Problem 5 :

Let b_1 : first ball drawn is ~~white~~ black w_1 : first ball drawn is white

w_2 : second ball drawn is white

w_3 : third "

w_4 : fourth "

a)

$$P(b_1|w_2) = \frac{P(w_2 \cap b_1)}{P(w_2)} = \frac{P(w_2|b_1).P(b_1)}{P(w_2|b_1).P(b_1) + P(w_2|w_1).P(w_1)}$$

$$= \frac{\frac{10}{18} \times \frac{9}{19}}{\frac{10}{18}\times\frac{9}{19} + \frac{9}{18} \times \frac{10}{19}} = \frac{1}{2}$$

b)

$$P(b_1|w_2w_3) = \frac{P(w_2w_3b_1)}{P(w_2w_3)} = \frac{P(w_2w_3|b_1).P(b_1)}{P(w_2w_3|b_1).P(b_1) + P(w_2w_3|w_1).P(w_1)}$$

$$= \frac{\frac{9}{17} \times \frac{10}{18} \times \frac{9}{19}}{\frac{9}{17}\times\frac{10}{18}\times\frac{9}{19} + \frac{8}{17}\times\frac{9}{18}\times\frac{10}{19}} = \frac{9}{17}$$

c)

$$P(w_4|w_2w_3) = \frac{P(w_2w_3w_4)}{P(w_2w_3)}$$

$$P(w_2w_3w_4) = P(w_2w_3w_4|b_1)\,P(b_1) + P(w_2w_3w_4|w_1).P(w_1)$$

$$= \frac{8}{16}\times\frac{9}{17}\times\frac{10}{18}\times\frac{9}{19} + \frac{7}{16}\times\frac{8}{17}\times\frac{9}{18}\times\frac{10}{19}$$

$$= \frac{10\times9\times8\,(9+7)}{19\times18\times17\times16} = \frac{10\times9\times8\times\cancel{16}}{19\times18\times17\times\cancel{16}} = \frac{10\times9\times8}{19\times18\times1}$$

$$P(w_2 w_3) = \frac{9\times10\times9 + 8\times9\times10}{19\times18\times17} \quad \text{(from part b)}$$

$$= \frac{9\times10\times17}{19\times18\times17}$$

$$\Rightarrow P(w_4 | w_2 w_3) = \frac{\frac{10\times9\times8}{19\times18\times17}}{\frac{10\times9\times17}{19\times18\times17}} = \frac{8}{17} = 0.47$$

Problem 6

Outcomes of two die which gives 7 on addition are

(1,6), (6,1), (2,5), (5,2), (3,4) (4,3)

Let E : sum is 7 $\Rightarrow$ $P(E) = \frac{6}{36} = \frac{1}{6}$

a) $P(\text{Gambler wins atleast twice in n trials}) = 1 - P(\text{Gambler wins atmost one time in n trials})$

$$= 1 - P(0 \text{ wins}) - P(1 \text{ win})$$

$$= 1 - (1 - P(n \text{ win})) - \binom{n}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1}$$

$$= \left(\frac{1}{6}\right)^n - n\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1}$$

b) Let P_k denote the probability that gambler wins k times out of n trials. Need to find k such that P_k is maximum

Consider

$$\frac{P_{k+1}}{P_k} = \frac{\binom{n}{k+1}\left(\frac{1}{6}\right)^{k+1}\left(\frac{5}{6}\right)^{n-k-1}}{\binom{n}{k}\left(\frac{1}{6}\right)^{k}\left(\frac{5}{6}\right)^{n-k}} = \frac{\binom{n}{k+1}\frac{1}{6}}{\binom{n}{k}\left(\frac{5}{6}\right)}$$

$$= \frac{n-k}{k+1} \times \frac{1}{5}$$

For $P_{k+1} \geq P_k$, $\Rightarrow$ $\frac{n-k}{k+1} \times \frac{1}{5} \geq 1 \Rightarrow \frac{n-5}{6} \geq k \Rightarrow \frac{n+1}{6} \geq k+1$

$$\Rightarrow \qquad x+1 \le \frac{n+1}{6}$$

$\therefore$ most likely times gambler wins is $\left\lfloor \frac{n+1}{6} \right\rfloor$