CS2601 Linear and Convex Optimization
Homework 3

Due: 2022.10.22

1. Let C_1, C_2 be nonempty, convex sets in $\mathbb{R}^n$. Suppose $\operatorname{int} C_1 \neq \emptyset$ and $(\operatorname{int} C_1) \cap C_2 = \emptyset$, where $\operatorname{int} C_1$ is the interior of C_1. Show that C_1 and C_2 can be separated by a hyperplane, i.e. there exists $\boldsymbol{w} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$, $b \in \mathbb{R}$ s.t.

$$\begin{aligned} \boldsymbol{w}^T \boldsymbol{x} \le b, \quad \forall \boldsymbol{x} \in C_1 \\ \boldsymbol{w}^T \boldsymbol{x} \ge b, \quad \forall \boldsymbol{x} \in C_2 \end{aligned}$$

Hint: You can assume the result in Problem 5(a) of HW2 and the fact that a point of C_1 is the limit of points in $\operatorname{int} C_1$ (by the lemma on slide 34 of §3).

Remark. In the right figure on slide 36 of §3, we assume the ball is open in order to apply the separating hyperplane theorem. This problem shows that the openness assumption is not necessary.

2. Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be an extended-valued convex function, i.e. for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\theta \in (0, 1)$,

$$f(\theta \boldsymbol{x} + \bar{\theta} \boldsymbol{y}) \le \theta f(\boldsymbol{x}) + \bar{\theta} f(\boldsymbol{y})$$

(a). For $\alpha \in (-\infty, +\infty]$, show $S_\alpha = \{\boldsymbol{x} : f(\boldsymbol{x}) < \alpha\}$ and $C_\alpha = \{\boldsymbol{x} : f(\boldsymbol{x}) \le \alpha\}$ are convex using definition.

(b). Show that the effective domain of f is convex.

(c). Let $X \subset \mathbb{R}^n$ be convex and M the set of global minima of f over X, i.e.

$$M = \{\boldsymbol{x}^* \in X : f(\boldsymbol{x}^*) \le f(\boldsymbol{x}), \forall \boldsymbol{x} \in X\}$$

Find an appropriate α such that $M = C_\alpha$ and conclude that M is convex.

3. Let f be convex and $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{dom} f$. Suppose $f(\theta_0 \boldsymbol{x} + \bar{\theta}_0 \boldsymbol{y}) < \theta_0 f(\boldsymbol{x}) + \bar{\theta}_0 f(\boldsymbol{y})$ for some $\theta_0 \in (0, 1)$. Show $f(\theta \boldsymbol{x} + \bar{\theta} \boldsymbol{y}) < \theta f(\boldsymbol{x}) + \bar{\theta} f(\boldsymbol{y})$ for all $\theta \in (0, 1)$. Hint: Consider the two cases $\theta \in (0, \theta_0)$ and $\theta \in (\theta_0, 1)$ separately. They are similar.

4. Let f be a differentiable convex function. Show that

$$\langle \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \ge 0, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \operatorname{dom} f.$$