AI2619 Digital Signal and Image Processing
Written Assignment 4

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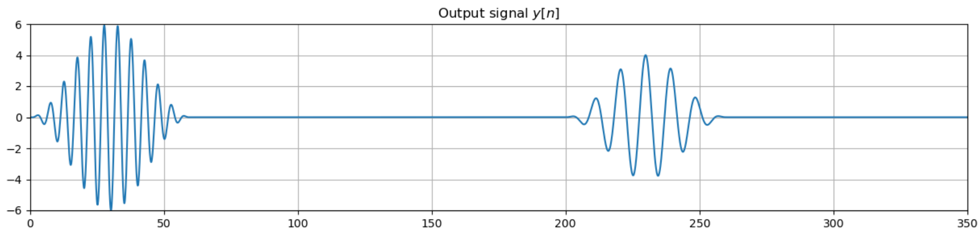
1. Divide $x[n]$ into 3 wave packets

$$\begin{aligned}x_1[n] &= x[n] \cdot (u[n] - u[n-60]) \\ x_2[n] &= x[n] \cdot (u[n-60] - u[n-120]) \\ x_3[n] &= x[n] \cdot (u[n-120] - u[n-180])\end{aligned}$$

which corresponds to 3 frequencies in $\left|X\left(e^{j\omega}\right)\right|$. According to the magnitude response $\left|H\left(e^{j\omega}\right)\right|$, the wave packets are approximately scaled by 4, 6 and 0 respectively. According to the phase response $\angle\left[H\left(e^{j\omega}\right)\right]$, the sinsoids are approximately biased by $-50°$, $-100°$ and $-100°$ respectively. According to the group delay response $\operatorname{grd}\left[H\left(e^{j\omega}\right)\right]$, the wave packets are approximately shifted by 140, 0 and 0 respectively.



Output signal $y[n]$ is roughly sketched in the figure above, where the parameters are estimated.

2. (a) By definition of impulse response

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{5}{4}z^{-1} - \frac{3}{2}z^{-2}}$$

which yields

$$Y(z) - \frac{5}{4}z^{-1}Y(z) - \frac{3}{2}z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

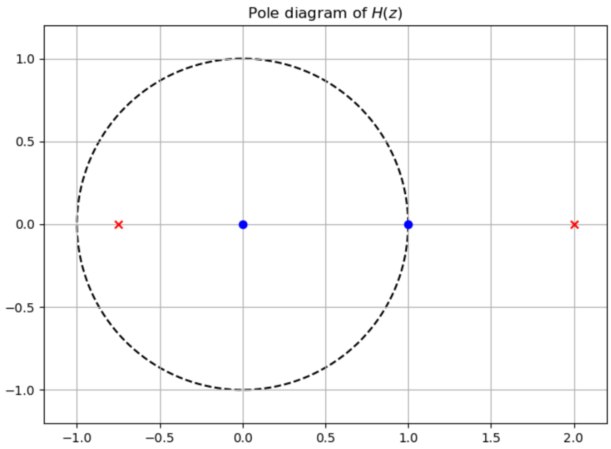
Therefore, the difference equation is

$$y[n] - \frac{5}{4}y[n-1] - \frac{3}{2}y[n-2] = x[n] - x[n-1]$$

(b) Reformulate the function as

$$H(z) = \frac{z(z-1)}{(z-2)(z+\frac{3}{4})}$$

The poles and zeros are shown in the figure below.



Since $\omega = \frac{\pi}{2}$, we have

$$H\left(e^{j\omega}\right) = H\left(z\right)|_{z=e^{j\frac{\pi}{2}}} = \frac{j(j-1)}{(j-2)(j+\frac{3}{4})} = \frac{12}{25} + j\frac{4}{25}$$

which yields

$$\left|H\left(e^{j\omega}\right)\right| = \frac{4\sqrt{10}}{25}$$
$$\angle\left[H\left(e^{j\omega}\right)\right] = \arctan\frac{1}{3}$$

Hence, the output is $y[n] = A\cos\left(\omega n + \phi\right)$, where $A = \frac{4\sqrt{10}}{25}$ and $\phi = \arctan\frac{1}{3}$.

3. (a) The phase response is

$$\angle\left[H_a\left(e^{j\omega}\right)\right] = -\arctan\frac{b\sin\omega}{a+b\cos\omega}$$

The group delay is

$$\operatorname{grd}\left[H_a\left(e^{j\omega}\right)\right] = -\frac{\mathrm{d}}{\mathrm{d}\omega}\left\{\angle\left[H_a\left(e^{j\omega}\right)\right]\right\} = \frac{b(a\cos\omega+b)}{a^2+b^2+2ab\cos\omega}$$

(b) The phase response is

$$\angle\left[H_b\left(e^{j\omega}\right)\right] = \arctan\frac{c\sin\omega}{1+c\cos\omega}$$

The group delay is

$$\operatorname{grd}\left[H_b\left(e^{j\omega}\right)\right] = -\frac{\mathrm{d}}{\mathrm{d}\omega}\left\{\angle\left[H_b\left(e^{j\omega}\right)\right]\right\} = -\frac{c(\cos\omega+c)}{1+c^2+2c\cos\omega}$$

(c) The phase response is

$$\angle\left[H_c\left(e^{j\omega}\right)\right] = -\arctan\frac{b\sin\omega}{a+b\cos\omega} + \arctan\frac{c\sin\omega}{1+c\cos\omega}$$

The group delay is

$$\operatorname{grd}\left[H_c\left(e^{j\omega}\right)\right] = -\frac{\mathrm{d}}{\mathrm{d}\omega}\left\{\angle\left[H_c\left(e^{j\omega}\right)\right]\right\} = \frac{b(a\cos\omega+b)}{a^2+b^2+2ab\cos\omega} - \frac{c(\cos\omega+c)}{1+c^2+2c\cos\omega}$$

(d) The phase response is

$$\angle\left[H_d\left(e^{j\omega}\right)\right]=\arctan\frac{c\sin\omega}{1+c\cos\omega}+\arctan\frac{d\sin\omega}{1+d\cos\omega}$$

The group delay is

$$\operatorname{grd}\left[H_d\left(e^{j\omega}\right)\right]=-\frac{\mathrm{d}}{\mathrm{d}\omega}\left\{\angle\left[H_d\left(e^{j\omega}\right)\right]\right\}=-\frac{c(\cos\omega+c)}{1+c^2+2c\cos\omega}-\frac{d(\cos\omega+d)}{1+d^2+2d\cos\omega}$$

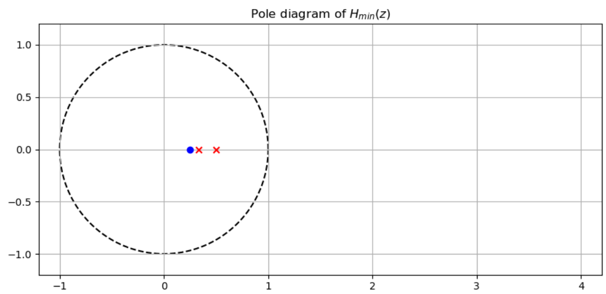
4. According to the pole diagram of $H(z)$, we have

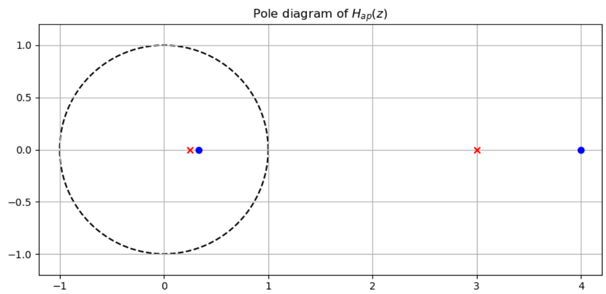
$$H(z)=\frac{z-4}{(z-\frac{1}{2})(z-3)}=\frac{(z-\frac{1}{3})(z-4)}{(z-\frac{1}{4})(z-3)}\cdot\frac{z-\frac{1}{4}}{(z-\frac{1}{3})(z-\frac{1}{2})}$$

which yields

$$H_{\min}(z)=\alpha\cdot\frac{z-\frac{1}{4}}{(z-\frac{1}{3})(z-\frac{1}{2})}$$
$$H_{\text{ap}}(z)=\beta\cdot\frac{(z-\frac{1}{3})(z-4)}{(z-\frac{1}{4})(z-3)}$$

Their pole diagrams are shown in the figures below.





We can see that all the poles and zeros have been included in $H_{\text{ap}}(z)$, and additional poles and zeros cannot be cancelled out by $H_{\min}(z)$. Therefore, the decomposition is unique up to a scale factor.

5. Assume $D(\cos\theta, \sin\theta)$ where $\theta \in [0, 2\pi)$ and $P(p, 0)$, $Z(q, 0)$, we have

$$|DP| = \sqrt{(\cos\theta - p)^2 + \sin^2\theta} = \sqrt{1 + p^2 - 2p\cos\theta}$$
$$|DZ| = \sqrt{(\cos\theta - q)^2 + \sin^2\theta} = \sqrt{1 + q^2 - 2q\cos\theta}$$

Since $\frac{|DZ|}{|DP|} = \frac{1}{\alpha}$, we have

$$\frac{1 + q^2 - 2q\cos\theta}{1 + p^2 - 2p\cos\theta} = \frac{1}{\alpha^2}$$

which is equivalent to

$$2\cos\theta\left(p - q\alpha^2\right) = \left(1 + p^2\right) - \alpha^2\left(1 + q^2\right)$$

The equation holds for any $\theta \in [0, 2\pi)$, which yields

$$\begin{cases} p = q\alpha^2 \\ 1 + p^2 = \alpha^2\left(1 + q^2\right) \end{cases}$$

Eliminate α and we have

$$(p - q)(pq - 1) = 0$$

Since $p \neq q$, we have $pq = 1$, namely

$$|OZ| \cdot |OP| = 1$$