AI3602 Data Mining: Homework 9

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1. (a) Let $p_x(z)$ denote the probability density function of $\mathcal{M}(x)$, then

$$\begin{aligned}\frac{p_x(z)}{p_y(z)} &= \frac{\frac{1}{2b}\exp\left(-\frac{|f(x)-z|}{b}\right)}{\frac{1}{2b}\exp\left(-\frac{|f(y)-z|}{b}\right)} \\ &= \exp\left(\frac{|f(y)-z|-|f(x)-z|}{b}\right) \\ &\le \exp\left(\frac{|f(y)-f(x)|}{b}\right) \\ &\le \exp\left(\frac{\Delta f}{b}\right)\end{aligned}$$

Since $\ln\frac{p_x(z)}{p_y(z)} \le \epsilon$, we have $b \ge \frac{\Delta f}{\epsilon}$. Therefore, $b = \frac{1}{\epsilon}$ for $\Delta f = 1$.

(b) Let $p_x(z)$ denote the probability density function of $\mathcal{M}(x)$, then

$$\begin{aligned}\frac{p_x(z)}{p_y(z)} &= \frac{\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(f(x)-z)^2}{2\sigma^2}\right)}{\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(f(y)-z)^2}{2\sigma^2}\right)} \\ &= \exp\left(\frac{(f(y)-z)^2-(f(x)-z)^2}{2\sigma^2}\right) \\ &\le \exp\left(\frac{(\Delta f)^2+2\Delta f|f(x)-z|}{2\sigma^2}\right)\end{aligned}$$

Since $\ln\frac{p_x(z)}{p_y(z)} \le \epsilon$, we have $\frac{(\Delta f)^2+2\Delta f|f(x)-z|}{2\sigma^2} \le \epsilon$, namely

$$|f(x)-z| \le \frac{\sigma^2\epsilon}{\Delta f} - \frac{\Delta f}{2}$$

Note that $f(x)-z \sim \mathcal{N}(0,\sigma^2)$, let $t = \frac{\sigma^2\epsilon}{\Delta f} - \frac{\Delta f}{2}$, then

$$P(|f(x)-z| > t) = 2P(f(x)-z > t) = 2\Phi\left(-\frac{t}{\sigma}\right) < \delta$$

An approximated solution is $\sigma > \frac{\Delta f\sqrt{2\ln\frac{1.25}{\delta}}}{\epsilon}$. Therefore, $\sigma = \frac{\sqrt{2\ln\frac{1.25}{\delta}}}{\epsilon}$ for $\Delta f = 1$.

2. Let $x, y \in \{1, 2, \ldots, m\}$ denote the truth and response respectively. By symmetry, we only need to consider a specific response (e.g. $y = 1$). We have

$$\frac{P(y=1|x=1)}{P(y=1|x=k\ne 1)} = \frac{p}{\frac{1-p}{m-1}} = \frac{(m-1)p}{1-p}$$

Since $\sup_y\left|\ln\frac{P(f(x)=y)}{P(f(x')=y)}\right| \le \epsilon$, we have $\frac{(m-1)p}{1-p} \le \exp(\epsilon)$. Therefore, $p = \frac{\exp(\epsilon)}{\exp(\epsilon)+m-1}$.