Mathematical Foundation

Ahmed Khaled

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Abstract

This Document contain my notes about Axioms, Definitions and basic theories.

1 Real Numbers

In rigorous mathematics real number is a set of numbers defined as a complete, ordered field

1.1 Fields

- Def. *Field* is a non-empty set on which two $\overline{\text{binary operation}}$ are defined

- Def. *Binary Operation* in field $\mathbb{F}$ is a function that "take" an ordered pair of element and "return" an element in $\mathbb{F}$, and it said to be the operation on the set whose both domain and co-domain in the same set.

refer to group theory and set theory

$$\forall a, b \in \mathbb{F}(\exists c \in \mathbb{F}) : (c = a \circ b)$$

- the 9 golden basic most primitive axioms:

  1. Axi. *Associative law for addition* $(a + b) + c = a + (a + c)$
  2. Axi. *Existence of additive identity* $\exists 0 : a + 0 = 0 + a = a$
  3. Axi. *Existence of additive inverse* $\forall a \in \mathbb{R}\exists(-a) : a + (-a) = (-a) + a = 0$
  4. Axi. *Commutative law of addition* $a + b = b + a$
  5. Axi. *Associative law for multiplication* $(a \cdot b) \cdot c = a \cdot (a \cdot c)$
  6. Axi. *Existence of multiplicative identity* $\exists 1 \neq 0 : a \cdot 1 = 1 \cdot a = a$
  7. Axi. *Existence of multiplicative inverse* $\forall a \neq 0 \in \mathbb{R}\exists(a^{-1}) : a + (a^{-1}) = (a^{-1}) + a = 0$
  8. Axi. *Commutative law of multiplication* $a \cdot b = b \cdot a$
  9. Axi. *Distributive law* $a \cdot (b + c) = a \cdot b + a \cdot c$

- Theorem

Theorem 1. $\forall a \in \mathbb{F} : a \cdot 0 = 0$

Proof. using axiom Num.9

$$\begin{aligned} a \cdot 0 &= a \cdot (0 + 0) \\ &= a \cdot 0 + a \cdot 0 \end{aligned}$$

by adding $-(a \cdot 0)$ to both side

$$a \cdot 0 = 0$$

■

1.2 Order

- **Def.** *Ordered field* $\mathbb{F}$. A field is said to be ordered if it has a distinguished subset $\overline{P \subset \mathbb{F}}$, that have the follow properties: Positive Numbers

  1. *Trichotomy*: which mean every element $a \in \mathbb{F}$ satisfied one and only one of the follow
     (a) $a \in P$
     (b) $-a \in P$
     (c) $a = 0$
  2. *Closure under addition* $\forall a, b(a, b \in P \implies a + b \in P)$
  3. *Closure under multiplication* $\forall a, b(a, b \in P \implies a \cdot b \in P)$

  and we said that $a < b$ means $b - a \in P$. which is clear if b bigger than a then the difference between them is positive number.

- Theorem

  Theorem 2. $\forall a, b \in \mathbb{F}$ *one fo the following hold*
  1. $a < b$ *2.* $a > b$ *3.* $a = b$

Proof. using Trichotomy one of these hold 1. $a, b \in P$ then it either 1. $a - b \in P$ then we say $a < b$ 2. $b - a \in P$ then we say $a > b$ 2. $a, -b \in p$ then by using Closure under addition $a + (-b) \in P$ then we say $a > b$ 3. the opposite of Num.2 ∎

Theorem 3. $a < b \implies a + c < b + c$

Proof. Suppose $a + c < b + c$, then it means $a + c - (b + c) \in P$ which deduce to $a < b$ ∎

Theorem 4. *Transitivity.* $a < b \land b < c \implies a < c$

Proof. $a < b$ means $b - a \in P$ and $b < c$ means $c - b \in P$. Thuh, $c - b - a + b \in P$ which means $a < c$ ∎

Theorem 5. $a, b < 0 \implies a \cdot b > 0$

Proof. Suppose $a, b < 0$ then $-a \cdot -b \in P$, thuh $a \cdot b > 0$ ∎

Corollary 6. $\forall a \neq 0 : a \cdot a \equiv a^2 > 0$

Proof. There is two cases 1. $a > 0$ in this case $a^2 > 0$ by closure under multiplication 2. $a < 0$ is a spicial case from previous theorem when $a = b$ ∎