

CS2601 Linear and Convex Optimization

Homework 10 Solution

Due: 2022.12.29

For this assignment, you should submit a **single** pdf file as well as your source code (.py or .ipynb files). **The pdf file should include all necessary figures, the outputs of your Python code, and your answers to the questions.** Do NOT submit your figures in separate files. Your answers in any of the .py or .ipynb files will NOT be graded.

1. Consider the LP in standard form,

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^n} \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

We are going to solve it using the barrier method.

- (a). Write down the approximating equality constrained problem.
- (b). Write down the gradient and Hessian matrix of the objective function you find in (a).
- (c). Implement the barrier method for solving a generic LP in standard form with a given feasible initial point. Complete the functions `centering_step` and `barrier` in `LP.py`. For the centering step, i.e. line 3 on slide 14 of §13, you can use your implementation of constrained Newton's method in HW 9 Problem 4(c) by defining the penalized objective function and its derivatives inside `centering_step`.
- (d). Consider the following LP

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^2} \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 8 \\ & x_1 - x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Convert it to standard form and then use your implementation in (c) to solve it. You can find a feasible initial point by setting $x_1 = 2, x_2 = 1$ and the slack variables appropriately. Show the output. Note `p2.py` also plots the projection of the iterates onto the x_1, x_2 coordinates.

2. Consider the LP in Problem 1(d).

- (a). Find the dual LP in the standard form, i.e. with all four dual variables.

- (b). Find the symmetric dual LP, i.e. without the dual variables for the primal nonnegativity constraints.
- (c). Solve the dual LP in (b) graphically. Compare the dual optimal value and the primal optimal value computed in Problem 1(d).
- (d). Solve the dual LP in (a) using your implementation in Problem 1(c). Note you need to convert the maximization problem into a minimization problem. You can use the feasible initial point $\mu_0 = (4, 1, 2, 6)^T$. Show the dual optimal solution and dual optimal value.

3. Consider the following optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}} \quad & f(x) = \log(2 + e^x) \\ \text{s.t.} \quad & x \geq 0 \end{aligned}$$

- (a). Find the optimal solution and the optimal value.
- (b). Find the dual function and the dual problem.
- (c). Find the dual optimal solution and the dual optimal value. Does strong duality hold?

4. Consider the optimization problem ,

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^2} \quad & f(\mathbf{x}) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & (x_1 - 2)^2 + (x_2 - 1)^2 \leq 1 \\ & (x_1 - 2)^2 + (x_2 + 1)^2 \leq 1 \end{aligned}$$

- (a). Find the Lagrange dual function and the dual problem.
- (b). Find the dual optimal value ϕ^* . Does strong duality hold?
- (c). Does Slater's condition hold? What can you conclude about the necessity of Slater's condition for strong duality?
- (d). Is the dual optimal value ϕ^* attained by any dual feasible point? What does this say about whether KKT conditions hold at the primal optimal solution? Explain your answer.

5. Consider the following minimization problem

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^2} \quad & f(\mathbf{x}) = \begin{cases} x_1^5 + x_2^5, & \text{if } \mathbf{x} \geq \mathbf{0} \\ +\infty, & \text{otherwise} \end{cases} \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \end{aligned} \tag{P1}$$

Note the domain of f is $\text{dom } f = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \geq \mathbf{0}\}$ and the domain of the problem is $D = \text{dom } f$.

- (a). Since D is not the entire space, the dual function of this problem is defined by

$$\phi(\mu) = \inf_{\mathbf{x} \in D} \{f(\mathbf{x}) + \mu(1 - x_1 - x_2)\}$$

Find the explicit expression of $\phi(\mu)$.

(b). Find the dual optimal solution.

(c). What is the primal optimal value? Hint: Note f is convex on its domain.

(d). Note the primal problem (P1) is equivalent to

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^2} \quad & f_1(\mathbf{x}) = x_1^5 + x_2^5 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{P2}$$

What's the dual function of this equivalent problem (P2)? Does strong duality hold for (P2)?

Remark. Note $\text{dom } f_1 = \mathbb{R}^2$ and f_1 is not convex. This problem shows that equivalent primal problems can have very different dual problems. Not all dual problems are equally useful.