CS2601 Linear and Convex Optimization
Homework 8

Due: 2022.12.7

1. Consider the equality constrained quadratic program

$$\begin{aligned} \min_{x_1,x_2} \quad & f(x_1,x_2) = x_1^2 + x_1x_2 + x_2^2 - x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 = 1 \end{aligned}$$

(a). Find the optimal solution x_1^*, x_2^* by reduction to an unconstrained problem.

(b). Find the optimal solution and the corresponding Lagrange multiplier λ^* using the Lagragian multipliers method.

2. Solve the following problem,

$$\begin{aligned} \min_{x_1,x_2} \quad & f(x_1,x_2) = x_1x_2 + x_1^2 \\ \text{s.t.} \quad & x_1^2 + \frac{1}{8}x_2^2 = 1 \end{aligned}$$

3. Consider the equality constrained quadratic program

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{g}^T\boldsymbol{x} + c \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

where $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, $\boldsymbol{Q} \succ \boldsymbol{O}$, $\boldsymbol{g} \in \mathbb{R}^n$, $c \in \mathbb{R}$, $\boldsymbol{A} \in \mathbb{R}^{k\times n}$ with rank $\boldsymbol{A} = k$, and $\boldsymbol{b} \in \mathbb{R}^k$.

(a). Write down the Lagrange condition for this problem.

(b). Find a closed form solution for the optimal solution $\boldsymbol{x}^*$ and the corresponding Lagrange multiplier $\boldsymbol{\lambda}^*$. Hint: Show $\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T \succ \boldsymbol{O}$ and hence is invertible.

(c). Use part (b) to find the projection $\mathrm{Proj}_S(\boldsymbol{x}_0)$ of a point $\boldsymbol{x}_0$ onto the the affine space $S = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$, i.e. solve

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}_0\|_2^2 \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

When $\boldsymbol{x}_0 = \boldsymbol{0}$, you should recover the result on slide 11 of §9.

(d). Consider a hyperplane $P = \{\boldsymbol{x} : \boldsymbol{w}^T \boldsymbol{x} = b\}$. Use the result in (c) to find the distance $\mathrm{d}(\boldsymbol{x}_0, P)$ between $\boldsymbol{x}_0$ and P. You should recover the result on slide 13 of §1.