Dynamic Pricing Competition under Product Differentiation with Posterior Price Matching

Abstract

With a posterior price matching policy (PM), a seller guarantees to reimburse consumers for any price difference if the seller marks down the product after the purchase. Should firms adopt a posterior PM policy in a competitive market? In this paper, we study a duopoly selling differentiated substitutable products to strategic customers, where firms can choose to commit to a posterior PM policy before the selling period, and dynamically adjust their prices during the selling period. Interestingly, we analytically show that an asymmetric equilibrium of PM decision may arise, in which only the firm offering high-quality product adopts a posterior PM.
Keywords: dynamic pricing, competition, posterior price matching

1. Introduction

Product differentiation and dynamic pricing are two widely used retailing strategies. Product differentiation allows firms to seek profitable niches in a competitive market by offering unique products or variations that appeal to specific segments of consumers. This strategy is prevalent across various industries, including electronics and food products, where companies tailor their offerings to meet diverse consumer preferences. In addition to product differentiation, firms often employ dynamic pricing, where prices are varied over time to better manage demand and maximize revenue. By adjusting prices dynamically, firms can optimize sales, manage inventory, and respond to market conditions more effectively. Product differentiation and dynamic pricing strategies are closely intertwined in competitive markets, where strategic consumers not only decide which products to purchase but also when to purchase them. This dual decision-making process makes it crucial for firms to consider both intratemporal demand competition and intertemporal demand substitution.

[4] investigated this interplay in a duopoly market, numerically showing that unilateral commitment to a fixed price by either firm generally improves the profits of both firms. Interestingly, they found that price commitment by the high-quality firm yields greater mutual benefits compared to commitment by the low-quality firm. These findings are particularly relevant in today's online retailing market, where established players like Amazon face competition from emerging e-commerce

platforms such as Temu, which offers lower prices but also lower product quality. Since Temu's launch in September 2022, its daily active users have grown by 51.4 million, while Amazon's daily active users have declined by 2.5 million over the same period [2]. In this context, [4]'s insights suggest that high-quality firms like Amazon could strategically benefit from committing to a fixed price strategy. Such a strategy may help high-quality firms stabilize their market position and effectively counter the appeal of lower-quality competitors.

Price commitment benefits the committing firm by eliminating customers' incentives to delay purchases, enabling the firm to set a higher initial price than would be possible with dynamic pricing. Additionally,unilateral price commitment softens competitive pressures, as it confines competition to the initial period. In this scenario, the committing firm sets a fixed price, allowing the rival firm to act with "monopoly" power in subsequent periods. However, this approach sacrifices pricing flexibility, potentially leading to unforeseen disadvantages. Moreover, price commitment is often not credible, as firms typically lack binding commitment mechanisms to enforce such commitments. This assumption of non-commitment is widely accepted in the durable goods literature [1, 5].

In this article, we focus on an alternative retail policy called posterior price matching (PM), a strategy that is also known to reduce customer waiting behavior. Under posterior PM, retailers guarantee compensation for future markdowns by refunding the price difference if a lower price is offered within a specified period after purchase. This policy, widely employed by retailers such as Target and Costco, is considered more credible and manageable than price commitment. Research suggests that posterior PM effectively mitigates the adverse effects of strategic customer behavior, making it an attractive alternative for retailers [3]. Unlike price commitment, posterior PM preserves pricing flexibility in subsequent periods, which may make it more appealing to retailers. These characteristics raise an important question: how does posterior PM influence dynamic pricing competition between firms with differentiated products, and is its adoption truly beneficial? In this article, we aim to address this question by examining the following aspects:

1. How does posterior PM influence firms' revenues in the competitive market?
2. Which firm—whether high-quality or low-quality—is more likely to benefit from posterior PM?

2. Model

Consider a market with two firms, firm H and firm L, each of which offers one product. The firms sell their products in two consecutive periods. The products can be sold at different prices in different time periods. The prices offered in period t by the two firms are denoted by $p_t = (p_{t,H}, p_{t,L})$, $t = 1, 2$. The period discount factor for each firm is α ($0 < \alpha < 1$). The firms determine prices simultaneously at the beginning of each period to maximize their respective total discounted revenues over the two selling periods. We assume the PM decisions are made and committed before the selling periods ($t = 0$) and their decisions are publicly known. Firm i's effective first-period price is

$$p_{1,i} - \mathbb{1}\{PM_i\} \cdot \alpha(p_{1,i} - p_{2,i})^+, \tag{1}$$

where $\mathbb{1}\{PM_i\}$ indicates whether firm i implements PM.

The two firms offer quality-differentiated products. The product quality of firm H (q_H) is higher than the product quality of firm L (q_L); i.e. $q_H > q_L$. Without loss of generality, (q_H, q_L) are normalized to $(1, \beta)$ with $0 < \beta < 1$. We assume the quality levels are exogenously given.

Customers have heterogeneous valuations v of the product. Their valuations follow a uniform distribution on $[0, 1]$, which is known by the firms. The total number of customers is normalized to one. If a customer with valuation v purchases product offered by firm i in period t, he earns a surplus of $vq_i - p_{t,i}$. Customers can also choose not to purchase and earn zero surplus. All customers arrive at the beginning of the selling season prior to period 1, and each of them purchase at most one unit of the product. Customers discount their utilities over time via a discount factor γ ($0 \leq \gamma < 1$). Assume $p_{i,t} = +\infty$ for $t > 2$. A customer with valuation v purchasing product i in period t has utility

$$u_i(t, v) = \gamma^{t-1}(q_i v - p_{t,i} + \mathbb{1}\{PM_i\} \cdot \gamma(p_{t,i} - p_{t+1,i})^+). \tag{2}$$

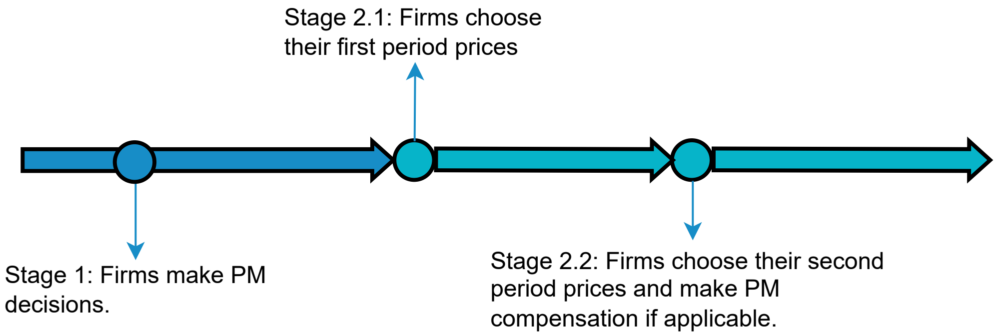
All customers are strategic and decide when and which product to purchase to maximize their utility.

Before the selling period begins, each firm decides whether to implement posterior PM. If a firm chooses to adopt PM, it commits to maintaining it until the end of the selling period. Both firms can dynamically adjust their prices during the selling period, but a firm adopting PM must compensate customers who purchase in the first period if the price is marked down in the second period. Specifically, we consider a multi-stage non-cooperative game between the two firms, structured as shown in Figure 1: At stage 0, the firms simultaneously decide whether to implement posterior PM as part of their retailing. At stage 1 and 2, the firms engage in competition to maximize revenue by setting prices based on the PM decisions and the current state.

We define the following values to simplify the notation of later discussion:

$$X = \beta - \gamma(1 - A_{2,H}), \quad Y = \frac{X - \gamma A_{2,L}}{1 - \gamma}, \quad Z = \beta - \gamma + \frac{\alpha(1-\gamma)}{1-\alpha} A_{2,H}, \quad k = 1 - \alpha(1 - \frac{A_{2,L}}{Y}). \tag{3}$$

Figure 1: Illustration of the multi-stage game.



2.1. Subgame Perfect Nash Equilibrium

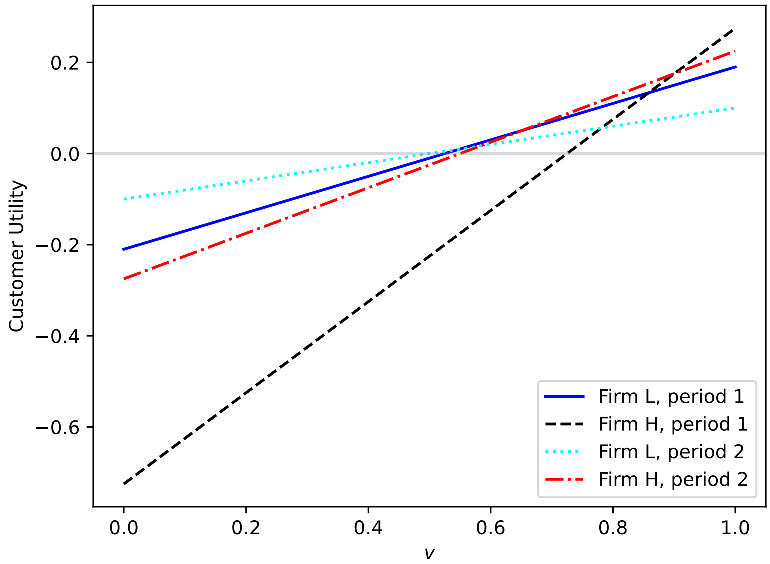
We adopt the concept of *subgame perfect Nash equilibrium* (SPNE). A SPNE is a strategy profile in which it is simultaneously a Nash equilibrium for every subgame of the initial game. In our model, there are three subgames that need to be analyzed. In the following, we first present a critical assumption for analyzing the equilibrium, and then define the SPNE for each subgame.

A critical assumption made in [4] for equilibrium analysis is $\beta > \gamma$ to ensure the remaining customers after the first period can be characterized by a continuous interval. The result an be extended to our case where the firms can choose to implement posterior PM, as stated in the following lemma:

Lemma 1. Suppose $\beta > \gamma$. If a customer with valuation v' purchases in period t, all customers in the market with valuations higher than v' will also purchase in period t regardless of the firms' PM decisions, for any expected future prices.

If this assumption is violated, the remaining customers after the first period may not form a continuous interval. Figure 2 provides an example of such a non-continuous interval, where only firm H implements PM when $\beta < \gamma$. In this case, some customers choose to buy from firm L in the first period, while others, with higher valuations, opt to purchase from firm H in the second period. This results in a non-continuous interval after the first period. Intuitively, this occurs because the product quality offered by firm L is too low, leading customers to prefer waiting until the second period to purchase the higher-quality product from firm H, rather than buying the low-quality product in the current period. Similar examples can be constructed for the cases where only firm L adopts PM and both firms adopt PM as shown in online appendix. When the state is a union of disjoint intervals, pure strategy Nash equilibrium may not exist even in a single-period dynamic pricing game. Online appendix also presents non-existence of pure-strategy equilibrium.

Figure 2: Utility of customers with different valuations when only firm H adopts PM ($\gamma = 0.5$, $\beta = 0.4$, $p_{1,H} = 0.9$, $p_{1,L} = 0.2$, $p_{2,H} = 0.55$, $p_{2,L} = 0.2$)



We first consider the last subgame (stage 2), in which the customers' behavior is the same regardless of the firms' PM decisions. Based on Lemma 1, the remaining customers' valuations after the first period form a continuous interval. Suppose the remaining customers in the second period have valuations over $[0, v_2]$. Given state v_2 and a price pair $p_2 = (p_{2,H}, p_{2,L})$, a customer with valuation $v \leq v_2$ will choose to purchase product H if doing so leads to a positive surplus higher than purchasing product L; i.e., $v - p_{2,H} > (\beta v - p_{2,L})^+$, which means

$$v > \frac{p_{2,H} - p_{2,L}}{1 - \beta} \quad \text{and} \quad v > p_{2,H}.$$

Similarly, the customer will purchase product L if $\beta v - p_{2,L} \geq (v - p_{2,H})^+$, which means

$$v \leq \frac{p_{2,H} - p_{2,L}}{1 - \beta} \quad \text{and} \quad v \geq \frac{p_{2,L}}{\beta}.$$

At equilibrium, both firms should have revenue, otherwise the equilibrium cannot be sustained. Therefore, at equilibrium, it must have

$$\frac{p_{2,L}}{\beta} < \frac{p_{2,H} - p_{2,L}}{1 - \beta} \implies p_{2,L} < \beta p_{2,H},$$

which further implies

$$\frac{p_{2,H} - p_{2,L}}{1 - \beta} > \frac{p_{2,H} - \beta p_{2,H}}{1 - \beta} = p_{2,H}.$$

Therefore, the customers with valuation $v \in (\frac{p_{2,H}-p_{2,L}}{1-\beta}, v_2]$ will purchase at H and the customers with valuation $v \in [\frac{p_{2,L}}{\beta}, \frac{p_{2,H}-p_{2,L}}{1-\beta}]$ will purchase at L. The subgame revenue of the firms are

$$r_{2,H}(p_{2,H}, p_{2,L}) = p_{2,H}(v_2 - \frac{p_{2,H} - p_{2,L}}{1-\beta}), \quad r_{2,L}(p_{2,H}, p_{2,L}) = p_{2,L}(\frac{p_{2,H} - p_{2,L}}{1-\beta} - \frac{p_{2,L}}{\beta}).$$

By definition, a tuple $p_2^* = (p_{2,L}^*, p_{2,H}^*)$ is a SPE if and only if

$$p_{2,H}^* = \arg\max_{p_{2,H}} r_{2,H}(p_{2,H}, p_{2,L}^*) \quad \text{and} \quad p_{2,L}^* = \arg\max_{p_{2,L}} r_{2,L}(p_{2,H}^*, p_{2,L}). \tag{4}$$

Knowing the SPE of the subgame in stage 2, we can now consider the whole dynamic pricing subgame which consists of both stage 1 and stage 2 by backward induction. Notice that for firm i, setting $p_{1,i} < p_{2,i}$ is equivalent to setting $p_{1,i} = p_{2,i}$, as no customer will purchase at firm i in the second period and $p_{2,i}$ does not affect the competition in this case. Therefore, we can freely assume $p_{1,i} \geq p_{2,i}$ for $i \in \{H, L\}$. Thus, the first-period effective price of firm i can be expressed as

$$e_{1,i}(p_{1,i}, p_{2,i}) = p_{1,i} - \mathbb{1}\{PM_i\} \cdot \alpha(p_{1,i} - p_{2,i}), \tag{5}$$

where $\mathbb{1}\{PM_i\}$ indicates whether firm i implements PM.

In the first period, each customer determines (i) whether to purchase in the first period, or wait until the next period; and (ii) in case he purchases in the first period, which product to choose (H or L). Customers' behavior is affected by firms' implementation of PM. Let $\hat{v}_2$ be the valuation of a marginal customer who is indifferent between purchasing in the first period and second period. The surpluses under different purchase options in the first period for a customer with valuation v are summarized in Table 1.

	$u_H(1, v)$	$u_L(1, v)$
Neither PM	$v - p_{1,H}$	$\beta v - p_{1,L}$
Only H PM	$v - p_{1,H} + \gamma(p_{1,H} - p_{2,H}^*(\hat{v}_2))$	$\beta v - p_{1,L}$
Only L PM	$v - p_{1,H}$	$\beta v - p_{1,L} + \gamma(p_{1,L} - p_{2,L}^*(\hat{v}_2))$
Both PM	$v - p_{1,H} + \gamma(p_{1,H} - p_{2,H}^*(\hat{v}_2))$	$\beta v - p_{1,L} + \gamma(p_{1,L} - p_{2,L}^*(\hat{v}_2))$

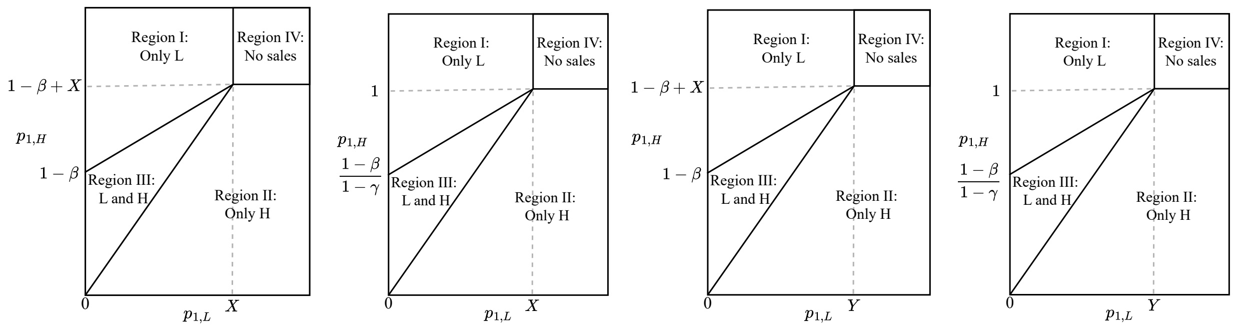
Table 1: The surpluses under different purchase options in the first period for a customer with valuation v.

Depending on p_1, α, β and γ, firm L may or may not incur demand in the first period. We can derive thresholds using the following steps:

- When firm L incurs no demand in the first period, the marginal valuation between the first and second periods satisfies the equation:

$$u_H(1, \hat{v}_2) = \gamma(\hat{v}_2 - p_{2,H}^*(\hat{v}_2)).$$

Figure 3: Strategy space under different PM decisions.



(a) Neither firm PM. (b) Only firm H PM. (c) Only firm L PM. (d) Both firms PM

The solution of $\hat{v}_2$ is denoted as t_1;

- When firm L incurs demand in the first period, the marginal valuation between the first and second periods satisfies:

$$u_L(1, \hat{v}_2) = \gamma(\hat{v}_2 - p^*_{2,H}(\hat{v}_2))$$

The solution of $\hat{v}_2$ is denoted as t_2;

- When firm L incurs demand in the first period, the indifferent customer's valuation between purchasing from firm H and firm L satisfies the equation:

$$u_H(1, v) = u_L(1, v).$$

The solution of v is denoted as t_3.

The resulting thresholds are summarized in Table 2.

	t_1	t_2	t_3
Neither PM	$\frac{p_{1,H}}{1-\beta+X}$	$\frac{p_{1,L}}{X}$	$\frac{p_{1,H}+p_{1,L}}{1-\beta}$
Only H PM	$p_{1,H}$	$\frac{p_{1,L}}{X}$	$\frac{1-\gamma}{1-\beta}p_{1,H} + \frac{\gamma-\beta}{1-\beta}\frac{p_{1,L}}{X}$
Only L PM	$\frac{p_{1,H}}{1-\beta+X}$	$\frac{p_{1,L}}{Y}$	$\frac{p_{1,H}}{1-\beta} + \frac{X}{1-\beta}\frac{p_{1,L}}{Y}$
Both PM	$p_{1,H}$	$\frac{p_{1,L}}{Y}$	$\frac{1-\gamma}{1-\beta}p_{1,H} + \frac{\gamma-\beta}{1-\beta}\frac{p_{1,L}}{Y}$

Table 2: The marginal valuation (thresholds) of purchase decision switch.

With the thresholds, we can divide firms' strategy space in the first period into four regions as shown in Figure 3. In Region I, only firm L incurs sales, while in Region II, only firm H incurs sales. In Region III, both firms incur sales, and in Region IV, neither firm incurs sales. Denote firm i's second period equilibrium revenue as $r^*_{2,i} = r_{2,i}(p^*_{2,H}, p^*_{2,L})$. Notice that $p^*_{2,H}$ and $p^*_{2,H}$ are actually determined by $\hat{v}_2$, which is determined by $p_1 = (p_{1,H}, p_{1,L})$. Therefore, we write the second-period equilibrium price as $p^*_{2,i}(p_1)$ and equilibrium revenue as $r^*_{2,i}(p_1)$ for clarity. The revenue of firm i from the whole selling periods is

$$r_i(p_1) = \begin{cases} \alpha r^*_{2,i}(p_1) & \text{region I,} \\ e_{1,i}(p_{1,i}, p^*_{2,i}(p_1))(1-t_1) + \alpha r^*_{2,i}(p_1) & \text{region II,} \\ e_{1,i}(p_{1,i}, p^*_{2,i}(p_1))(1-t_3) + \alpha r^*_{2,i}(p_1) & \text{region III,} \\ \alpha r^*_{2,i}(p_1) & \text{region IV.} \end{cases}$$

By definition, a tuple $p^*_1 = (p^*_{1,L}, p^*_{1,H})$ is a SPE if and only if

$$p^*_{1,H} = \arg\max_{p_{1,H}} r_H(p_{1,H}, p^*_{1,L}) \quad \text{and} \quad p^*_{1,L} = \arg\max_{p_{1,L}} r_L(p^*_{1,H}, p_{1,L}). \tag{6}$$

Notice that the equilibrium prices and revenues are affected by the firms' PM decisions. In the remainder of this paper, we denote the revenue and price of firm i under price-matching decision m in period t as $r^{(m)}_{t,i}$ and $p^{(m)}_{t,i}$, respectively. Specifically, $m = h$ indicates that only firm H implements PM, $m = l$ indicates that only firm L implements PM, $m = b$ means that both firms implement PM, and $m = 0$ means that neither firm implements PM. Additionally, we sometimes use $r^{(m)}_i$ to represent the total revenue firm i generates during the selling period under the PM decision m.

3. Equilibrium Behavior

3.1. Stage 2 Equilibrium

By Equation (4), we can easily solve the equilibrium prices and revenue

$$\begin{aligned} p^*_{2,H}(v_2) &= A_{2,H} v_2, \quad r^*_{2,H}(v_2) = B_{2,H} v_2, \\ p^*_{2,L}(v_2) &= A_{2,L} v_2, \quad r^*_{2,L}(v_2) = B_{2,L} v_2. \end{aligned} \tag{7}$$

where

$$A_{2,H} = \frac{2(1-\beta)}{4-\beta}, \quad A_{2,L} = \frac{\beta(1-\beta)}{4-\beta},$$
$$B_{2,H} = \frac{4(1-\beta)}{(4-\beta)^2}, \quad B_{2,L} = \frac{\beta(1-\beta)}{(4-\beta)^2}.$$

This holds for all cases regardless of the firms' PM decisions as long as the remaining customers' valuations after the first period forms a continuous interval (i.e. $\beta > \gamma$) ranging from 0 to v_2.

3.2. Stage 1 Equilibrium

3.2.1. When Both Firms Choose No Price Matching

This is the exactly the same case as the work of [4]. The unique first-period equilibrium prices and revenues are given by

$$p_{1,H}^{(0)} = \frac{1-\beta+p_{1,L}^{(0)}}{2}, \quad p_{1,L}^{(0)} = \frac{(1-\beta)X^2}{3X^2+4(1-\beta)X-4\alpha(1-\beta)B_{2,L}},$$
$$r_H^{(0)} = \frac{1-\beta}{4}(1+\frac{p_L^{(0)}}{1-\beta})^2 + \alpha B_{2,H}(\frac{p_{1,L}^{(0)}}{X})^2, \quad r_L^{(0)} = p_{1,L}^{(0)}\frac{[X+(1-\beta)]X-\alpha(1-\beta)B_{1,L}^{(0)}}{3X^2+4(1-\beta)X-4\alpha(1-\beta)B_{2,L}} \tag{8}$$

3.2.2. When Firms Choose Price Matching

The following proposition establishes the existence of a unique subgame perfect Nash equilibrium (SPNE) for each of these scenarios.

Proposition 1. If $\beta > \gamma$, there exists a unique SPNE in the region III under all posterior PM cases. The first-period prices and revenues at the equilibria are as follows:

- Only firm H PM:

$$\begin{aligned}
p_{1,H}^{(h)} &= \frac{1-\beta+\frac{Z}{X}p_{1,L}^{(h)}}{2(1-\gamma)}, \\
p_{1,L}^{(h)} &= \frac{(1-\beta)X^2}{[2(\beta-\gamma)+Z+4(1-\beta)]X-4\alpha(1-\beta)B_{2,L}}, \\
r_H^{(h)} &= \frac{(1-\alpha)(1-\beta)}{4(1-\gamma)}(1+\frac{Zp_{1,L}^{(h)}}{(1-\beta)X})^2 + \alpha B_{2,H}(\frac{p_{1,L}^{(h)}}{X})^2 \\
r_L^{(h)} &= p_{1,L}^{(h)}\frac{(1-\gamma)X-\alpha(1-\beta)B_{2,L}}{[2(\beta-\gamma)+Z+4(1-\beta)]X-4\alpha(1-\beta)B_{2,L}}
\end{aligned}$$

- Only firm L PM:

$$\begin{aligned}
p_{1,H}^{(l)} &= \frac{1-\beta+\frac{X}{Y}p_{1,L}^{(l)}}{2}, \\
p_{1,L}^{(l)} &= \frac{(1-\beta)kY^2}{[3X+4(1-\beta)]kY-4\alpha(1-\beta)B_{2,L}}, \\
r_H^{(l)} &= \frac{1-\beta}{4}(1+\frac{Xp_{1,L}^{(l)}}{(1-\beta)Y})^2 + \alpha B_{2,H}(\frac{p_{1,L}^{(l)}}{Y})^2, \\
r_L^{(l)} &= kp_{1,L}^{(l)}\frac{[X+(1-\beta)]kY-\alpha(1-\beta)B_{2,L}}{[3X+4(1-\beta)]kY-4\alpha(1-\beta)B_{2,L}}.
\end{aligned}$$

- Both firms PM:

$$
\begin{aligned}
p_{1,H}^{(b)} &= \frac{1-\beta+\frac{Z}{Y}p_{1,L}^{(b)}}{2(1-\gamma)}, \\
p_{1,L}^{(b)} &= \frac{(1-\beta)kY^2}{[2(\beta-\gamma)+Z+4(1-\beta)]kY-4\alpha(1-\beta)B_{2,L}}, \\
r_H^{(b)} &= \frac{(1-\alpha)(1-\beta)}{4(1-\gamma)}(1+\frac{Zp_{1,L}^{(b)}}{(1-\beta)Y})^2+\alpha B_{2,H}(\frac{p_{1,L}^{(b)}}{Y})^2, \\
r_L^{(b)} &= kp_{1,L}^{(b)}\frac{(1-\gamma)kY-\alpha(1-\beta)B_{2,L}}{[2(\beta-\gamma)+Z+4(1-\beta)]kY-4\alpha(1-\beta)B_{2,L}}.
\end{aligned}
$$

3.3. Pricing Competition Equilibrium Results

We first point out an interesting finding that, while PM has been widely believed as an effective policy to incentivize strategic customers' purchasing and mitigate the negative impact of their waiting behavior, it can also cause more customers to delay their purchases in our competition context:

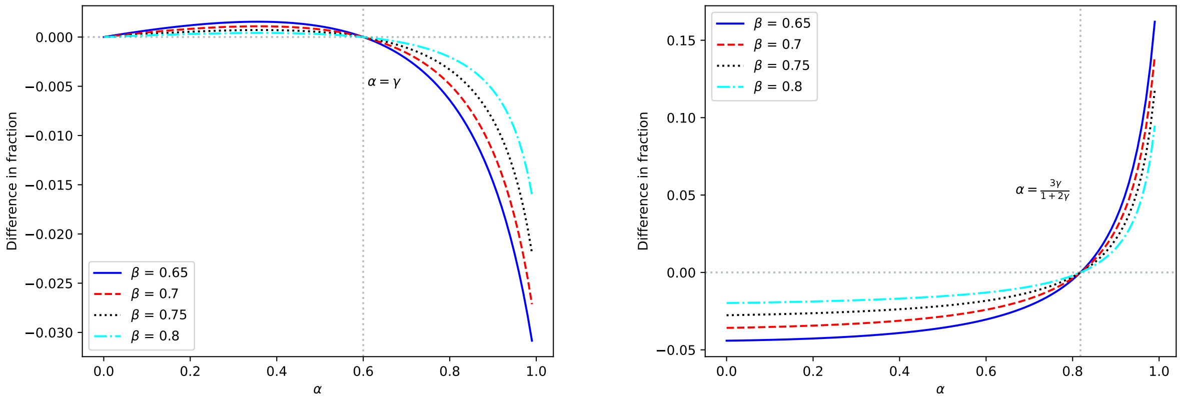
Proposition 2. Suppose $\beta > \gamma$. Regardless of the opposite firm's decision,

- if $\alpha < \gamma$, firm L's PM will incentivize more customers to purchase in the first period; if $\alpha = \gamma$, firm L's PM has no effect on customers' waiting behavior; if $\alpha > \gamma$, firm L's PM will lead more customers to delay their purchases to the second period.

- if $\alpha < \frac{3\gamma}{1+2\gamma}$, firm H's PM will lead more customers to delay their purchase to the second period; if $\alpha = \frac{3\gamma}{1+2\gamma}$, firm H's PM has no effect on customers' waiting behavior; if $\alpha > \frac{3\gamma}{1+2\gamma}$, firm H's PM will incentivize more customers to purchase in the first period.

We now give some intuitions about the results. If firm H's PM decision is given, firm L's implementation of PM policy encourages more customers to purchase in the first period, as they anticipate potential compensation for price differences in the second period. However, firm L faces a trade-off: while PM increases first-period demand, it also requires firm L to compensate customers for any future price reductions. This creates an incentive for firm L to raise its first-period price after implementing PM to balance the additional cost of compensation. When $\alpha < \gamma$, firm L's loss due to PM, represented by $\alpha(p_{1,L} - p_{2,L})$, is smaller than customers' utility gain from PM, represented by $\gamma(p_{1,L} - p_{2,L})$. This creates a situation where the price increase by firm L after implementing PM does not fully "offset" the customers' utility gain from the policy. This is exactly the case at equilibrium. To compete with firm L, firm H will lower its price. As a result, more customers will be incentivized to purchase in the first period. However, when $\alpha > \gamma$, firm L becomes more sensitive to the potential loss from PM compared to the customers' sensitive to

Figure 4: The difference in the fraction of customers purchasing in the first period after unilateral price matching ($\gamma = 0.6$)



(a) Only firm L PM.

(b) Only firm H PM.

the utility gain. In this scenario, firm L is incentivized to set a much higher first-period price in an attempt to protect itself from future compensation costs. This increase in price outweighs the utility gain customers expect from PM. Also, firm H gets a chance to increase its price in this scenario. These jointly cause more customers to delay their purchase to the second period. A representative experiment is shown in Figure 4a, where we fix $\gamma = 0.6$ and vary quality parameter $\beta \in \{0.65, 0.7, 0.75, 0.8\}$.

Suppose firm L's PM decision is fixed. Firm H similarly has the incentive to increase its first-period price after implementing PM. However, instead of lowering its price to compete firm H's PM, firm L will increase its price. This response arises because firm L's quality disadvantage makes it unprofitable to compete with firm H in the first period by lowering its price. On the contrary, firm L finds it more beneficial to raise its price in the first period to generate higher revenue in the second period when α is not overly large, that is $\alpha < \frac{3\gamma}{1+2\gamma}$ based on our finding. This results more customers to delay their purchase. However, if α becomes overly large, firm H's effective first-period will approach its second period price, which is determined by the first-period price of firm L as shown in Equation (7) and Table 2. Consequently, firm H cares more about its first-period market share. It will lower its first-period price, leading to an intensified price competition between the two firms. This price competition drives both firms' first-period prices down, resulting more customers to buy in the first period. A representative experiment is shown in Figure 4b.

Before analyzing the effect of PM policies on revenue, we first discuss their impact on product prices in the first period. Intuitively, since firms' PM policies compensate customers for any price difference in the second period, PM enables firms to increase their first-period price while still maintaining customer interest. However, PM also imposes additional costs on firms, which could lower the effective price (5) in the first period and influence overall revenue. The following proposition fully characterizes firms' effective prices under different PM decisions at equilibrium:

Proposition 3. Suppose $\beta > \gamma$. Regardless of the opposite firm's decision,

- if $\alpha < \gamma$, firm L's PM can increase its first-period effective price; if $\alpha = \gamma$, firm L's PM has no effect on its first-period effective price; if $\alpha > \gamma$, firm L's PM decreases its first-period effective price.

- there exists $\alpha_p \in (\gamma, 1)$ such that, if $\alpha < \alpha_p$, firm H's PM can increase its first-period effective price; if $\alpha = \alpha_p$, firm H's PM has no effect on its first-period effective price; if $\alpha > \alpha_p$, firm H's PM decreases its first-period effective price. The value of α_p may be different under different PM decision of firm L. Moreover, $\alpha_p < \frac{3\gamma}{(1+2\gamma)}$.

We now give some intuitions regarding these results. Suppose the PM decision of firm H is given. When $\alpha < \gamma$, firm L's discounted loss from implementing PM is less than the customers' discounted utility gain. This discrepancy allows firm L to set a higher first-period effective price through implementing PM while still maintain customers' utility. In contrast, when $\gamma \leq \alpha$, firm L faces a high discounted loss from PM. In this scenario, setting a high first-period price, which results in the same or even higher effective price than before, will discourage more customers from purchasing. The trade-off between higher prices and customer retention leads firm L to opt for a lower effective price compared to the scenario without PM.

Suppose firm L's PM decision is given. When $\alpha < \gamma$, firm H's discounted loss from implementing PM is smaller than customers' discounted utility gain from it, enabling it to set a higher first-period effective price through PM. When $\alpha = \gamma$, firm H's implementation of PM leads firm L to increase its price due to its quality disadvantage, as discussed previously. Consequently, firm H can potentially set a higher effective price after implementing PM, as firm L's higher price prevents it from significantly cannibalizing firm H's market share. When firm H becomes more forward-looking, its effective price in the first period approaches its second-period price, which is determined by firm L's first-period price. Consequently, firm H will care more about its market share, and it will lower its first-period price to compete for more market share, prompting firm L to respond by also reducing its price. This results in increased price competition, which gradually drives their prices down to zero. Thus, there exists a threshold α_p larger than γ, where firm H's effective price after implementing PM equals its price before implementing PM.

PM also has impact on each firm's equilibrium market share in the first period, which is summarized in the following proposition:

Proposition 4. Suppose $\beta > \gamma$. Regardless of the opposite firm's decision,

- if $\alpha < \gamma$, firm L's PM can increase its first-period market share; if $\alpha = \gamma$, firm L's PM has no effect on its first-period market share; if $\alpha > \gamma$, firm L's PM decreases its first-period market share.

- there exists $\alpha_m \in (0, \gamma)$ such that, if $\alpha < \alpha_m$, firm H's PM can decrease its first-period market share; if $\alpha = \alpha_m$, firm H's PM has no effect on its first-period market share; if $\alpha > \alpha_m$, firm H's PM increases its first-period market share. The value of α_m may be different under different PM decision of firm L.

We now provide intuitions for these results. Assume firm H's PM decision is fixed. When $\alpha < \gamma$, customers' discounted utility gain from PM exceeds firm L's discounted loss. Although firm L may raise its first-period price to offset the PM loss, this increase does not entirely counteract customers' utility gain from buying at L in the first period. Consequently, firm L's captures a larger market share in the first period. When $\alpha > \gamma$, firm L becomes more sensitive to the potential PM loss, which surpasses customers' sensitivity to utility gain. In this case, firm L sets a higher price to compensate itself, which results in a reduced first-period market share. Now suppose firm H's PM decision is given. When $\alpha < \alpha_m$, firm H's PM policy, although making its product more attractive, coincides with a drop in the total number of first-period customers (as noted in Proposition 2). This reduction in customer numbers outweighs the PM's attractiveness, leading to a decrease in firm H's first-period market share. However, as α surpasses α_m, firm L's price decreases, attracting more customers in the first period. Consequently, the appeal of PM increases, with α_m acting as the threshold.

The effects of PM on firm's revenues are summarized the following theorem:

Theorem 1. Suppose $\beta > \gamma$. Regardless of the opposite firm's decision:

- if $\alpha < \gamma$, firm L's PM increases its revenue; if $\alpha = \gamma$, firm L's PM has no effect on its revenue; if $\alpha > \gamma$, firm L's PM decreases its revenue.

- there exists $\alpha_r \in (\gamma, 1)$ such that, if $\alpha < \alpha_r$, firm H's PM increases its revenue; if $\alpha = \alpha_r$, firm

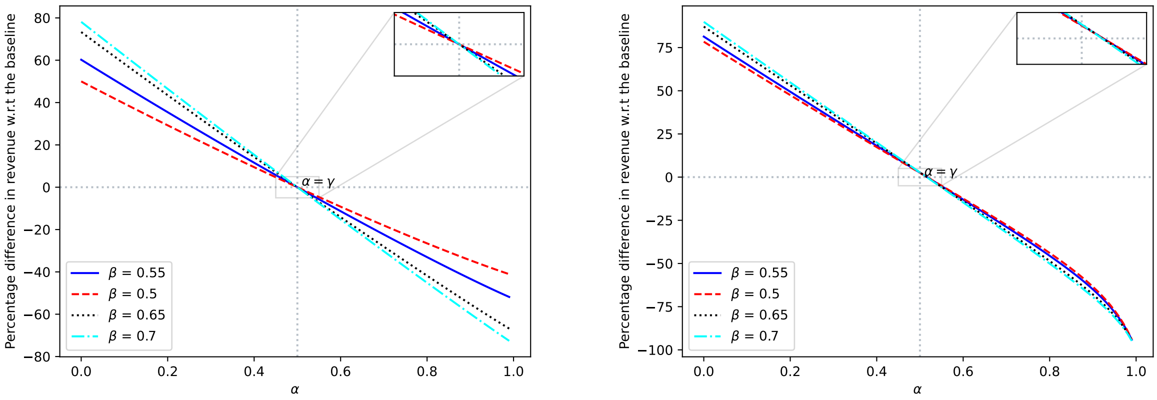
H's PM has no effect on its revenue; if $\alpha > \alpha_r$, firm H's PM decreases its revenue. The value of α_r may be different under different PM decision of firm L. Moreover, $\alpha_p < \alpha_r < \frac{3\gamma}{(1+2\gamma)}$.

	Firm L	Firm H
$\alpha < \gamma$	$r_L^{(l)} > r_L^{(0)}$, $r_L^{(b)} > r_L^{(h)}$	$r_H^{(h)} > r_H^{(0)}$, $r_H^{(b)} > r_H^{(l)}$
$\alpha = \gamma$	$r_L^{(l)} = r_L^{(0)}$, $r_L^{(b)} = r_L^{(h)}$	$r_H^{(h)} > r_H^{(0)}$, $r_H^{(b)} > r_H^{(l)}$
$\gamma < \alpha < \alpha_r$	$r_L^{(l)} < r_L^{(0)}$, $r_L^{(b)} < r_L^{(h)}$	$r_H^{(h)} > r_H^{(0)}$, $r_H^{(b)} > r_H^{(l)}$
$\alpha = \alpha_r$	$r_L^{(l)} < r_L^{(0)}$, $r_L^{(b)} < r_L^{(h)}$	$r_H^{(h)} = r_H^{(0)}$, $r_H^{(b)} = r_H^{(l)}$
$\alpha > \alpha_r$	$r_L^{(l)} < r_L^{(0)}$, $r_L^{(b)} < r_L^{(h)}$	$r_H^{(h)} < r_H^{(0)}$, $r_H^{(b)} < r_H^{(l)}$

Table 3: Comparison of the equilibrium revenues.

The results of Theorem 1 are summarized in Table 3. These results can be explained by our previous observations of market share and effective price. Suppose firm H's PM decision is fixed. When $\alpha < \gamma$, firm L's effective price and market share in the first period increase after implementing PM, as shown in Proposition 3 and Proposition 4. Although the number of customers purchasing from L in the second period decreases (Proposition 2), this reduction is outweighed by the first-period gain. Consequently, firm L achieves higher revenue by implementing PM. Conversely, when $\alpha > \gamma$, Proposition 3 and Proposition 4 indicate that firm L's market share and effective price increase in the first period due to PM. so firm L's first-period revenue increases. However, because of discounting, the additional revenue in the second period does not offset the first-period revenue decline, leading to a decrease in firm L's total revenue. Suppose firm L's PM decision is fixed. When $\alpha < \alpha_m$, Proposition 4 shows that firm H' first-period market share decreases due to PM. However, firm H compensates for this reduced market share by setting a high first-period price, resulting in a higher effective first-period price and increased first-period revenue. In addition, more customers purchase in the second period, which also boosts its second-period revenue, leading to an overall revenue increase. When $\alpha_m < \alpha < \alpha_p$, PM increases firm H's effective first-period price, market share, and second-period revenue, raising total revenue. When $\alpha_p < \alpha < \alpha_r$, even though the effective first-period price decreases due to PM, this decrease is offset by the gains from a larger first-period market share and increased second-period revenue. However, when $\alpha > \alpha_r$, the low effective price, driven by firm H's focus on PM's future loss, outweighs the benefits of the larger first-period market share and second-period revenue gains (if $\alpha < \frac{3\gamma}{1+2\gamma}$). Therefore, PM ultimately reduces firm H's total revenue.

Figure 5: Firms' percentage gain comparing to the baseline under different PM decisions ($\beta = 0.6, \gamma = 0.5$)



(a) Percentage gain of firm L when only firm L PM.

(b) Percentage gain of firm H when only firm H PM.

Setting $\gamma = 0.5$, we conduct numerical experiments where only one of the firm implements PM. The percentage revenue gain comparing to the baseline is shown in Figure 5. Figure 5a shows that firm L's PM can only increase firm L's revenue when α is smaller that γ, while Figure 5b shows that firm H can still make more revenue by implementing PM when α is larger than γ. This observation is consistent with our theorem and demonstrates how quality advantage can help a firm to get more revenue by implementing PM. Figure 6 provides a numerical illustration where only firm H implements PM with $\alpha = \gamma$. It can be seen that firm H's PM benefits itself in general. Moreover, firm H's PM benefits firm H more than firm L, and as customers become more forward-looking, the maximum benefit of PM increases.

3.4. Stage 0 Equilibrium

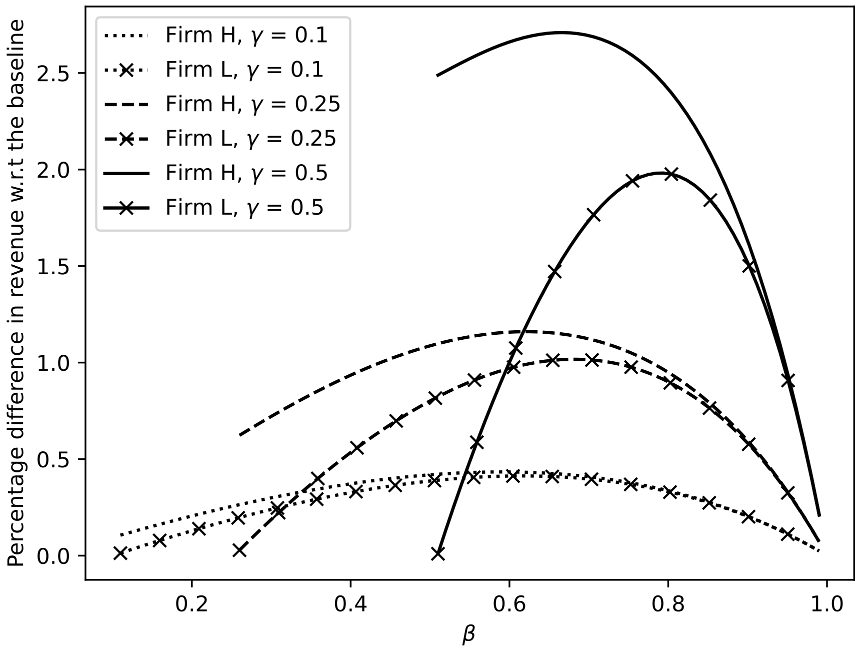
Based on Theorem 1, we can conclude firms' equilibrium PM decision in stage 1.

Corollary 1. Suppose $\beta > \gamma$. At the equilibrium of the price matching game (stage 1),

- if $\alpha < \gamma$, both firms will choose to implement PM.
- if $\gamma \leq \alpha < \alpha_r$, only firm H will choose to implement PM.
- if $\alpha \geq \alpha_r$, no firm will implement PM.

It is interesting to see that the asymmetric equilibrium (namely, one firm chooses to implement posterior PM and the other firm does not) arises due to the difference in product quality.

Figure 6: Firms' percentage gain (%) comparing to baseline with firm H's unilateral PM ($\alpha = \gamma$)



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Appendix A. Useful Lemmas

The following lemma will be useful for later proof.

Lemma 2. If $\beta > \gamma$, the parameters and constants have the following properties:

- $X > 2A_{2,L}$, $X < Y$, $kY \geq A_{2,L}$;
- kY is strictly decreasing w.r.t α and $kY = X$ only when $\alpha = \gamma$;
- Z is strictly increasing w.r.t α and $Z = X$ only when $\alpha = \gamma$.

Proof. X is defined as $\beta - \gamma(1 - A_{2,H})$, so we have

$$X - 2A_{2,L} = \beta - \gamma(1 - A_{2,H}) - \beta A_{2,H} = (\beta - \gamma)(1 - A_{2,H}) > 0.$$

Therefore, $X - \gamma A_{2,L} > X - \gamma X$, and we have

$$Y = \frac{X - \gamma A_{2,L}}{1 - \gamma} > X.$$

kY (respectively, Z) is strictly decreasing (respectively, increasing) w.r.t α can be directly seen from their definition. If $\alpha = \gamma$,

$$\begin{aligned}
kY &= [1 - \gamma(1 - \frac{A_{2,L}}{Y})]Y = (1 - \gamma)Y + \gamma A_{2,L} = X, \\
Z &= \beta - \gamma + \gamma A_{2,L} = X.
\end{aligned}$$

Since kY is decreasing, the minimizer of kY is $\alpha = 1$ and minimum value is $A_{2,L}$. $\square$

Lemma 3. $\frac{(1-\beta)m}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}} < \frac{1}{2}$ for any $m \geq A_{2,L}$.

Proof.

$$\begin{aligned}
&\frac{(1-\beta)m}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}} \\
=&\frac{1}{4 + \frac{2(\beta-\gamma)+Z}{1-\beta} - \frac{4\alpha B_{2,L}}{m}} \\
\leq&\frac{1}{4 + \frac{2(\beta-\gamma)+Z}{1-\beta} - \frac{4\alpha B_{2,L}}{A_{2,L}}} \\
=&\frac{1}{4 + \frac{2(\beta-\gamma)+Z}{1-\beta} - \frac{4\alpha}{4-\beta}} \\
<&\frac{1}{4 - \frac{4}{4-\beta}} \\
<&\frac{1}{2}.
\end{aligned}$$

$\square$

Lemma 4. $f(\alpha, m) = \frac{Z(1-\beta)m}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}$ in strictly increasing w.r.t α for any $m \geq A_{2,L}$.

Proof. Taking the derivative we have

$$\begin{aligned}
\frac{\partial f}{\partial \alpha} &= \frac{f(\alpha, m)}{Z} \frac{\frac{1-\gamma}{(1-\alpha)^2} A_{2,H}(4(1-\beta)(m-\alpha B_{2,L}) + 2(\beta-\gamma)m) + 4(1-\beta)B_{2,L}Z}{[4(1-\beta) + Z + 2(\beta-\gamma)]m - 4\alpha(1-\beta)B_{2,L}} \\
&= \frac{f(\alpha, m)}{Z} \frac{\frac{1-\gamma}{(1-\alpha)^2} A_{2,H}(4(1-\beta)(m-\alpha B_{2,L}) + 2(\beta-\gamma)m) + 4(1-\beta)B_{2,L}Z}{4(1-\beta)(m-\alpha B_{2,L}) + [Z + 2(\beta-\gamma)]m} \\
&> 0
\end{aligned}$$

because $A_{2,L} > B_{2,L}$. Therefore, $f(\alpha, m)$ is increasing. $\square$

Lemma 5. Let $n_1 = 2(\beta-\gamma) + Z + 4(1-\beta) - \frac{4\alpha(1-\beta)B_{2,L}}{m}$, $n_2 = 3X + 4(1-\beta) - \frac{4\alpha(1-\beta)B_{2,L}}{m}$, and $f(m, n) = \frac{(1-\beta)m}{n}(\frac{1}{2} - \frac{Z}{2(1-\beta)} \cdot \frac{1-\beta}{n} - \frac{1-\beta}{n}) + \alpha B_{2,L}(\frac{1-\beta}{n})^2$. Both $f(m, n_1)$ and $f(m, n_2)$ are increasing on $m \in (A_{2,L}, Y)$.

Proof. Taking partial derivative, we have

$$\begin{aligned}
\frac{\partial f}{\partial m} =& \frac{(1-\beta)(n - \frac{4\alpha(1-\beta)B_{2,L}}{m})}{n^2}(\frac{1}{2} - \frac{Z}{2(1-\beta)} \cdot \frac{1-\beta}{n} - \frac{1-\beta}{n}) \\
&+ \frac{4\alpha(1-\beta)^3 B_{2,L}}{n^3 m}(\frac{Z}{2(1-\beta)} + 1 - \frac{2\alpha B_{2,L}}{m}).
\end{aligned}$$

According to Lemma 2, $1 - \frac{2\alpha B_{2,L}}{m} > 1 - \frac{2\alpha B_{2,L}}{A_{2,L}} > 0$. Also, both n_1 and n_2 is larger than $4(1-\beta)(1 - \frac{\alpha B_{2,L}}{m})$, so we have

$$n - \frac{4\alpha(1-\beta)B_{2,L}}{m} > 4(1-\beta)(1 - \frac{2\alpha B_{2,L}}{m}) > 0.$$

Therefore, we have $\frac{\partial f}{\partial m} > 0$ for $n = n_1$ and $n = n_2$. $\square$

Lemma 6. $g(\alpha, m) = \frac{(1-\alpha)(1-\beta)}{4(1-\gamma)}(1 + \frac{f(\alpha,m)}{1-\beta})^2$, where $f(\alpha, m)$ is defined the same as Lemma 4, is strictly decreasing w.r.t $\alpha \in (0, 1)$ for any $m \geq A_{2,L}$

Proof. Taking derivative, we have

$$\frac{\partial g}{\partial \alpha} = \frac{1-\beta}{4(1-\gamma)}(1 + \frac{f(\alpha, m)}{1-\beta})(2\frac{1-\alpha}{1-\beta}\frac{\partial f}{\partial \alpha} - \frac{f(\alpha, m)}{1-\beta} - 1).$$

According to the proof of Lemma 4, we have

$$2\frac{1-\alpha}{1-\beta}\frac{\partial f}{\partial \alpha} = \frac{f(\alpha, m)}{Z} \frac{\frac{2(1-\gamma)A_{2,H}}{(1-\alpha)(1-\beta)}(4(1-\beta)(m-\alpha B_{2,L}) + 2(\beta-\gamma)m) + 8(1-\alpha)B_{2,L}Z}{[4(1-\beta) + Z + 2(\beta-\gamma)]m - 4\alpha(1-\beta)B_{2,L}},$$

and

$$
\begin{aligned}
&2\frac{1-\alpha}{1-\beta}\frac{\partial f}{\partial \alpha}-\frac{f(\alpha,m)}{1-\beta}\\
=&\frac{f(\alpha,m)}{Z}\frac{\frac{2(1-\gamma)A_{2,H}}{(1-\alpha)(1-\beta)}(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)+8(1-\alpha)B_{2,L}Z}{[4(1-\beta)+Z+2(\beta-\gamma)]m-4\alpha(1-\beta)B_{2,L}}-\frac{f(\alpha,m)}{1-\beta}\\
=&\frac{f(\alpha,m)}{Z}\frac{(\frac{2(1-\gamma)A_{2,H}}{(1-\alpha)(1-\beta)}-\frac{Z}{1-\beta})(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)+(8(1-\alpha)B_{2,L}-\frac{Zm}{1-\beta})Z}{[4(1-\beta)+Z+2(\beta-\gamma)]m-4\alpha(1-\beta)B_{2,L}}.
\end{aligned}
$$

Denote $h_1(\alpha)=\frac{f(\alpha,m)}{Z}(\frac{2(1-\gamma)A_{2,H}}{(1-\alpha)(1-\beta)}-\frac{Z}{1-\beta})(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)$ and $h_2(\alpha)=\frac{f(\alpha,m)}{Z}(8(1-\alpha)B_{2,L}-\frac{Zm}{1-\beta})Z$, we have

$$
\begin{aligned}
&h_1(\alpha)-(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)\\
=&(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)[\frac{f(\alpha,m)}{Z}(\frac{2(1-\gamma)A_{2,H}}{(1-\alpha)(1-\beta)}-\frac{Z}{1-\beta})-1]\\
=&(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)[\frac{m(\frac{2(1-\gamma)A_{2,H}}{1-\alpha}-Z)}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}-1]\\
<&(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)\frac{2m(1-\gamma)(A_{2,H}-1)-2(1-\beta)(m-2\alpha B_{2,L})-2(\beta-\gamma)m}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}.
\end{aligned}
$$

It can be verified that $(1-\gamma)A_{2,H}-(1-\gamma)<0$ and $m-2\alpha B_{2,L}\geq A_{2,L}-2\alpha B_{2,L}>0$. Therefore,

$$h_1(\alpha)-(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)<0.$$

In addition,

$$
\begin{aligned}
&h_2(\alpha)-Zm\\
=&\frac{f(\alpha,m)}{Z}[8(1-\alpha)B_{2,L}-\frac{Zm}{1-\beta}]Z-Zm\\
=&\frac{[8(1-\alpha)(1-\beta)B_{2,L}-Zm]Zm}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}-Zm\\
=&\frac{8(1-\alpha)(1-\beta)B_{2,L}-Zm-[2(\beta-\gamma)+Z+4(1-\beta)]m+4\alpha(1-\beta)B_{2,L}}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}Zm\\
=&\frac{4(1-\alpha)(1-\beta)B_{2,L}-2Zm-2(\beta-\gamma)m-4(1-\beta)m+4(1-\beta)B_{2,L}}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}Zm\\
<&\frac{-4(1-\beta)(m-2B_{2,L})-2Zm-2(\beta-\gamma)m}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}Zm\\
<&0.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
2\frac{1-\alpha}{1-\beta}\frac{\partial f}{\partial \alpha} - \frac{f(\alpha,m)}{1-\beta} - 1 = & \frac{h_1(\alpha) - (4(1-\beta)(m-\alpha B_{2,L}) + 2(\beta-\gamma)m) + h_2(\alpha) - Zm}{[2(\beta-\gamma)+Z+4(1-\beta)]m - 4\alpha(1-\beta)B_{2,L}} \\
& < 0,
\end{aligned}
$$

which implies $\frac{\partial g}{\partial \alpha} < 0$. □

Lemma 7. $h(\alpha, m) = g(\alpha, m) + \alpha B_{2,H}(\frac{f(\alpha,m)}{Z})^2$, where $g(\alpha, m)$ and $f(\alpha, m)$ are defined the same as Lemma 6, is strictly decreasing w.r.t $\alpha \in [\gamma, 1)$ for any m satisfying $m \geq A_{2,L}$.

Proof. Taking derivative, we have

$$
\begin{aligned}
\frac{\partial h}{\partial \alpha} =& \frac{1-\beta}{4(1-\gamma)}(1+\frac{f(\alpha,m)}{1-\beta})(2\frac{1-\alpha}{1-\beta}\frac{\partial f}{\partial \alpha} - \frac{f(\alpha,m)}{1-\beta} - 1) \\
&+ B_{2,H}\frac{f(\alpha,m)}{Z}[\frac{f(\alpha,m)}{Z} + 2\alpha(\frac{1}{Z}\cdot\frac{\partial f}{\partial \alpha} - \frac{1}{Z^2}\frac{(1-\gamma)A_{2,H}}{(1-\alpha)^2}f(\alpha,m))].
\end{aligned}
$$

Since $\alpha \geq \gamma$, it has $Z \geq X > 2A_{2,L}$. By Lemma 3, $\frac{f(\alpha,m)}{Z} < \frac{1}{2}$. Therefore, it has

$$
\begin{aligned}
\frac{1-\beta}{4(1-\gamma)}(1+\frac{f(\alpha,m)}{1-\beta}) =& \frac{1-\beta}{4(1-\gamma)} + \frac{f(\alpha,m)}{4(1-\gamma)} \\
>& \frac{2(1-\beta)}{4(1-\gamma)}\frac{f(\alpha,m)}{Z} + \frac{f(\alpha,m)}{4(1-\gamma)} \\
=& \frac{f(\alpha,m)}{Z}\frac{2(1-\beta)+Z}{4(1-\gamma)} \\
>& \frac{f(\alpha,m)}{Z}\frac{2(1-\beta)+2A_{2,L}}{4(1-\gamma)} \\
=& \frac{f(\alpha,m)}{Z}\frac{2(1-\beta)}{(4-\beta)(1-\gamma)} \\
=& \frac{f(\alpha,m)}{Z}B_{2,H}\frac{4-\beta}{2(1-\gamma)}.
\end{aligned}
$$

We can bound the derivative as follows

$$
\begin{aligned}
\frac{\partial h}{\partial \alpha} < \frac{1-\beta}{4(1-\gamma)}(1+\frac{f(\alpha,m)}{1-\beta})[2\frac{1-\alpha}{1-\beta}\frac{\partial f}{\partial \alpha} - \frac{f(\alpha,m)}{1-\beta} - 1 + \frac{2(1-\gamma)}{4-\beta}\frac{f(\alpha,m)}{Z} \\
+ \frac{4\alpha(1-\gamma)}{4-\beta}(\frac{1}{Z}\cdot\frac{\partial f}{\partial \alpha} - \frac{1}{Z^2}\frac{(1-\gamma)A_{2,H}}{(1-\alpha)^2}f(\alpha,m))].
\end{aligned}
$$

Rearranging the inequality, we have

$$
\frac{\partial h}{\partial \alpha} < \frac{1-\beta}{4(1-\gamma)}(1+\frac{f(\alpha,m)}{1-\beta})(M_1\frac{\partial f}{\partial \alpha} - M_2\frac{f(\alpha,m)}{Z} - 1),
$$

where

$$
\begin{aligned}
M_1 &= 2\frac{1-\alpha}{1-\beta} + \frac{4\alpha(1-\gamma)}{(4-\beta)Z}, \\
M_2 &= \frac{Z}{1-\beta} - \frac{2(1-\gamma)}{4-\beta} + \frac{4\alpha(1-\gamma)}{(4-\beta)Z}\cdot\frac{(1-\gamma)A_{2,H}}{(1-\alpha)^2}.
\end{aligned}
$$

By calculation,

$$\begin{aligned}
&M_1\frac{\partial f}{\partial \alpha}-M_2\frac{f(\alpha,m)}{Z}\\
=&\frac{f(\alpha,m)}{Z}\cdot\frac{N_1(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)+N_2Zm}{[4(1-\beta)+Z+2(\beta-\gamma)]m-4\alpha(1-\beta)B_{2,L}}
\end{aligned}$$

where

$$\begin{aligned}
N_1&=\frac{2(1-\gamma)A_{2,H}}{(1-\alpha)(1-\beta)}-\frac{Z}{1-\beta}+\frac{2(1-\gamma)}{4-\beta}\\
N_2&=8(1-\alpha)B_{2,L}-\frac{Zm}{1-\beta}+\frac{2(1-\gamma)m}{4-\beta}+\frac{4\alpha(1-\gamma)}{(4-\beta)Z}(4(1-\beta)B_{2,L}-\frac{(1-\gamma)A_{2,H}}{(1-\alpha)^2}m)\\
&<8(1-\alpha)B_{2,L}-\frac{Zm}{1-\beta}+\frac{2(1-\gamma)m}{4-\beta},
\end{aligned}$$

where the last inequality is due to our assumption that $m\geq A_{2,L}$. Let $h_1(\alpha)=\frac{f(\alpha,m)}{Z}N_1(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)$ and $h_2(\alpha)=\frac{f(\alpha,m)}{Z}N_2Z$.

$$\begin{aligned}
&h_1(\alpha)-(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)\\
=&(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)[\frac{f(\alpha,m)}{Z}(\frac{2(1-\gamma)A_{2,H}}{(1-\alpha)(1-\beta)}-\frac{Z}{1-\beta}+\frac{2(1-\gamma)}{4-\beta})-1]\\
=&(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)[\frac{m(\frac{2(1-\gamma)A_{2,H}}{1-\alpha}-Z+\frac{2(1-\gamma)(1-\beta)}{4-\beta})}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}-1]\\
<&(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)\frac{2m(1-\gamma)(A_{2,H}-1+\frac{1-\beta}{4-\beta})-2(1-\beta)(m-2\alpha B_{2,L})-2(\beta-\gamma)m}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}.
\end{aligned}$$

It can be verified that $A_{2,H}-1+\frac{1-\beta}{4-\beta}=-\frac{1+2\beta}{4-\beta}<0$, and $m-2\alpha B_{2,L}\geq A_{2,L}-2\alpha B_{2,L}>0$. So we have

$$h_1(\alpha)-(4(1-\beta)(m-\alpha B_{2,L})+2(\beta-\gamma)m)<0.$$

In addition,

$$
\begin{aligned}
&h_2(\alpha) - Zm \\
=&\frac{f(\alpha, m)}{Z}[8(1-\alpha)B_{2,L} - \frac{Zm}{1-\beta} + \frac{2(1-\gamma)m}{4-\beta}]Z - Zm \\
=&\frac{[8(1-\alpha)(1-\beta)B_{2,L} - Zm + \frac{2(1-\gamma)(1-\beta)m}{4-\beta}]Zm}{[2(\beta-\gamma) + Z + 4(1-\beta)]m - 4\alpha(1-\beta)B_{2,L}} - Zm \\
=&\frac{4(1-\alpha)(1-\beta)B_{2,L} + \frac{2(1-\gamma)(1-\beta)m}{4-\beta} - 2Zm - 2(1-\gamma)m - 2(1-\beta)(m - 2B_{2,L})}{[2(\beta-\gamma) + Z + 4(1-\beta)]m - 4\alpha(1-\beta)B_{2,L}}Zm \\
=&\frac{4(1-\alpha)(1-\beta)\frac{\beta A_{2,H}}{2(4-\beta)} + \frac{2(1-\gamma)(1-\beta)m}{4-\beta} - 2Zm - 2(1-\gamma)m - 2(1-\beta)(m - 2B_{2,L})}{[2(\beta-\gamma) + Z + 4(1-\beta)]m - 4\alpha(1-\beta)B_{2,L}}Zm \\
=&\frac{2(1-\alpha)A_{2,L}A_{2,H} + \frac{2(1-\gamma)(1-\beta)m}{4-\beta} - 2Zm - 2(1-\gamma)m - 2(1-\beta)(m - 2B_{2,L})}{[2(\beta-\gamma) + Z + 4(1-\beta)]m - 4\alpha(1-\beta)B_{2,L}}Zm \\
\leq&\frac{2(1-\gamma)mA_{2,H} + \frac{2(1-\gamma)(1-\beta)m}{4-\beta} - 2Zm - 2(1-\gamma)m - 2(1-\beta)(m - 2B_{2,L})}{[2(\beta-\gamma) + Z + 4(1-\beta)]m - 4\alpha(1-\beta)B_{2,L}}Zm \\
=&\frac{2(1-\gamma)m(A_{2,H} + \frac{1-\beta}{4-\beta} - 1) - 2Zm - 2(1-\beta)(m - 2B_{2,L})}{[2(\beta-\gamma) + Z + 4(1-\beta)]m - 4\alpha(1-\beta)B_{2,L}}Zm \\
<&0.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
M_1\frac{\partial f}{\partial \alpha} - M_2\frac{f(\alpha, m)}{Z} - 1 &= \frac{h_1(\alpha) - (4(1-\beta)(m - \alpha B_{2,L}) + 2(\beta-\gamma)m) + h_2(\alpha) - mZ}{2(\beta-\gamma) + Z + 4(1-\beta)]m - 4\alpha(1-\beta)B_{2,L}} \\
&< 0,
\end{aligned}
$$

which implies $\frac{\partial h}{\partial \alpha} < 0$. □

Appendix B. Proof of Proposition 1

Lemma 8. Given two linear functions $y = f(x) = ax + b$, $y = g(x) = cx + d$ satisfying $a > c > 0$ and $d \geq b \geq 0$, the two linear functions have a unique intersection $(\hat{x}, \hat{y})$. For any $k \in \mathbb{R}$, $\hat{y} \leq k$ if $g(f^{-1}(k)) \leq k$.

Proof. Calculating intersection of $y = f(x)$ and $y = g(x)$ gives

$$\hat{y} = \frac{d-b}{a-c}.$$

If $g(f^{-1}(k)) \leq k$, it has $f^{-1}(k) \leq g^{-1}(k)$, which is

$$\frac{k-b}{a} \leq \frac{k-d}{c} \implies k \geq \frac{d - \frac{c}{a}b}{a-c} > \frac{d-b}{a-c}.$$

Therefore, $k > \hat{y}$. □

Appendix B.1. Only firm H PM

Proof. We use backward induction to establish a unique SPNE in period 1. The revenue of firm H is

$$r_H(p_1) = \begin{cases} \alpha B_{2,H}(\frac{p_{1,L}}{X})^2 & \text{region I,} \\ [p_{1,H} - \alpha(p_{1,H} - A_{2,H}p_{1,H})](1 - p_{1,H}) + \alpha B_{2,H}(p_{1,H})^2 & \text{region II,} \\ [p_{1,H} - \alpha(p_{1,H} - A_{2,H}\frac{p_{1,L}}{X})](1 - \frac{1-\gamma}{1-\beta}p_{1,H} - \frac{\gamma-\beta}{1-\beta}\frac{p_{1,L}}{X}) + \alpha B_{2,H}(\frac{p_{1,L}}{X})^2 & \text{region III,} \\ \alpha B_{2,H} & \text{region IV.} \end{cases} \tag{B.1}$$

Similarly, the revenue function of firm L is

$$r_L(p_1) = \begin{cases} p_{1,L}(1 - \frac{p_{1,L}}{X}) + \alpha B_{2,L}(\frac{p_{1,L}}{X})^2 & \text{region I,} \\ \alpha B_{2,L}(p_{1,H})^2 & \text{region II,} \\ p_{1,L}(\frac{1-\gamma}{1-\beta}p_{1,H} + \frac{\gamma-\beta}{1-\beta}\frac{p_{1,L}}{X} - \frac{p_{1,L}}{X}) + \alpha B_{2,L}(\frac{p_{1,L}}{X})^2 & \text{region III,} \\ \alpha B_{2,L} & \text{region IV.} \end{cases} \tag{B.2}$$

Taking partial derivatives, we have

$$\frac{\partial r_H(p_1)}{\partial p_{1,H}} = \begin{cases} 0 & \text{region I,} \\ [1 - \alpha(1 - A_{2,H})](1 - 2p_{1,H}) + 2\alpha B_{2,H}p_{1,H} & \text{region II,} \\ (1-\alpha) - \frac{2(1-\alpha)(1-\gamma)}{1-\beta}p_{1,H} + [\frac{(\beta-\gamma)(1-\alpha)}{1-\beta} - \frac{\alpha(1-\gamma)A_{2,H}}{1-\beta}]\frac{p_{1,L}}{X} & \text{region III,} \\ 0 & \text{region IV.} \end{cases} \tag{B.3}$$

and

$$\frac{\partial r_L(p_1)}{\partial p_{1,L}} = \begin{cases} 1 - \frac{2p_{1,L}}{X} + 2\alpha B_{2,L}\frac{p_{1,L}}{X^2} & \text{region I,} \\ 0 & \text{region II,} \\ \frac{1-\gamma}{1-\beta}p_{1,H} + \frac{2(\gamma-1)X + 2\alpha(1-\beta)B_{2,L}}{(1-\beta)X^2}p_{1,L} & \text{region III,} \\ 0 & \text{region IV.} \end{cases} \tag{B.4}$$

We next show that there exists a unique Nash equilibrium in region III. The proof proceeds in three steps. We first calculate the first-order conditions in each region, and verify that the solution in region III is valid. We then show that the solution in region III is a Nash equilibrium. Finally, we show that the solutions in regions I, II, and IV cannot sustain as Nash equilibria, establishing the uniqueness of the equilibrium.

Step 1: Solve the first-order conditions in each region. We first solve the first-order conditions in each region, ignoring the boundary conditions. We obtain the following solutions:

$$\hat{p}_{1,H} = \begin{cases} \text{any value in range} & \text{region I,} \\ \frac{1-\alpha(1-A_{2,H})}{2[1-\alpha(1-A_{2,H})]-2\alpha B_{2,H}} & \text{region II,} \\ \frac{1-\beta}{2(1-\gamma)} + [\frac{\beta-\gamma}{2(1-\gamma)} - \frac{\alpha A_{2,H}}{2(1-\alpha)}]\frac{p_{1,L}}{X} & \text{region III,} \\ \text{any value in range} & \text{region IV.} \end{cases} \tag{B.5}$$

$$\hat{p}_{1,L} = \begin{cases} \frac{X^2}{2(X-\alpha B_{2,L})} & \text{region I,} \\ \text{any value in range} & \text{region II,} \\ \frac{(1-\beta)X^2}{[4(1-\beta)+\frac{\alpha(1-\gamma)}{1-\alpha}A_{2,H}+3(\beta-\gamma)]X-4\alpha(1-\beta)B_{2,L}} & \text{region III,} \\ \text{any value in range} & \text{region IV.} \end{cases} \tag{B.6}$$

We then show that the solution is valid in region III. The best response functions in region III for H and L are, respectively,

$$\begin{aligned} p_{1,H}(p_{1,L}) &= \frac{1-\beta}{2(1-\gamma)} + [\frac{\beta-\gamma}{2(1-\gamma)} - \frac{\alpha A_{2,H}}{2(1-\alpha)}]\frac{p_{1,L}}{X}, \\ p_{1,L}(p_{1,H}) &= \frac{X^2}{2(X - \frac{1-\beta}{1-\gamma}\alpha B_{2,L})}p_{1,H} \end{aligned} \tag{B.7}$$

The solution of the first-order conditions in region II is the intersection of the two best response functions. Because the response function $p_{1,H}(p_{1,L})$ is below the boundary of regions I and III in Figure 3b, we only need to show the response function $p_{1,L}(p_{1,H})$ is above the boundary of regions II and III, requiring that $\frac{X^2}{2(X-\frac{1-\beta}{1-\gamma}\alpha B_{2,L})} \leq X$. By Lemma 2 we have $X > A_{2,L} > 2\alpha B_{2,L}$, therefore we have

$$\frac{X^2}{2(X - \frac{1-\beta}{1-\gamma}\alpha B_{2,L})} < \frac{X^2}{2(X-\alpha B_{2,L})} = \frac{X^2}{X+(X-2\alpha B_{2,L})} < X \tag{B.8}$$

We conclude that the solution derived from first-order conditions in region III is valid.

Step 2: Prove that the solution in region III is a Nash equilibrium. We first show that firm L has no incentive to deviate. Note that firm L cannot increase revenue by deviating to the boundary between region II and III, because it is no longer a best response to the current $\hat{p}_{1,H}$. Therefore, L cannot increase revenue by deviating to any point in region II, in which L's revenue only depends on $\hat{p}_{1,H}$. Now we show that L cannot deviate to region I. To show this, we only need to show

$\hat{p}_{1,H} \leq \frac{1-\beta}{1-\gamma}$ so that L can never deviate to region I. By the best response functions (B.7), we have

$$
\begin{aligned}
p_{1,H}(p_{1,L}(\frac{1-\beta}{1-\gamma})) &= \frac{1-\beta}{2(1-\gamma)} + [\frac{\beta-\gamma}{2(1-\gamma)} - \frac{\alpha A_{2,H}}{2(1-\alpha)}]\frac{X}{2(\frac{1-\gamma}{1-\beta}X - \alpha B_{2,L})} \\
&= \frac{1-\beta}{2(1-\gamma)} + \frac{1-\beta}{2(1-\gamma)}[\frac{\beta-\gamma}{2(1-\gamma)} - \frac{\alpha A_{2,H}}{2(1-\alpha)}]\frac{X}{X - \frac{1-\beta}{1-\gamma}\alpha B_{2,L}} \\
&\leq \frac{1-\beta}{2(1-\gamma)} + \frac{1-\beta}{2(1-\gamma)}[\frac{\beta-\gamma}{2(1-\gamma)} - \frac{\alpha A_{2,H}}{2(1-\alpha)}]\frac{1}{1 - \frac{1-\beta}{1-\gamma}} \\
&= \frac{1-\beta}{2(1-\gamma)} + \frac{1-\beta}{2(1-\gamma)}[\frac{1}{2} - \frac{\alpha A_{2,H}(1-\gamma)}{2(1-\alpha)(\beta-\gamma)}] \\
&< \frac{1-\beta}{1-\gamma}.
\end{aligned}
$$

By Lemma 8, we have $\hat{p}_{1,H} \leq \frac{1-\beta}{1-\gamma}$. Therefore, L cannot deviate to region I.

We next show that firm H has no incentive to deviate. Notice that firm H cannot increase revenue be deviating to the boundary between region I and III, because it is no longer a best response to the current $\hat{p}_{1,L}$. Therefore, H cannot increase revenue by deviating to any point in region I, in which H's revenue only depends on $\hat{p}_{1,L}$. We then show that firm H has no incentive to deviate to region II. Given firm L's equilibrium price $\hat{p}_{1,L}$, the optimal price of firm H in region II can be solved by the first order conditions:

$$
\tilde{p}_{1,H} = \frac{1}{2 - \frac{2\alpha B_{2,H}}{1-\alpha(1-A_{2,H})}}. \tag{B.9}
$$

Therefore, $\tilde{p}_{1,H} \geq \frac{1}{2}$. Also, it has

$$
\frac{\hat{p}_{1,L}}{X} = \frac{(1-\beta)X}{[4(1-\beta) + \frac{\alpha(1-\gamma)}{1-\alpha}A_{2,H} + 3(\beta-\gamma)]X - 4\alpha(1-\beta)B_{2,L}}.
$$

By Lemma 3 and Lemma 2 we know that $\frac{\hat{p}_{1,L}}{X} < \frac{1}{2}$, which means $\tilde{p}_{1,H} > \frac{\hat{p}_{1,L}}{X}$. Therefore, the point $(\tilde{p}_{1,H}, \hat{p}_{1,L})$ lies above region II, implying that the revenue of firm H is maximized at the boundary of regions II and III within region II. Combining the arguments above shows that firm H has no incentive to deviate from region III to region II.

Step 3: Prove that solutions in region I, II, and IV cannot sustain as Nash equilibria. In region I, firm H can always deviate from region I to region III by lowering its price. Because the marginal valuation, at which a customer is indifferent in purchasing in the first and second period, is determined by the price of L in the first period in this case, firm H can earn higher revenue in period I and the same revenue in the future by deviating to region III.

The solution in region II cannot be a Nash equilibrium either. For a fixed price of firm H in the first period, firm L can always lower its price and set it to the best response (B.7) in region

III. Since it is the best response of L in region III, it will provide higher revenue for L than on the boundary of region II and region III. When prices are on the boundary of region II and region III, the proof in Step 2 shows that firm L earns higher profit than any circumstances in region II. Therefore, firm L can make higher revenue by deviating to region III.

The solution in region IV cannot be a Nash equilibrium. In region IV, both firms have zero sales in period 1, thus the number of remaining customers is the same as in the second period. Because both firms discount revenues over time, they would incur sales in period 1 rather than in period 2 unless there is no sales in the future anymore. But we have seen that there are positive sales in the last period. Therefore, no Nash equilibrium exists in region IV.

Finally, using the equilibrium prices in region III, we can calculate the revenues. □

Appendix B.2. Only firm L PM

Proof. We use backward induction to establish a unique SPNE in period 1. The revenue of firm H is

$$r_H(p_1) = \begin{cases} \alpha B_{2,H}(\frac{p_{1,L}}{Y})^2 & \text{region I,} \\ p_{1,H}(1 - \frac{p_{1,H}}{1-\beta+X}) + \alpha B_{2,H}(\frac{p_{1,H}}{1-\beta+X})^2 & \text{region II,} \\ p_{1,H}(1 - \frac{p_{1,H}-\frac{X}{Y}p_{1,L}}{1-\beta}) + \alpha B_{2,H}(\frac{p_{1,L}}{Y})^2 & \text{region III,} \\ \alpha B_{2,H} & \text{region IV.} \end{cases} \tag{B.10}$$

Similarly, the revenue function of firm L is

$$r_L(p_1) = \begin{cases} kp_{1,L}(1 - \frac{p_{1,L}}{Y}) + \alpha B_{2,L}(\frac{p_{1,L}}{Y})^2 & \text{region I,} \\ \alpha B_{2,L}(\frac{p_{1,H}}{1-\beta+X})^2 & \text{region II,} \\ kp_{1,L}(\frac{p_{1,H}-\frac{X}{Y}p_{1,L}}{1-\beta} - \frac{p_{1,L}}{Y}) + \alpha B_{2,L}(\frac{p_{1,L}}{Y})^2 & \text{region III,} \\ \alpha B_{2,L} & \text{region IV.} \end{cases} \tag{B.11}$$

Taking partial derivatives, we have

$$\frac{\partial r_H(p_1)}{\partial p_{1,H}} = \begin{cases} 0 & \text{region I,} \\ 1 - \frac{2p_{1,H}}{1-\beta+X} + \frac{2\alpha B_{2,H}p_{1,H}}{(1-\beta+X)^2} & \text{region II,} \\ 1 - \frac{2p_{1,H}-\frac{X}{Y}p_{1,L}}{1-\beta} & \text{region III,} \\ 0 & \text{region IV.} \end{cases} \tag{B.12}$$

and

$$\frac{\partial r_L(p_1)}{\partial p_{1,L}} = \begin{cases} k(1-\frac{2p_{1,L}}{Y}) + 2\alpha B_{2,L}\frac{p_{1,L}}{Y^2} & \text{region I,} \\ 0 & \text{region II,} \\ k(\frac{p_{1,H}-\frac{2X}{Y}p_{1,L}}{1-\beta} - \frac{2p_{1,L}}{Y}) + 2\alpha B_{2,L}\frac{p_{1,L}}{Y^2} & \text{region III,} \\ 0 & \text{region IV.} \end{cases} \tag{B.13}$$

We next show that there exists a unique Nash equilibrium in region III. The proof proceeds in three steps. We first calculate the first-order conditions in each region, and verify that the solution in region III is valid. We then show that the solution in region III is a Nash equilibrium. Finally, we show that the solutions in regions I, II, and IV cannot sustain as Nash equilibria, establishing the uniqueness of the equilibrium.

Step 1: Solve the first-order conditions in each region. We first solve the first-order conditions in each region, ignoring the boundary conditions. We obtain the following solutions:

$$\hat{p}_{1,H} = \begin{cases} \text{any value in range} & \text{region I,} \\ \frac{(1-\beta+X)^2}{2(1-\beta+X)-2\alpha B_{2,H}} & \text{region II,} \\ \frac{1-\beta+\frac{X}{Y}p_{1,L}}{2} & \text{region III,} \\ \text{any value in range} & \text{region IV.} \end{cases} \tag{B.14}$$

$$\hat{p}_{1,L} = \begin{cases} \frac{kY^2}{2(kY-\alpha B_{2,L})} & \text{region I,} \\ \text{any value in range} & \text{region II,} \\ \frac{(1-\beta)kY^2}{[3X+4(1-\beta)]kY-4\alpha(1-\beta)B_{2,L}} & \text{region III,} \\ \text{any value in range} & \text{region IV.} \end{cases} \tag{B.15}$$

We then show that the solution is valid in region III. The best response functions in region III for H and L are, respectively,

$$\begin{aligned} p_{1,H}(p_{1,L}) &= \frac{1-\beta+\frac{X}{Y}p_{1,L}}{2}, \\ p_{1,L}(p_{1,H}) &= \frac{kY^2}{2kY(1-\beta+X)-2\alpha(1-\beta)B_{2,L}}p_{1,H} \end{aligned} \tag{B.16}$$

The solution of the first-order conditions in region II is the intersection of the two best response functions. Because the response function $p_{1,H}(p_{1,L})$ is below the boundary of regions I and III in Figure 3c, we only need to show the response function $p_{1,L}(p_{1,H})$ is above the boundary of regions II and III, requiring that $\frac{kY^2}{2kY(1-\beta+X)-2\alpha(1-\beta)B_{2,L}} \leq \frac{Y}{1-\beta+X}$. By Lemma 2 we have $kY > A_{2,L} >$

$2\alpha B_{2,L}$, therefore we have

$$\frac{kY^2}{2kY(1-\beta+X)-2\alpha(1-\beta)B_{2,L}} < \frac{kY^2}{kY(1-\beta+X)+(1-\beta)(kY-2\alpha B_{2,L})} < \frac{Y}{1-\beta+X} \tag{B.17}$$

We conclude that the solution derived from first-order conditions in region III is valid.

Step 2: Prove that the solution in region III is a Nash equilibrium. We first show that firm L has no incentive to deviate. Note that firm L cannot increase revenue by deviating to the boundary between region II and III, because it is no longer a best response to the current $\hat{p}_{1,H}$. Therefore, L cannot increase revenue by deviating to any point in region II, in which L's revenue only depends on $\hat{p}_{1,H}$. Now we show that L cannot deviate to region I. To show this, we only need to show $\hat{p}_{1,H} \leq 1-\beta$ so that L can never deviate to region I. By the best response functions (B.16), we have

$$\begin{aligned}
p_{1,H}(p_{1,L}(1-\beta)) &= \frac{1-\beta}{2} + \frac{X}{2Y}\frac{(1-\beta)kY^2}{2kY(1-\beta+X)-2\alpha(1-\beta)B_{2,L}} \\
&= \frac{1-\beta}{2} + \frac{1}{2}\cdot\frac{(1-\beta)X}{2(1-\beta+X)-\frac{2\alpha(1-\beta)B_{2,L}}{kY}} \\
&= \frac{1-\beta}{2} + \frac{1}{2}\cdot\frac{(1-\beta)X}{X+\frac{2(1-\beta)}{kY}(kY-\alpha B_{2,L})+X} \\
&< \frac{1-\beta}{2} + \frac{1}{2}\cdot\frac{(1-\beta)X}{X} \\
&< 1-\beta.
\end{aligned}$$

By Lemma 8, we have $\hat{p}_{1,H} \leq 1-\beta$. Therefore, L cannot deviate to region I.

We next show that firm H has no incentive to deviate. Notice that firm H cannot increase revenue be deviating to the boundary between region I and III, because it is no longer a best response to the current $\hat{p}_{1,L}$. Therefore, H cannot increase revenue by deviating to any point in region I, in which H's revenue only depends on $\hat{p}_{1,L}$. We then show that firm H has no incentive to deviate to region II. Given firm L's equilibrium price $\hat{p}_{1,L}$, the optimal price of firm H in region II can be solved by the first order conditions:

$$\tilde{p}_{1,H} = \frac{(1-\beta+X)^2}{2(1-\beta+X)-2\alpha B_{2,H}}. \tag{B.18}$$

Therefore, $\tilde{p}_{1,H} \geq \frac{1-\beta+X}{2}$. Also, it has

$$\begin{aligned}
\frac{1-\beta+X}{Y}\hat{p}_{1,L} &= \frac{(1-\beta+X)\hat{p}_{1,H}}{2(1-\beta+X)-\frac{2\alpha(1-\beta)B_{2,L}}{kY}} \\
&= \frac{(1-\beta+X)\hat{p}_{1,H}}{2(1-\beta+X)-\frac{2\alpha\beta(1-\beta)B_{2,H}}{4kY}}.
\end{aligned}$$

Also it has

$$\frac{\beta(1-\beta)}{4kY} \leq \frac{\beta(1-\beta)}{4A_{2,L}} = \frac{4-\beta}{4} \leq 1,$$

which implies

$$\frac{1-\beta+X}{Y}\hat{p}_{1,L} \leq \frac{(1-\beta+X)\hat{p}_{1,H}}{2(1-\beta+X)-2\alpha B_{2,H}} \leq \frac{(1-\beta+X)^2}{2(1-\beta+X)-2\alpha B_{2,H}} = \tilde{p}_{1,H}.$$

Therefore, the point $(\tilde{p}_{1,H}, \hat{p}_{1,L})$ lies above region II, implying that the revenue of firm H is maximized at the boundary of regions II and III within region II. Combining the arguments above shows that firm H has no incentive to deviate from region III to region II.

Step 3: Prove that solutions in region I, II, and IV cannot sustain as Nash equilibria. This can be proved by repeating the argument of the case where only firm H PM.

Finally, using the equilibrium prices in region III, we can calculate the revenues. □

Appendix B.3. Both firms PM

Proof. We use backward induction to establish a unique SPNE in period 1. The revenue of firm H is

$$r_H(p_1) = \begin{cases} \alpha B_{2,H}(\frac{p_{1,L}}{Y})^2 & \text{region I,} \\ [p_{1,H} - \alpha(p_{1,H} - A_{2,H}p_{1,H})](1-p_{1,H}) + \alpha B_{2,H}(p_{1,H})^2 & \text{region II,} \\ [p_{1,H} - \alpha(p_{1,H} - A_{2,H}\frac{p_{1,L}}{Y})](1 - \frac{1-\gamma}{1-\beta}p_{1,H} - \frac{\gamma-\beta}{1-\beta}\frac{p_{1,L}}{Y}) + \alpha B_{2,H}(\frac{p_{1,L}}{Y})^2 & \text{region III,} \\ \alpha B_{2,H} & \text{region IV.} \end{cases} \tag{B.19}$$

Similarly, the revenue function of firm L is

$$r_L(p_1) = \begin{cases} kp_{1,L}(1 - \frac{p_{1,L}}{Y}) + \alpha B_{2,L}(\frac{p_{1,L}}{Y})^2 & \text{region I,} \\ \alpha B_{2,L}(p_{1,H})^2 & \text{region II,} \\ kp_{1,L}(\frac{1-\gamma}{1-\beta}p_{1,H} + \frac{\gamma-\beta}{1-\beta}\frac{p_{1,L}}{Y} - \frac{p_{1,L}}{Y}) + \alpha B_{2,L}(\frac{p_{1,L}}{Y})^2 & \text{region III,} \\ \alpha B_{2,L} & \text{region IV.} \end{cases} \tag{B.20}$$

Taking partial derivatives, we have

$$\frac{\partial r_H(p_1)}{\partial p_{1,H}} = \begin{cases} 0 & \text{region I,} \\ k(1-2p_{1,H}) + 2\alpha B_{2,H}p_{1,H} & \text{region II,} \\ (1-\alpha) - \frac{2(1-\alpha)(1-\gamma)}{1-\beta}p_{1,H} + [\frac{(\beta-\gamma)(1-\alpha)}{1-\beta} - \frac{\alpha(1-\gamma)A_{2,H}}{1-\beta}]\frac{p_{1,L}}{Y} & \text{region III,} \\ 0 & \text{region IV.} \end{cases} \tag{B.21}$$

and

$$\frac{\partial r_L(p_1)}{\partial p_{1,L}} = \begin{cases} k(1-\frac{2p_{1,L}}{Y}) + 2\alpha B_{2,L}\frac{p_{1,L}}{Y^2} & \text{region I,} \\ 0 & \text{region II,} \\ \frac{1-\gamma}{1-\beta}kp_{1,H} + \frac{2(\gamma-1)kY+2\alpha(1-\beta)B_{2,L}}{(1-\beta)Y^2}p_{1,L} & \text{region III,} \\ 0 & \text{region IV.} \end{cases} \tag{B.22}$$

We next show that there exists a unique Nash equilibrium in region III. The proof proceeds in three steps. We first calculate the first-order conditions in each region, and verify that the solution in region III is valid. We then show that the solution in region III is a Nash equilibrium. Finally, we show that the solutions in regions I, II, and IV cannot sustain as Nash equilibria, establishing the uniqueness of the equilibrium.

Step 1: Solve the first-order conditions in each region. We first solve the first-order conditions in each region, ignoring the boundary conditions. We obtain the following solutions:

$$\hat{p}_{1,H} = \begin{cases} \text{any value in range} & \text{region I,} \\ \frac{1-\alpha(1-A_{2,H})}{2[1-\alpha(1-A_{2,H})]-2\alpha B_{2,H}} & \text{region II,} \\ \frac{1-\beta}{2(1-\gamma)} + [\frac{\beta-\gamma}{2(1-\gamma)} - \frac{\alpha A_{2,H}}{2(1-\alpha)}]\frac{p_{1,L}}{Y} & \text{region III,} \\ \text{any value in range} & \text{region IV.} \end{cases} \tag{B.23}$$

$$\hat{p}_{1,L} = \begin{cases} \frac{kY^2}{2(kY-\alpha B_{2,L})} & \text{region I,} \\ \text{any value in range} & \text{region II,} \\ \frac{(1-\beta)kY^2}{[4(1-\beta)+\frac{\alpha(1-\gamma)}{1-\alpha}A_{2,H}+3(\beta-\gamma)]kY-4\alpha(1-\beta)B_{2,L}} & \text{region III,} \\ \text{any value in range} & \text{region IV.} \end{cases} \tag{B.24}$$

We then show that the solution is valid in region III. The best response functions in region III for H and L are, respectively,

$$\begin{aligned} p_{1,H}(p_{1,L}) &= \frac{1-\beta}{2(1-\gamma)} + [\frac{\beta-\gamma}{2(1-\gamma)} - \frac{\alpha A_{2,H}}{2(1-\alpha)}]\frac{p_{1,L}}{Y}, \\ p_{1,L}(p_{1,H}) &= \frac{kY^2}{2(kY-\frac{1-\beta}{1-\gamma}\alpha B_{2,L})}p_{1,H} \end{aligned} \tag{B.25}$$

The solution of the first-order conditions in region II is the intersection of the two best response functions. Because the response function $p_{1,H}(p_{1,L})$ is below the boundary of regions I and III in Figure 3d, we only need to show the response function $p_{1,L}(p_{1,H})$ is above the boundary of regions II and III, requiring that $\frac{kY^2}{2(kY-\frac{1-\beta}{1-\gamma}\alpha B_{2,L})} \le Y$. By Lemma 2 we have $kY > A_{2,L} > 2\alpha B_{2,L}$, therefore we have

$$\frac{kY^2}{2(kY - \frac{1-\beta}{1-\gamma}\alpha B_{2,L})} < \frac{kY^2}{2(kY - \alpha B_{2,L})} = \frac{kY^2}{kY + (kY - 2\alpha B_{2,L})} < Y. \tag{B.26}$$

We conclude that the solution derived from first-order conditions in region III is valid.

Step 2: Prove that the solution in region III is a Nash equilibrium. We first show that firm L has no incentive to deviate. Note that firm L cannot increase revenue by deviating to the boundary between region II and III, because it is no longer a best response to the current $\hat{p}_{1,H}$. Therefore, L cannot increase revenue by deviating to any point in region II, in which L's revenue only depends on $\hat{p}_{1,H}$. Now we show that L cannot deviate to region I. To show this, we only need to show $\hat{p}_{1,H} \le \frac{1-\beta}{1-\gamma}$ so that L can never deviate to region I. By the best response functions (B.25), we have

$$\begin{aligned}
p_{1,H}(p_{1,L}(\frac{1-\beta}{1-\gamma})) &= \frac{1-\beta}{2(1-\gamma)} + [\frac{\beta-\gamma}{2(1-\gamma)} - \frac{\alpha A_{2,H}}{2(1-\alpha)}]\frac{kY}{2(\frac{1-\gamma}{1-\beta}kY - \alpha B_{2,L})} \\
&= \frac{1-\beta}{2(1-\gamma)} + \frac{1-\beta}{2(1-\gamma)}[\frac{\beta-\gamma}{2(1-\gamma)} - \frac{\alpha A_{2,H}}{2(1-\alpha)}]\frac{kY}{kY - \frac{1-\beta}{1-\gamma}\alpha B_{2,L}} \\
&\le \frac{1-\beta}{2(1-\gamma)} + \frac{1-\beta}{2(1-\gamma)}[\frac{\beta-\gamma}{2(1-\gamma)} - \frac{\alpha A_{2,H}}{2(1-\alpha)}]\frac{1}{1 - \frac{1-\beta}{1-\gamma}} \\
&= \frac{1-\beta}{2(1-\gamma)} + \frac{1-\beta}{2(1-\gamma)}[\frac{1}{2} - \frac{\alpha A_{2,H}(1-\gamma)}{2(1-\alpha)(\beta-\gamma)}] \\
&< \frac{1-\beta}{1-\gamma}.
\end{aligned}$$

By Lemma 8, we have $\hat{p}_{1,H} \le \frac{1-\beta}{1-\gamma}$. Therefore, L cannot deviate to region I.

We next show that firm H has no incentive to deviate. Notice that firm H cannot increase revenue be deviating to the boundary between region I and III, because it is no longer a best response to the current $\hat{p}_{1,L}$. Therefore, H cannot increase revenue by deviating to any point in region I, in which H's revenue only depends on $\hat{p}_{1,L}$. We then show that firm H has no incentive to deviate to region II. Given firm L's equilibrium price $\hat{p}_{1,L}$, the optimal price of firm H in region II can be solved by the first order conditions:

$$\tilde{p}_{1,H} = \frac{1}{2 - \frac{2\alpha B_{2,H}}{1-\alpha(1-A_{2,H})}}. \tag{B.27}$$

Therefore, $\tilde{p}_{1,H} \ge \frac{1}{2}$. Also, it has

$$\frac{\hat{p}_{1,L}}{Y} = \frac{(1-\beta)kY}{[4(1-\beta) + \frac{\alpha(1-\gamma)}{1-\alpha}A_{2,H} + 3(\beta-\gamma)]kY - 4\alpha(1-\beta)B_{2,L}}.$$

By Lemma 2 and Lemma 3 we know that $\frac{\hat{p}_{1,L}}{Y} < \frac{1}{2}$, which means $\tilde{p}_{1,H} > \frac{\hat{p}_{1,L}}{Y}$. Therefore, the point $(\tilde{p}_{1,H}, \hat{p}_{1,L})$ lies above region II, implying that the revenue of firm H is maximized at the boundary of regions II and III within region II. Combining the arguments above shows that firm H has no incentive to deviate from region III to region II.

Step 3: Prove that solutions in region I, II, and IV cannot sustain as Nash equilibria. This can be proved by repeating the argument of the case where only firm H PM.

Finally, using the equilibrium prices in region III, we can calculate the revenues. □

Appendix C. Proof of Proposition 2

Based on proposition 1, firm L's first-period prices under different PM decisions are:

$$
\begin{aligned}
p_{1,L}^{(0)} &= \frac{(1-\beta)X^2}{[3X+4(1-\beta)]X-4\alpha(1-\beta)B_{2,L}},\\
p_{1,L}^{(h)} &= \frac{(1-\beta)X^2}{[2(\beta-\gamma)+Z+4(1-\beta)]X-4\alpha(1-\beta)B_{2,L}},\\
p_{1,L}^{(l)} &= \frac{(1-\beta)kY^2}{[3X+4(1-\beta)]kY-4\alpha(1-\beta)B_{2,L}},\\
p_{1,L}^{(b)} &= \frac{(1-\beta)kY^2}{[2(\beta-\gamma)+Z+4(1-\beta)]kY-4\alpha(1-\beta)B_{2,L}}.
\end{aligned}
$$

And the corresponding thresholds between purchasing at L in the first period and at H in the second period are

$$
\begin{aligned}
t^{(0)} &= \frac{p_{1,L}^{(0)}}{X} = \frac{1-\beta}{3X+4(1-\beta)-\frac{4\alpha(1-\beta)B_{2,L}}{X}},\\
t^{(h)} &= \frac{p_{1,L}^{(h)}}{X} = \frac{1-\beta}{2(\beta-\gamma)+Z+4(1-\beta)-\frac{4\alpha(1-\beta)B_{2,L}}{X}},\\
t^{(l)} &= \frac{p_{1,L}^{(l)}}{Y} = \frac{1-\beta}{3X+4(1-\beta)-\frac{4\alpha(1-\beta)B_{2,L}}{kY}},\\
t^{(b)} &= \frac{p_{1,L}^{(b)}}{Y} = \frac{1-\beta}{2(\beta-\gamma)+Z+4(1-\beta)-\frac{4\alpha(1-\beta)B_{2,L}}{kY}}.
\end{aligned}
$$

First consider firm L. When $\alpha<\gamma$, we have $kY>X$ by Lemma 2. Therefore, $t^{(l)}<t^{(0)}, t^{(b)}<t^{(h)}$. So firm L's PM incentivize more customers to purchase at the first period. When $\alpha=\gamma$, we have $kY=X$ by Lemma 2. Therefore, $t^{(l)}=t^{(0)}, t^{(b)}=t^{(h)}$. So firm L's PM has no effect. When $\alpha>\gamma$, we have $kY<X$ by Lemma 2. Therefore, $t^{(l)}>t^{(0)}, t^{(b)}>t^{(h)}$. So firm L's PM leads more customers to delay their purchase.

Then we consider firm H. It can be seen that

$$2(\beta-\gamma)+Z\begin{cases} < 3X, & \text{if } \alpha < \frac{3\gamma}{1+2\gamma}, \\ = 3X, & \text{if } \alpha = \frac{3\gamma}{1+2\gamma}, \\ > 3X, & \text{if } \alpha > \frac{3\gamma}{1+2\gamma}. \end{cases}$$

So we have

$$\begin{cases} t^{(h)} > t^{(0)}, t^{(b)} > t^{(l)}, & \text{if } \alpha < \frac{3\gamma}{1+2\gamma}, \\ t^{(h)} = t^{(0)}, t^{(b)} = t^{(l)}, & \text{if } \alpha = \frac{3\gamma}{1+2\gamma}, \\ t^{(h)} < t^{(0)}, t^{(b)} < t^{(l)}, & \text{if } \alpha > \frac{3\gamma}{1+2\gamma}. \end{cases}$$

The results are proved.

Appendix D. Proof of Proposition 3

We first introduce two Lemmas.

Lemma 9. $f(m) = \frac{(1-\beta)m^2}{[3X+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}$ and $g(m) = \frac{(1-\beta)m^2}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}$ are both strictly increasing on $[A_{2,L}, Y)$.

Proof. Taking derivative, we have

$$\begin{aligned} f'(m) &= \frac{(1-\beta)m[(3X+4(1-\beta))m-8\alpha(1-\beta)B_{2,L}]}{[(3X+4(1-\beta))m-4\alpha(1-\beta)B_{2,L}]^2} \\ &\geq \frac{(1-\beta)m[3Xm+4(1-\beta)(A_{2,L}-2\alpha B_{2,L})]}{[(3X+4(1-\beta))m-4\alpha(1-\beta)B_{2,L}]^2} \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} g'(m) &= \frac{(1-\beta)m[(2(\beta-\gamma)+Z+4(1-\beta))m-8\alpha(1-\beta)B_{2,L}]}{[(3X+4(1-\beta))m-4\alpha(1-\beta)B_{2,L}]^2} \\ &\geq \frac{(1-\beta)m[2(\beta-\gamma)m+Zm+4(1-\beta)(A_{2,L}-2\alpha B_{2,L})]}{[(3X+4(1-\beta))m-4\alpha(1-\beta)B_{2,L}]^2} \\ &> 0 \end{aligned}$$

because $A_{2,L} > 2\alpha B_{2,L}$ by Lemma 2. $\square$

Lemma 10. Let $f(\alpha) = \frac{Z(1-\beta)m}{[2(\beta-\gamma)+Z+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}$, then $g(\alpha) = \frac{(1-\alpha)(1-\beta)}{2(1-\gamma)}(1+\frac{f(\alpha)}{1-\beta})$ in strictly decreasing on $(0,1)$ for any $m \geq A_{2,L}$.

Proof. Notice that $f(\alpha), g(\alpha) > 0$. Let $h(\alpha) = f(\alpha)g(\alpha)$, we have

$$h'(\alpha) = f'(\alpha)g(\alpha) + f(\alpha)g'(\alpha) \implies g'(\alpha) = \frac{1}{f(\alpha)}(h'(\alpha) - f'(\alpha)g(\alpha))$$

Lemma 4 shows that $f(\alpha)$ is strictly increasing and Lemma 6 show that $h(\alpha)$ is strictly decreasing for any $m > A_{2,L}$. So we have $g'(\alpha) < 0$. □

The following is the proof of the proposition:

Proof. Based on proposition 1, firm L's first period effective prices under different PM decisions are:

$$\begin{aligned}
e_{1,L}^{(0)} &= p_{1,L}^{(0)} = \frac{(1-\beta)X^2}{[3X + 4(1-\beta)]X - 4\alpha(1-\beta)B_{2,L}}, \\
e_{1,L}^{(h)} &= p_{1,L}^{(h)} = \frac{(1-\beta)X^2}{[2(\beta-\gamma) + Z + 4(1-\beta)]X - 4\alpha(1-\beta)B_{2,L}}, \\
e_{1,L}^{(l)} &= kp_{1,L}^{(l)} = \frac{(1-\beta)(kY)^2}{[3X + 4(1-\beta)]kY - 4\alpha(1-\beta)B_{2,L}}, \\
e_{1,L}^{(b)} &= kp_{1,L}^{(b)} = \frac{(1-\beta)(kY)^2}{[2(\beta-\gamma) + Z + 4(1-\beta)]kY - 4\alpha(1-\beta)B_{2,L}}.
\end{aligned}$$

By Lemma 2, kY is strictly decreasing on α and $kY = X$ only when $\alpha = \gamma$. By Lemma 9, we have

$$e_{1,L}^{(l)} \begin{cases} > e_{1,L}^{(0)}, & \text{if } \alpha < \gamma \\ = e_{1,L}^{(0)}, & \text{if } \alpha = \gamma \\ < e_{1,L}^{(0)}, & \text{if } \alpha > \gamma \end{cases} \quad \text{and} \quad e_{1,L}^{(b)} \begin{cases} > e_{1,L}^{(h)}, & \text{if } \alpha < \gamma \\ = e_{1,L}^{(h)}, & \text{if } \alpha = \gamma \\ < e_{1,L}^{(h)}, & \text{if } \alpha > \gamma \end{cases}$$

Firm H's first period effective prices under different PM decisions are:

$$\begin{aligned}
e_{1,H}^{(0)} &= p_{1,H}^{(0)} = \frac{1-\beta+p_{1,L}^{(0)}}{2}, \\
e_{1,H}^{(h)} &= (p_{1,H}^{(h)} - \alpha(p_{1,H}^{(h)} - p_{2,H}^{(h)})) = \frac{(1-\alpha)(1-\beta)}{2(1-\gamma)}(1 + \frac{Zp_{1,L}^{(h)}}{(1-\beta)X}), \\
e_{1,H}^{(l)} &= p_{1,H}^{(l)} = \frac{1-\beta+\frac{X}{Y}p_{1,L}^{(l)}}{2}, \\
e_{1,H}^{(b)} &= (p_{1,H}^{(b)} - \alpha(p_{1,H}^{(b)} - p_{2,H}^{(b)})) = \frac{(1-\alpha)(1-\beta)}{2(1-\gamma)}(1 + \frac{Zp_{1,L}^{(b)}}{(1-\beta)Y}).
\end{aligned}$$

When $\alpha = \gamma$, we have

$$\begin{aligned}
e_{1,H}^{(h)} &= \frac{1-\beta+p_{1,L}^{(h)}}{2} > \frac{1-\beta+p_{1,L}^{(0)}}{2} = e_{1,H}^{(0)}, \\
e_{1,H}^{(b)} &= \frac{1-\beta+p_{1,L}^{(b)}}{2} > \frac{1-\beta+p_{1,L}^{(l)}}{2} = e_{1,H}^{(l)}.
\end{aligned}$$

By Lemma 10, $e_{1,H}^{(h)}$ and $e_{1,H}^{(b)}$ are decreasing on α. Besides, it is easy to see $e_{1,H}^{(0)}$ and $e_{1,H}^{(l)}$ are increasing on α. Therefore, there exists $\alpha_p \in (\gamma, 1)$ such that

$$e_{1,H}^{(h)} \begin{cases} > e_{1,H}^{(0)}, & \text{if } \alpha < \alpha_p \\ = e_{1,H}^{(0)}, & \text{if } \alpha = \alpha_p \\ < e_{1,H}^{(0)}, & \text{if } \alpha > \alpha_p \end{cases} \quad \text{and} \quad e_{1,H}^{(b)} \begin{cases} > e_{1,H}^{(l)}, & \text{if } \alpha < \alpha_p \\ = e_{1,H}^{(l)}, & \text{if } \alpha = \alpha_p \\ < e_{1,H}^{(l)}, & \text{if } \alpha > \alpha_p \end{cases}$$

□

Appendix E. Proof of Theorem 1

Proof. Let $n_1 = 2(\beta - \gamma) + Z + 4(1-\beta) - \frac{4\alpha(1-\beta)B_{2,L}}{m}$, $n_2 = 3X + 4(1-\beta) - \frac{4\alpha(1-\beta)B_{2,L}}{m}$, and $f(m,n) = \frac{(1-\beta)m}{n}(\frac{1}{2} - \frac{Z}{2(1-\beta)} \cdot \frac{1-\beta}{n} - \frac{1-\beta}{n}) + \alpha B_{2,L}(\frac{1-\beta}{n})^2$. By the definition of firm L's revenue,

$$r_L^{(0)} = f(X, n_2), r_L^{(l)} = f(kY, n_2), r_L^{(h)} = f(X, n_1), r_L^{(b)} = f(kY, n_1).$$

According to Lemma 5, we have

$$\begin{cases} r_L^{(0)} < r_L^{(l)} & \text{if } kY > X \\ r_L^{(0)} = r_L^{(l)} & \text{if } kY = X \\ r_L^{(0)} > r_L^{(l)} & \text{if } kY < X \end{cases} \quad and \quad \begin{cases} r_L^{(h)} < r_L^{(b)} & \text{if } kY > X \\ r_L^{(h)} = r_L^{(b)} & \text{if } kY = X \\ r_L^{(h)} > r_L^{(b)} & \text{if } kY < X \end{cases} \cdot$$

Lemma 2 says that kY is strictly decreasing w.r.t. α and $kY = X$ if and only if $\alpha = \gamma$. Therefore, we have

$$\begin{cases} r_L^{(0)} < r_L^{(l)} & \text{if } \alpha < \gamma \\ r_L^{(0)} = r_L^{(l)} & \text{if } \alpha = \gamma \\ r_L^{(0)} > r_L^{(l)} & \text{if } \alpha > \gamma \end{cases} \quad and \quad \begin{cases} r_L^{(h)} < r_L^{(b)} & \text{if } \alpha < \gamma \\ r_L^{(h)} = r_L^{(b)} & \text{if } \alpha = \gamma \\ r_L^{(h)} > r_L^{(b)} & \text{if } \alpha > \gamma \end{cases} \cdot$$

Let $f(\alpha, m) = \frac{Z(1-\beta)m}{[2(\beta-\gamma)+Z+4(1-\beta)]m - 4\alpha(1-\beta)B_{2,L}}$ and $g(\alpha, m) = \frac{(1-\alpha)(1-\beta)}{4(1-\gamma)}(1 + \frac{f(\alpha,m)}{1-\beta})^2$. By the definition of firm H's revenue,

$$r_H^{(h)} = g(\alpha, X) + \alpha B_{2,H}(\frac{f(\alpha, X)}{Z})^2, \quad r_H^{(b)} = g(\alpha, kY) + \alpha B_{2,H}(\frac{f(\alpha, kY)}{Z})^2.$$

First note that when $\alpha \leq \gamma$, it has

$$\begin{aligned} \frac{f(\alpha, X)}{Z} &= \frac{(1-\beta)X}{[2(\beta-\gamma) + Z + 4(1-\beta)]X - 4\alpha(1-\beta)B_{2,L}} \\ &> \frac{(1-\beta)X}{[3X + 4(1-\beta)]X - 4\alpha(1-\beta)B_{2,L}} \\ &= \frac{p_{1,L}^{(0)}}{X} \end{aligned}$$

and

$$
\begin{aligned}
\frac{f(\alpha, kY)}{Z} &= \frac{(1-\beta)kY}{[2(\beta-\gamma)+Z+4(1-\beta)]kY-4\alpha(1-\beta)B_{2,L}} \\
&> \frac{(1-\beta)kY}{[3X+4(1-\beta)]kY-4\alpha(1-\beta)B_{2,L}} \\
&= \frac{p_{1,L}^{(l)}}{Y}.
\end{aligned}
$$

According to Lemma 6, both $f(\alpha, m)$ is strictly decreasing with respect to α for any $m \geq A_{2,L}$. When $\alpha \leq \gamma$, we have

$$
\begin{aligned}
r_H^{(h)} =& g(\alpha, X) + \alpha B_{2,H}(\frac{f(\alpha, X)}{Z})^2 \\
\geq& g(\gamma, X) + \alpha B_{2,H}(\frac{f(\alpha, X)}{Z})^2 \\
=& \frac{1-\beta}{4}(1+\frac{X^2}{[2(\beta-\gamma)+X+4(1-\beta)]X-4\gamma(1-\beta)B_{2,L}})^2 + \alpha B_{2,H}(\frac{f(\alpha, X)}{Z})^2 \\
>& \frac{1-\beta}{4}(1+\frac{X^2}{[3X+4(1-\beta)]X-4\gamma(1-\beta)B_{2,L}})^2 + \alpha B_{2,H}(\frac{p_{1,L}^{(0)}}{X})^2 \\
\geq& \frac{1-\beta}{4}(1+\frac{X^2}{[3X+4(1-\beta)]X-4\alpha(1-\beta)B_{2,L}})^2 + \alpha B_{2,H}(\frac{p_{1,L}^{(0)}}{X})^2 \\
=& r_H^{(0)},
\end{aligned}
$$

and

$$
\begin{aligned}
r_H^{(b)} =& g(\alpha, kY) + \alpha B_{2,H}(\frac{f(\alpha, kY)}{Z})^2 \\
\geq& g(\gamma, kY) + \alpha B_{2,H}(\frac{f(\alpha, kY)}{Z})^2 \\
=& \frac{1-\beta}{4}(1+\frac{X\cdot kY}{[2(\beta-\gamma)+X+4(1-\beta)]kY-4\gamma(1-\beta)B_{2,L}})^2 + \alpha B_{2,H}(\frac{f(\alpha, kY)}{Z})^2 \\
>& \frac{1-\beta}{4}(1+\frac{X\cdot kY}{[3X+4(1-\beta)]kY-4\gamma(1-\beta)B_{2,L}})^2 + \alpha B_{2,H}(\frac{p_{1,L}^{(l)}}{Y})^2 \\
\geq& \frac{1-\beta}{4}(1+\frac{X\cdot kY}{[3X+4(1-\beta)]kY-4\alpha(1-\beta)B_{2,L}})^2 + \alpha B_{2,H}(\frac{p_{1,L}^{(l)}}{Y})^2 \\
=& r_H^{(l)},
\end{aligned}
$$

Therefore, we have $r_H^{(h)} > r_H^{(0)}$ and $r_H^{(b)} > r_H^{(l)}$ when $\alpha \leq \gamma$.

Now consider the case where $\alpha > \gamma$. By Lemma 7, $h(\alpha, m) = g(\alpha, m) + \alpha B_{2,H}(\frac{f(\alpha,m)}{Z})^2$ is decreasing w.r.t α for any $m \geq A_{2,L}$. Let $h_0 = g_0(\alpha, m) + \alpha B_{2,H}(\frac{f_0(\alpha,m)}{X})^2$ where $f_0(\alpha, m) = \frac{X(1-\beta)m}{[3X+4(1-\beta)]m-4\alpha(1-\beta)B_{2,L}}$ and $g_0(\alpha, m) = \frac{(1-\beta)}{4}(1+\frac{f_0(\alpha,m)}{1-\beta})^2$, it is easy to see $h_0(\alpha, m)$ is increasing w.r.t α. In addition, since $h_0(\alpha, m) > 0$ and $\lim_{\alpha\to 1^-} h(\alpha, m) = 0$, $h_0(\alpha, m)$ and $h(\alpha, m)$ must intersect at some $\alpha \in (\gamma, 1)$. Notice that $r_H^{(h)} = h(\alpha, X)$, $r_H^{(0)} = h_0(\alpha, X)$, $r_H^{(b)} = h(\alpha, kY)$, and $r_H^{(l)} = h_0(\alpha, kY)$. Therefore, there exists $\alpha_r \in (\gamma, 1)$ such that

$$\begin{cases} r_H^{(0)} < r_H^{(h)} & \text{if } \alpha < \alpha_r \\ r_H^{(0)} = r_H^{(h)} & \text{if } \alpha = \alpha_r \\ r_H^{(0)} > r_H^{(h)} & \text{if } \alpha > \alpha_r \end{cases} \quad and \quad \begin{cases} r_H^{(l)} < r_H^{(b)} & \text{if } \alpha < \alpha_r \\ r_H^{(l)} = r_H^{(b)} & \text{if } \alpha = \alpha_r \\ r_H^{(l)} > r_H^{(b)} & \text{if } \alpha > \alpha_r \end{cases}$$

□