Introduction to Group Theory

What is a “group” ?

– A group “**G**” is a set of elements {a, b, c,….} together with a binary composition law that has the following properties:
  - – 1. **Closure**.
  - – 2. **Associativity**.
  - – 3. **Identity.**
  - – 4. **Inversibility.**

1. Closure

If a, b are elements of a group **G**, and their composition $a \cdot b = c$, c must also be an element of **G**.

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	e	d	c	a
c	c	d	f	e	a	b
d	d	c	a	b	f	e
f	f	b	c	a	e	d

Fig 1. Multiplication table of $\mathbf{S_3}$ group.

2. Associativity

For any elements a, b, c …. in the group **G**, *(ab)c=a(bc)*.

* $a \cdot b \neq b \cdot a$ in general. If all the elements in the group satisfy the equation above, then the group **G** is said to be Abelian.

3. Identity

There exist an identity **e**, such that $e \cdot a = a$ and $a \cdot e = a$.

4. Inversibility

Every element a in **G** has an inverse a^{-1} , which is also in **G**, and satisfies $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Properties of Groups

- 1. The identity element of a group is unique.
- 2. Cancellation law holds. i.e., $ab=ac$ implies $b=c$, $bd=cd$ implies $b=c$.
- 3. The inverse of an element is unique.
- 4. For a finite group, its **order** (the number of elements it contains)is denoted by |G|.
- 5. For a recuring product $g = a^q$ and $g^n = e$, the **order** of the element g is denoted by $|g| = n$. The **period** of g is a collection of $\{e, g^1, \ldots, g^{n-1}\}$.
- 6. For a set of elements $\{e, g^1, \ldots, g^{n-1}\}$, if the inverse of an element g^k follows $(g^k)^{-1} = g^{n-k}$, this set forms a group and is called cyclic group.

Example:

Within $\mathbf{S_3}$, $|\mathbf{S_3}| = 6$,

$|a| = |b| = |c| = 2$, and $|d| = |f| = 3$. The periods are $\{e, a\}$, $\{e, b\}$, $\{e, c\}$, $\{e, d, f = d^2\}$, and $\{e, f, d = f^2\}$.

Multiplication Table

The rearrangement theorem :

If $\{e, g_1, g_2, \ldots, g_{|G|}\}$ are the elements of a group **G**, and g_k is an element of **G**, then the set of elements $\boldsymbol{G}g_k = \{eg_k, g_1g_k, g_2g_k, \ldots, g_{|G|}g_k\}$ contains each group element once and only once.

(This means that each element in a group only appear once in each row and each column of a multiplication table.)

Example:

Consider a group with three distinct elements $\{e, a, b\}$.

Need to calculate ab, ba, a^2, b^2.

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Subgroups

- If we can select a sets of elements from a group **G**, which themselves also form a group **H**. **H** is then a **subgroup** of **G**.
- {e} itself forms an **improper subgroup** of any group **G**. The other subgroups are called **proper subgroups**.
- Lagrange's Theorem : the order of a subgroup **H** of a finite group **G** is a divisor of |**G**|. i.e., $\frac{|H|}{|G|} = integer$.

Example:

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	e	d	c	a
c	c	d	f	e	a	b
d	d	c	a	b	f	e
f	f	b	c	a	e	d

What are the proper subgroups of the $\mathbf{S_3}$ group?

{e, *a*} {e, *b*} {e, *c*} {e, *d*, *f*}

Cosets

- Definition: if $\boldsymbol{H} = \{e, h_1, h_2, \ldots, h_{|H|}\}$ is a subgroup of **G** and g is an element of **G**, then the set $g\boldsymbol{H} = \{ge, gh_1, gh_2, \ldots, gh_{|H|}\}$ is called a **left coset**. Similarly the set $\boldsymbol{H}g = \{eg, h_1g, h_2g, \ldots, h_{|H|}g\}$ is called a **right coset**.
- Theorem: two cosets of the same group either contain exactly the same elements or else have no common elements

* A coset does not need to be a group

Example: consider again the $\mathbf{S_3}$ group and its subgroup $H = \{e, a\}$, what are all of the right cosets?

$$\{e,a\}e = \{e,a\}, \qquad \{e,a\}a = \{a,e\}, \qquad \{e,a\}b = \{b,d\},$$

$$\{e,a\}c = \{c,f\}, \qquad \{e,a\}d = \{d,b\}, \qquad \{e,a\}f = \{f,c\}.$$

Classes

- Two elements *a* and *b* of a group **G** are said to be conjugate if there is a group element *g*, called the conjugating element, such that $a = gbg^{-1}$.
- Subsets of elements that are conjugate to each other are called **conjugate classes**.
- Theorem : all of the elements in a conjugate class have the same order.

Example: consider again the S_3 group, what are the classes of it?

$\{e\}$, $\{a, b, c\}$ $\{d, f\}$

Self-Conjugate Subgroups

- A subgroup **H** of **G** is self-conjugate if the elements gHg^{-1} are identical with those of **H** for all elements *g* of **G.** In other words, $gH = Hg$. This is also termed as **invariant group** or **normal group**.
- a subgroup **H** of **G** is self-conjugate if and only if it contains elements of **G** in complete classes, i.e., **H** contains either all or none of the elements of classes of **G.**

Product Groups

- A group **G** is a product group if there are two proper subgroups H_a and H_b of **G** such that
  - All elements of **G** are commuting products of $\mathbf{H_a}$ and $\mathbf{H_b}$. i.e., for all g in **G**, there exist elements h_a in $\mathbf{H_a}$ and h_b in $\mathbf{H_b}$ such that $g = h_a h_b = h_b h_a$.
  - H_a and H_b have only the identity in common.
- $H_a \times H_b$ is defined to be all the distinct pairs of (a, b), with a multiplication law

$$(a_1, b_1)\,(a_2, b_2) = (a_1 a_2,\, b_1 b_2)$$

Example:

Let $\mathbf{H_a} = \{e_a, a\}$ and $\mathbf{H_b} = \{e_b, b\}$ be two copies of the group $\mathbf{Z_2}$, construct the multiplication table for $\mathbf{Z_2} \times \mathbf{Z_2}$.

$$E \equiv (e_a, e_b)\,, \quad A \equiv (a, e_b)\,, \quad B \equiv (e_a, b)\,, \quad C \equiv (a, b)\,,$$

$$AB = (a, e_b)(e_a, b) = (ae_a, e_b b) = (a, b) = C\,.$$

Factor Groups

– The factor group (also called the quotient group) of a self-conjugate subgroup **H** of a group **G** is the collection of cosets, with each being considered a group element.

– The order of the factor group is equal to the index of the self-conjugate subgroup. The quotient group is denoted by G/H.

Example:

Consider the subgroup {*e, d, f*} of $\mathbf{S_3}$.

This subgroup is self-conjugate. i.e., *g**H*** = ***H**g*.

What are the cosets of **H**?

$$\{e, d, f\} \text{ and } \{a, b, c\}$$

Therefore, these two cosets are the factor groups of $\mathbf{S_3}$ group, with an order of two.

Homomorphisms and Isomorphisms

Consider two finite groups **G** and **G′** with elements $\{e, a, b, \ldots\}$ and $\{e', a', b', \ldots\}$ which need not be of the same order. Suppose there is a mapping φ between the elements of **G** and **G′** that preserves their composition rules, i.e.,

$$a' = \varphi(a) \text{ and } b' = \varphi(b), \text{ then } \varphi(ab) = \varphi(a)\varphi(b) = a'b' .$$

If the two groups have the same order, then this mapping is said to be an **isomorphism**, **G** ≅ **G′**. Otherwise, the mapping is called a **homomorphism**.

*an isomorphism is a one-to-one correspondence between two groups, while a homomorphism is a many-to-one correspondence.

Examples:

1. Consider the correspondence between the elements of $\mathbf{S_3}$ and the elements of the factor group of $\mathbf{S_3}$.

$$\{e, d, f\} \mapsto \{\mathcal{E}\}, \qquad \{a, b, c\} \mapsto \{\mathcal{A}\}$$

2. Look at the multiplication tables below and determine whether they are isomorphic.

G

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

G'

	e'	a'	b'	c'
e'	e'	a'	b'	c'
a'	a'	b'	c'	e'
b'	b'	c'	e'	a'
c'	c'	e'	a'	b'

	e	d	f	a	b	c
e	e	d	f	a	b	c
d	d	f	e	c	a	b
f	f	e	d	b	c	a
a	a	b	c	e	d	f
b	b	c	a	f	e	d
c	c	a	b	d	f	e

	$\mathcal{E}$	$\mathcal{A}$
$\mathcal{E}$	$\mathcal{E}$	$\mathcal{A}$
$\mathcal{A}$	$\mathcal{A}$	$\mathcal{E}$

Representations

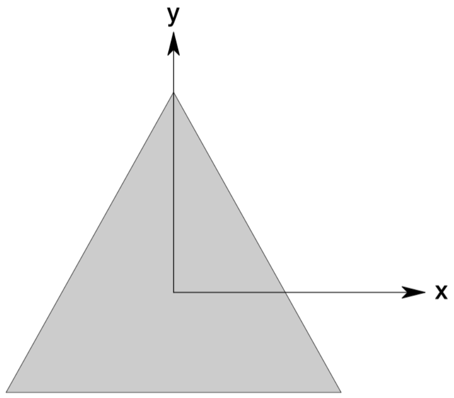
Definition: A representation of dimension n of an abstract group **G** is a homomorphism or isomorphism $\varphi : \mathbf{G} \rightarrow \mathbf{GL}(n, C)$ between **G** and the group **GL**(n, C) of non-singular $n \times n$ matrices, with complex entries and with ordinary matrix multiplication as the composition law.

Example: consider the isomorphism and homomorphism between the elements of $\mathbf{S_3}$ and the planar symmetry operations of an equilateral triangle. What are the representations of this?

1.

$$\mathsf{D}_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{D}_a = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad \mathsf{D}_b = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

$$\mathsf{D}_c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{D}_d = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad \mathsf{D}_f = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}.$$



2.

$$\{e, d, f\} \mapsto \{1\}, \quad \{a, b, c\} \mapsto \{-1\}.$$

3. Identical representation

$$\{e, a, b, c, d, f\} \mapsto 1$$

Similarity Transformation

Problem:

Given a representation $\{D_e , D_a , D_b , \ldots\}$ of an abstract group **G** = {*e, a, b, …*}, consider the set of matrices obtained from what is known variously as a similarity, equivalence, or canonical transformation:

$$\{SD_eS^{-1}, Sd_aS^{-1}, SD_bS^{-1}, \ldots\},$$

generated by any non-singular matrix S, also forms a representation of **G**.

Solution:

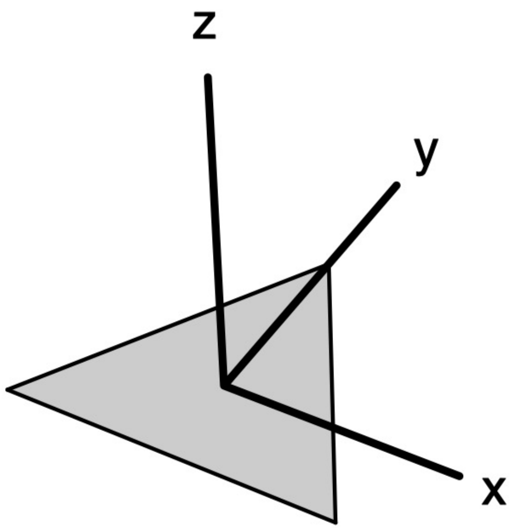
consider the sum of the diagonal elements of an *n* × *n* matrix A, called the **trace** of A and denoted by “Tr”:

$$\mathrm{Tr}\{A\} = \sum_{i=1}^{n} A_{ii}.$$

$$\mathrm{Tr}\{A\} = \mathrm{Tr}\{BAB^{-1}\} .$$

Although there is an infinite variety of representations related by similarity transformations, each such representation has the same trace associated with each of its elements.

Direct Sum



How to obtain a faithful three-dimensional representation of $\mathbf{S_3}$?

Direct sum of the representation in 2D and the identical representation.

$$\mathsf{D}_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathsf{D}_c = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathsf{D}_a = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} & 0 \\ -\sqrt{3} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathsf{D}_d = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\mathsf{D}_b = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathsf{D}_f = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} & 0 \\ -\sqrt{3} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Reducible and Irreducible Representations

Example：
Consider the 2D and the 3D representation of the **S_3** group.

If the same similarity transformation brings all of the matrices of a representation into the same block form (with matrices of the same dimension in the same positions), then this representation is said to be reducible. Otherwise, the representation is said to be irreducible. Irreducible representations cannot be expressed in terms of representations of lower dimensionality.

*One-dimensional representations are, by definition, always irreducible.

Unitary Representations

- Theorem: every representation of a finite group can be brought into unitary form by a similarity transformation.
- Hence, without any loss of generality, we may always assume that a representation is unitary.

Unitary matrices:

A complex square matrix is unitary if its conjugate transpose is also its inverse, i.e.,

$$U^{*}U = UU^{*} = UU^{-1} = 1$$