The multiplication of $X[k]$ and $H[k]$ is

$$Y[k] = X[k]H[k] = \{0, -6 + j12, 0, -6 - j12\}$$

Hence

$$\begin{aligned} y[n] &= \mathscr{F}^{-1}\{Y[k]\} \\ &= \frac{1}{4}\sum_{k=0}^{3} Y[k]W_4^{-kn} \\ &= \frac{1}{2}\left[(-3 + j6)e^{j\frac{\pi}{2}n} + (-3 - j6)e^{j\frac{\pi}{2}3n}\right] \end{aligned}$$

which can be written as

$$y[n] = \{-3, -6, 3, 6\}$$

where $n = 0, 1, 2, 3$.

2. (a) Since $-\frac{N}{2} : \frac{N}{2}$ forms an $(N + 1)$-point circle, we have

$$\begin{aligned} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} W_{N+1}^{-n(m-k)} &= \sum_{n=0}^{N} W_{N+1}^{-n(m-k)} \\ &= \begin{cases} \sum_{n=0}^{N} 1, & W_{N+1}^{-(m-k)} = 1 \\ \dfrac{1 - W_{N+1}^{-(N+1)(m-k)}}{1 - W_{N+1}^{-(m-k)}}, & W_{N+1}^{-(m-k)} \neq 1 \end{cases} \\ &= \begin{cases} N + 1, & (m - k) \mid (N + 1) \\ 0, & (m - k) \nmid (N + 1) \end{cases} \\ &= (N + 1) \sum_{r=-\infty}^{+\infty} \delta\left[m - k - r(N + 1)\right] \end{aligned}$$

which verifies the orthogonality property.

(b) Since the forward DFT is defined as

$$F_k = \frac{1}{N + 1} \sum_{p=-\frac{N}{2}}^{\frac{N}{2}} f_p W_{N+1}^{-pk}$$

Multiply both sides by W_{N+1}^{nk} and sum over k from $-\frac{N}{2}$ to $\frac{N}{2}$, we have

$$\begin{aligned}
\sum_{k=-\frac{N}{2}}^{\frac{N}{2}} F_k W_{N+1}^{nk} &= \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{N+1} \sum_{p=-\frac{N}{2}}^{\frac{N}{2}} f_p W_{N+1}^{-pk} W_{N+1}^{nk} \\
&= \frac{1}{N+1} \sum_{p=-\frac{N}{2}}^{\frac{N}{2}} f_p \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} W_{N+1}^{-k(p-n)} \\
&= \frac{1}{N+1} \sum_{p=-\frac{N}{2}}^{\frac{N}{2}} f_p \cdot (N+1) \sum_{r=-\infty}^{+\infty} \delta\left[p-n-r(N+1)\right] \\
&= \sum_{p=-\frac{N}{2}}^{\frac{N}{2}} f_p \cdot \delta\left[p-n\right] = f_n
\end{aligned}$$

Hence, the inverse DFT is specified as

$$f_n = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} F_k W_{N+1}^{nk}$$

which holds for $n = -\frac{N}{2} : \frac{N}{2}$.

(c) For any $n = -\frac{N}{2} : \frac{N}{2}$, we have

$$\begin{aligned}
\max\left\{x_n\right\} &= x_{\frac{N}{2}} = \frac{N}{N+1} \cdot \frac{A}{2} < \frac{A}{2} \\
\min\left\{x_n\right\} &= x_{-\frac{N}{2}} = -\frac{N}{N+1} \cdot \frac{A}{2} > -\frac{A}{2}
\end{aligned}$$

Hence, $x_n = \frac{n}{N+1} \cdot A$ will not include either $-\frac{A}{2}$ or $\frac{A}{2}$. Note that

$$\begin{aligned}
\lim_{N\to\infty} x_{\frac{N}{2}} &= \lim_{N\to\infty} \frac{N}{N+1} \cdot \frac{A}{2} = \frac{A}{2} \\
\lim_{N\to\infty} x_{-\frac{N}{2}} &= \lim_{N\to\infty} -\frac{N}{N+1} \cdot \frac{A}{2} = -\frac{A}{2}
\end{aligned}$$

Hence, $x_{\pm\frac{N}{2}}$ will approach $\pm\frac{A}{2}$ as N increases.

3. The 8-point DFT of $\bar{f}_n$ is

$$\bar{F}_k = \sum_{n=-3}^{4} \bar{f}_n W_8^{kn} = \sum_{n=-3}^{4} \bar{f}_n e^{-j\frac{\pi}{4}kn}$$

which yields

$$
\begin{aligned}
\bar{F_{-3}} &= \sum_{n=-3}^{4} \bar{f}_n e^{j\frac{3\pi}{4}n} = -1 + j\left(2\sqrt{2} - 2\right) \\
\bar{F_{-2}} &= \sum_{n=-3}^{4} \bar{f}_n e^{j\frac{\pi}{2}n} = 1 \\
\bar{F_{-1}} &= \sum_{n=-3}^{4} \bar{f}_n e^{j\frac{\pi}{4}n} = -1 + j\left(2\sqrt{2} + 2\right) \\
\bar{F_0} &= \sum_{n=-3}^{4} \bar{f}_n = 1 \\
\bar{F_1} &= \sum_{n=-3}^{4} \bar{f}_n e^{-j\frac{\pi}{4}n} = -1 - j\left(2\sqrt{2} + 2\right) \\
\bar{F_2} &= \sum_{n=-3}^{4} \bar{f}_n e^{-j\frac{\pi}{2}n} = 1 \\
\bar{F_3} &= \sum_{n=-3}^{4} \bar{f}_n e^{-j\frac{3\pi}{4}n} = -1 - j\left(2\sqrt{2} - 2\right) \\
\bar{F_4} &= \sum_{n=-3}^{4} \bar{f}_n e^{-j\pi n} = 1
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
\bar{F_k} = \Big\{&-1 + j\left(2\sqrt{2} - 2\right), 1, -1 + j\left(2\sqrt{2} + 2\right), 1, \\
&-1 - j\left(2\sqrt{2} + 2\right), 1, -1 - j\left(2\sqrt{2} - 2\right), 1\Big\}
\end{aligned}
$$

which holds for $k = -3 : 4$. Since $\bar{F_0} = 1$, we can claim that $\bar{F_k}$ is neither odd nor imaginary. This error comes from the unreasonable sampling. Note that the sampling points are not symmetric, making its periodic extension look like

$$
\{\ldots, 1, -1, -1, -1, 0, 1, 1, 1, 1, \ldots\}
$$

which is not odd and real. To fix the error, the input sequence can be defined as

$$
\hat{f_n} = \{-1, -1, -1, -1, 0, 1, 1, 1, 1\}
$$

where $n = -4 : 4$ and $x_n = \frac{n}{9}$. Then we have

$$
\begin{aligned}
\hat{F_{-k}} &= \sum_{n=-4}^{4} \hat{f}_n W_9^{-kn} = -\sum_{n=-4}^{4} \hat{f}_n W_9^{kn} = -\hat{F}_k \\
\hat{F_k}^* &= \sum_{n=-4}^{4} \hat{f}_n \left(W_9^{kn}\right)^* = \sum_{n=-4}^{4} \hat{f}_n W_9^{-kn} = -\hat{F}_k
\end{aligned}
$$

which verifies that $\hat{F_k}$ is odd and imaginary.

4. (a) Let $L = 50$ and $P = 10$ denote the length of $x[n]$ and $h[n]$ respectively. We might as well let $x[n] = 1, 0 \leq n \leq 49$ and $h[n] = 1, 0 \leq n \leq 9$. The maximum number of nonzero values in $x[n] * h[n]$ is

$$L + P - 1 = 59$$

(b) Let $y_c[n] = x[n] \circledcirc y[n]$ and $y_l[n] = x[n] * h[n]$ denote the 50-point circular convolution and the linear convolution respectively. The circular convolution can be expressed as time aliasing of the shifted linear convolution, namely

$$y_c[n] = \sum_{r=-\infty}^{+\infty} y_l[n - rL]$$

Since the maximum length of $y_l[n]$ is $L + P - 1$, the possible time aliasing happens when $0 \leq n \leq P - 2$. Then the circular convolution can be expressed as

$$y_c[n] = \begin{cases} y_l[n] + y_l[n + 50], & 0 \leq n \leq 8 \\ y_l[n], & 9 \leq n \leq 49 \end{cases}$$

For $9 \leq n \leq 49$, we have

$$y_l[n] = y_c[n] = 10$$

For $0 \leq n \leq 4$, we have

$$y_l[n + 50] = y_c[n] - y_c[n + 50] = 10 - 5 = 5$$

Therefore, we can conclude that

$$y_l[n] = \begin{cases} 5, & 0 \leq n \leq 4 \\ 10, & 9 \leq n \leq 49 \\ 5, & 50 \leq n \leq 54 \end{cases}$$

Note that the value of $y_l[n]$ for $5 \leq n \leq 8$ and $55 \leq n \leq 58$ cannot be determined.