AI2613 Stochastic Processes Homework 1

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1 Probability Space of Tossing Coins

1. Consider an arbitrary $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots) \in \Omega$, it falls into a unique set $C_{\boldsymbol{s}}$ where $\boldsymbol{s} = (\omega_1, \omega_2, \dots, \omega_n) \in \{0,1\}^n$, indicating that

   - For any $\boldsymbol{s}_1 \neq \boldsymbol{s}_2$, it holds that $C_{\boldsymbol{s}_1} \cap C_{\boldsymbol{s}_2} = \varnothing$
   - $\bigcup\limits_{\boldsymbol{s} \in \{0,1\}^n} C_{\boldsymbol{s}} = \Omega$

   Since $C_{\boldsymbol{s}}$ are disjoint and cover Ω, we claim that $\{C_{\boldsymbol{s}}\}_{\boldsymbol{s} \in \{0,1\}^n}$ forms a partition of Ω.

2. Construct a map $f : \mathcal{F}_n \to 2^{\{0,1\}^n}$ as follow

   - For any $F \in \mathcal{F}_n$, let $f(F) = \{(\omega_1, \omega_2, \dots, \omega_n) \,|\, \boldsymbol{\omega} \in F\}$, which promises that f is injective
   - For any $S \in 2^{\{0,1\}^n}$ containing finite elements, let $F = \bigcup\limits_{\boldsymbol{s} \in S} C_{\boldsymbol{s}}$, we have $f(F) = S$, which promises that f is surjective

   Therefore, f is a bijection between $\mathcal{F}_n$ and $2^{\{0,1\}^n}$.

3. Let $f : \mathcal{F}_n \to 2^{\{0,1\}^n}$ be the bijection in 2. Consider an arbitrary $F \in \mathcal{F}_n$, we have

   $$F = \bigcup_{\boldsymbol{s} \in f(F)} C_{\boldsymbol{s}}$$

   For any $\boldsymbol{s} = (s_1, s_2, \dots, s_n) \in \{0,1\}^n$, let $\boldsymbol{s}_0 = (s_1, s_2, \dots, s_n, 0), \boldsymbol{s}_1 = (s_1, s_2, \dots, s_n, 1) \in \{0,1\}^{n+1}$, we have $C_{\boldsymbol{s}} = C_{\boldsymbol{s}_0} \cup C_{\boldsymbol{s}_1}$, thus

   $$F = \bigcup_{\boldsymbol{s} \in f(F)} (C_{\boldsymbol{s}_0} \cup C_{\boldsymbol{s}_1}) \in \mathcal{F}_{n+1}$$

   which indicates that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

   Let $\hat{\boldsymbol{s}} \in \{0,1\}^{n+1}$, we have $C_{\hat{\boldsymbol{s}}} \in \mathcal{F}_{n+1}$ but $C_{\hat{\boldsymbol{s}}} \notin \mathcal{F}_n$, so it holds that $\mathcal{F}_n \subsetneq \mathcal{F}_{n+1}$.

   Since such $n \in \mathbb{N}$ is arbitrary, we can conclude that $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \dots$ is increasing.

4. It has been proved that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_\infty$ in 3, indicating that

   - For any $F \in \mathcal{F}_\infty$, there exists smallest $i \in \mathbb{N}$ such that $F \in \mathcal{F}_i$, so $F^c \in \mathcal{F}_i \subset \mathcal{F}_\infty$
   - For any $F, G \in \mathcal{F}_\infty$, there exists smallest $i, j \in \mathbb{N}$ such that $F \in \mathcal{F}_i, G \in \mathcal{F}_j$, so $F \cup G \in \mathcal{F}_{\max\{i,j\}} \subset \mathcal{F}_\infty$

Therefore, $\mathcal{F}_\infty$ is an algebra.

Note that $\mathcal{F}_\infty = \bigcup\limits_{n=1}^{\infty} \mathcal{F}_n$ where $\mathcal{F}_n$ is finite, thus $|\mathcal{F}_\infty| = \aleph_0$ is countable. However, $|\Omega| = \aleph_0$, so 2^Ω is not countable, which promises that $\mathcal{F}_\infty \neq 2^\Omega$.

5. Consider an arbitrary $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots) \in \Omega$, let $\boldsymbol{s}_k = (\omega_1, \omega_2, \dots, \omega_k)$. For any $n \in \mathbb{N}$, we have $\boldsymbol{\omega} \in C_{\boldsymbol{s}_n}$, indicating that

$$\{\boldsymbol{\omega}\} = \bigcap_{n=1}^{\infty} C_{\boldsymbol{s}_n} = \left(\bigcup_{n=1}^{\infty} C_{\boldsymbol{s}_n}^c\right)^c \in \mathcal{B}(\Omega)$$

For any $n \in \mathbb{N}$, we have $\{\boldsymbol{\omega}\} \notin \mathcal{F}_n$ since $\boldsymbol{\omega}$ contains infinite components, thus $\{\boldsymbol{\omega}\} \notin \mathcal{F}_\infty$. Therefore, we can conclude that $\{\boldsymbol{\omega}\} \in \mathcal{B}(\Omega) \setminus \mathcal{F}_\infty$.

6. Consider an arbitrary $A \in \mathcal{F}_\infty$. Since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_\infty$, there exists smallest $n \in \mathbb{N}$ such that $A \in \mathcal{F}_n$. Let $f : \mathcal{F}_n \to 2^{\{0,1\}^n}$ be the bijection in 2. Suppose $f(A) = S = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_k\}$, we have $A = \bigcup\limits_{i=1}^{k} C_{\boldsymbol{s}_i}$.

Let $n' = n + 1$, we construct $S' = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_{k'}\}$ as follow

$$S = \{\boldsymbol{s}_0 = (s_1, s_2, \dots, s_n, 0), \boldsymbol{s}_1 = (s_1, s_2, \dots, s_n, 1) \mid \boldsymbol{s} = (s_1, s_2, \dots, s_n) \in f(A)\}$$

where $k' = |S'| = 2|S| = 2k$, thus $\frac{k'}{2^{n'}} = \frac{2k}{2^{n+1}} = \frac{k}{2^n}$. By induction, this equation holds for any $n' > n$. Therefore, the value $\frac{k}{2^n}$ only depends on A.

7. Define a measure $\mathcal{P}(C_{\boldsymbol{s}}) = \frac{1}{2^n}$ where $\boldsymbol{s} \in \{0,1\}^n$ on $\mathcal{F}_\infty$. Then $\mathcal{P}$ satisfies

   - Since $\mathcal{F}_\infty$ is an algebra, it holds that $\mathcal{P}(A^c) = 1 - \sum\limits_{\boldsymbol{s} \in f(A)} \mathcal{P}(C_{\boldsymbol{s}}) = 1 - \mathcal{P}(A)$
   - For countable disjoint sets $A_1, A_2, \dots \in \mathcal{F}_\infty$, it holds that $\mathcal{P}\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mathcal{P}(A_n)$

Hence, $\mathcal{P}(A) = \sum\limits_{\boldsymbol{s} \in f(A)} \mathcal{P}(C_{\boldsymbol{s}}) = \frac{k}{2^n}$ for any $A \in \mathcal{F}_\infty$, where k and n are defined in 6.

By Carathéodory's extension theorem, we can extend the measure $\mathcal{P}$ on $\mathcal{F}_\infty$ to a unique measure $P : \mathcal{B}(\Omega) \to [0,1]$ on $\sigma(\mathcal{F}_\infty)$ where $P(A) = \frac{k}{2^n}$ for any $A \in \mathcal{F}_\infty$. Since $P(\varnothing) = 0$ and $P(\Omega) = \sum\limits_{\boldsymbol{s} \in 2^{\{0,1\}^n}} P(C_{\boldsymbol{s}}) = 1$, we can conclude that P is a probability measure.

8. Random variable $X : \boldsymbol{\omega} \to \mathbb{N}$ is defined as follow

$$X(\boldsymbol{\omega}) = \min_{n \in \mathbb{N}} \{n \mid \omega_n = 1\}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots)$.

2 Conditional Expectation

1. For any $B \in \mathcal{B}$, we have $f^{-1}(B) \in \mathcal{B}$, hence

$$(f \circ X)^{-1}(B) = X^{-1}\left[f^{-1}(B)\right] \in \sigma(X)$$

Therefore, $f(X)$ is $\sigma(X)$-measurable.

2. Since $\sigma(Y) = \sigma(Y')$, for any $\omega \in \Omega$, it holds that $Y^{-1}[Y(\omega)] = Y'^{-1}[Y'(\omega)]$. Otherwise, we have $Y^{-1}[Y(\omega)] \cap Y'^{-1}[Y'(\omega)] \subset \sigma(Y)$, indicating that $\sigma(Y)$ is not minimal. Hence,

$$\mathbb{E}\left[X|Y^{-1}[Y(\omega)]\right] = \mathbb{E}\left[X|Y'^{-1}\left[Y'(\omega)\right]\right]$$

holds for any $\omega \in \Omega$, namely $\mathbb{E}[X|Y] = \mathbb{E}[X|Y']$.

3. **Lemma.** If X is $\mathcal{F}$-measurable, then $\mathbb{E}[X|\mathcal{F}] = X$.

Let $Y = \mathbb{E}[X|\mathcal{F}_1]$, which is $\mathcal{F}_1$-measurable. Since $\mathcal{F}_1 \subset \mathcal{F}_2$, Y is also $\mathcal{F}_2$-measurable. By lemma, we have $\mathbb{E}[Y|\mathcal{F}_2] = Y$, namely $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_1]|\mathcal{F}_2] = \mathbb{E}[X|\mathcal{F}_1]$.[1]

For any $A \in \mathcal{F}_1$, we have $A \in \mathcal{F}_2$. By definition of conditional expectation for general random variables, it holds that

$$\int_A \mathbb{E}[\mathbb{E}[X|\mathcal{F}_2]|\mathcal{F}_1]\,\mathrm{d}\mathbb{P} = \int_A \mathbb{E}[X|\mathcal{F}_2]\,\mathrm{d}\mathbb{P} = \int_A X\,\mathrm{d}\mathbb{P} = \int_A \mathbb{E}[X|\mathcal{F}_1]\,\mathrm{d}\mathbb{P}$$

which yields $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_2]|\mathcal{F}_1] = \mathbb{E}[X|\mathcal{F}_1]$.[2]

Therefore, we can conclude that $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_1]|\mathcal{F}_2] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_2]|\mathcal{F}_1] = \mathbb{E}[X|\mathcal{F}_1]$.

References

[1] Conditional expectation. Samy Tindel. Purdue University.

[2] Martingales through measure theory. Alison Etheridge. University of Oxford.