Tutorial 5

3.45. Suppose we have 10 coins such that if the ith coin is flipped, heads will appear with probability $i/10, i = 1, 2, \ldots, 10$. When one of the coins is randomly selected and flipped, it shows heads. What is the conditional probability that it was the fifth coin?

Solution: Let C_i denote the event that the ith coin is selected. Then, we know that $P(H \mid C_i) = i/10$, and $P(C_i) = 1/10$. Therefore,

$$\begin{aligned} P(C_5 \mid H) &= \frac{P(C_5 \cap H)}{P(H)} \\ &= \frac{P(H \mid C_5)P(C_5)}{\sum_{i=1}^{10} P(H \mid C_i)P(C_i)} \\ &= \frac{(5/10) \times (1/10)}{\sum_{i=1}^{10} (i/10) \times (1/10)} \\ &= \frac{5}{1 + \cdots + 10} = \frac{1}{11}. \end{aligned}$$

The last line follows from the fact that $(1 + \cdots + 10) = \binom{11}{2} = 55$. Can you think of a combinatorial proof for this?) More generally, for all $\ell = 1, \ldots, 10$ we have

$$P(C_\ell \mid H) = \frac{\ell}{1 + \cdots + 10} = \frac{\ell}{55}.$$

3.59. Independent flips of a coin that lands on heads with probability p are made. What is the probability that the first four outcomes are

(a) H, H, H, H?

(b) T, H, H, H?

(c) What is the probability that the pattern T, H, H, H occurs before the pattern H, H, H, H?

Hint for part (c): How can the pattern H, H, H, H occur first?

(a) According to independence, the probability for H, H, H, H is p^4.

(b) According to independence, the probability for T, H, H, H is $p^3(1-p)$.

(c) The trick is that the only possible outcome such that a $HHHH$ pattern appears before a $THHH$ is a sequence start with $HHHH$ (think about it), and this probability is p^4. Therefore, the probability that the pattern T, H, H, H occurs before the pattern H, H, H, H is $1 - p^4$.

3.12. Maria will take two books with her on a trip. Suppose that the probability that she will like book 1 is .6, the probability that she will like book 2 is .5, and the probability that she will like both books is .4. Find the conditional probability that she will like book 2 given that she did not like book 1.

3.12. Let L_i be the event that Maria likes book $i, i = 1, 2$. Then

$$P(L_2|L_1^c) = \frac{P(L_1^c L_2)}{P(L_1^c)} = \frac{P(L_1^c L_2)}{.4}$$

Using that L_2 is the union of the mutually exclusive events $L_1 L_2$ and $L_1^c L_2$, we see that

$$.5 = P(L_2) = P(L_1 L_2) + P(L_1^c L_2) = .4 + P(L_1^c L_2)$$

Thus,

$$P\left(L_2|L_1^c\right) = \frac{.1}{.4} = .25$$

3.21. If A flips $n+1$ and B flips n fair coins, show that the probability that A gets more heads than B is $\frac{1}{2}$. *Hint*: Condition on which player has more heads after each has flipped n coins. (There are three possibilities.)

3.21. Let $E = \{A \text{ gets more heads than } B\}$; then

$$\begin{aligned}P(E) &= P(E|A \text{ leads after both flip } n)P(A \text{ leads after both flip } n)\\&\quad + P(E|\text{ even after both flip } n)P(\text{ even after both flip } n)\\&\quad + P(E|B \text{ leads after both flip } n)P(B \text{ leads after both flip } n)\\&= P(A \text{ leads}) + \frac{1}{2}P(\text{ even})\end{aligned}$$

Now, by symmetry,

$$\begin{aligned}P(A \text{ leads}) &= P(B \text{ leads})\\&= \frac{1 - P(\text{even})}{2}\end{aligned}$$

Hence,

$$P(E) = \frac{1}{2}$$

3.22. Prove or give counterexamples to the following statements:

(a) If E is independent of F and E is independent of G, then E is independent of $F \cup G$.

(b) If E is independent of F, and E is independent of G, and $FG = \emptyset$, then E is independent of $F \cup G$.

(c) If E is independent of F, and F is independent of G, and E is independent of FG, then G is independent of EF.

3.22. (a) Not true: In rolling 2 dice, let $E = \{\text{sum is } 7\}$, $F = \{\text{1st die does not land on 4}\}$, and $G = \{\text{2nd die does not land on 3}\}$. Then

$$P(E|F \cup G) = \frac{P\{7, \text{not } (4,3)\}}{P\{\text{not } (4,3)\}} = \frac{5/36}{35/36} = 5/35 \neq P(E)$$

(b)

$$\begin{aligned} P(E(F \cup G)) &= P(EF \cup EG) \\ &= P(EF) + P(EG) \qquad \text{since } EFG = \emptyset \\ &= P(E)[P(F) + P(G)] \\ &= P(E)P(F \cup G) \qquad \text{since } FG = \emptyset \end{aligned}$$

(c)

$$\begin{aligned} P(G|EF) &= \frac{P(EFG)}{P(EF)} \\ &= \frac{P(E)P(FG)}{P(EF)} \qquad \text{since } E \text{ is independent of } FG \\ &= \frac{P(E)P(F)P(G)}{P(E)P(F)} \qquad \text{by independence} \\ &= P(G). \end{aligned}$$

3.26. Show that if $P(A|B) = 1$, then $P(B^c|A^c) = 1$.

3.26. We are given that $P(AB) = P(B)$ and must show that this implies that $P(B^cA^c) = P(A^c)$. One way is as follows:

$$\begin{aligned} P(B^cA^c) &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(AB) \\ &= 1 - P(A) \\ &= P(A^c) \end{aligned}$$

3.30. Show that, for any events E and F,

$$P(E|E \cup F) \geq P(E|F)$$

Hint: Compute $P(E|E \cup F)$ by conditioning on whether F occurs.

$$P(E|E \cup F) = P(E|F(E \cup F))P(F|E \cup F) + P(E|F^c(E \cup F))P(F^c|E \cup F)$$

Using

$$F(E \cup F) = F \quad \text{and} \quad F^c(E \cup F) = F^c E$$

gives

$$\begin{aligned} P(E|E \cup F) &= P(E|F)P(F|E \cup F) + P(E|EF^c)P(F^c|E \cup F) \\ &= P(E|F)P(F|E \cup F) + P(F^c|E \cup F) \\ &\geq P(E|F)P(F|E \cup F) + P(E|F)P(F^c|E \cup F) \\ &= P(E|F) \end{aligned}$$

3.31

There is a 60% chance that event A will occur, if A does not occur, then there is a 10% percent chance that B will occur.

What is the probability that at least one of the events A or B will occur?
