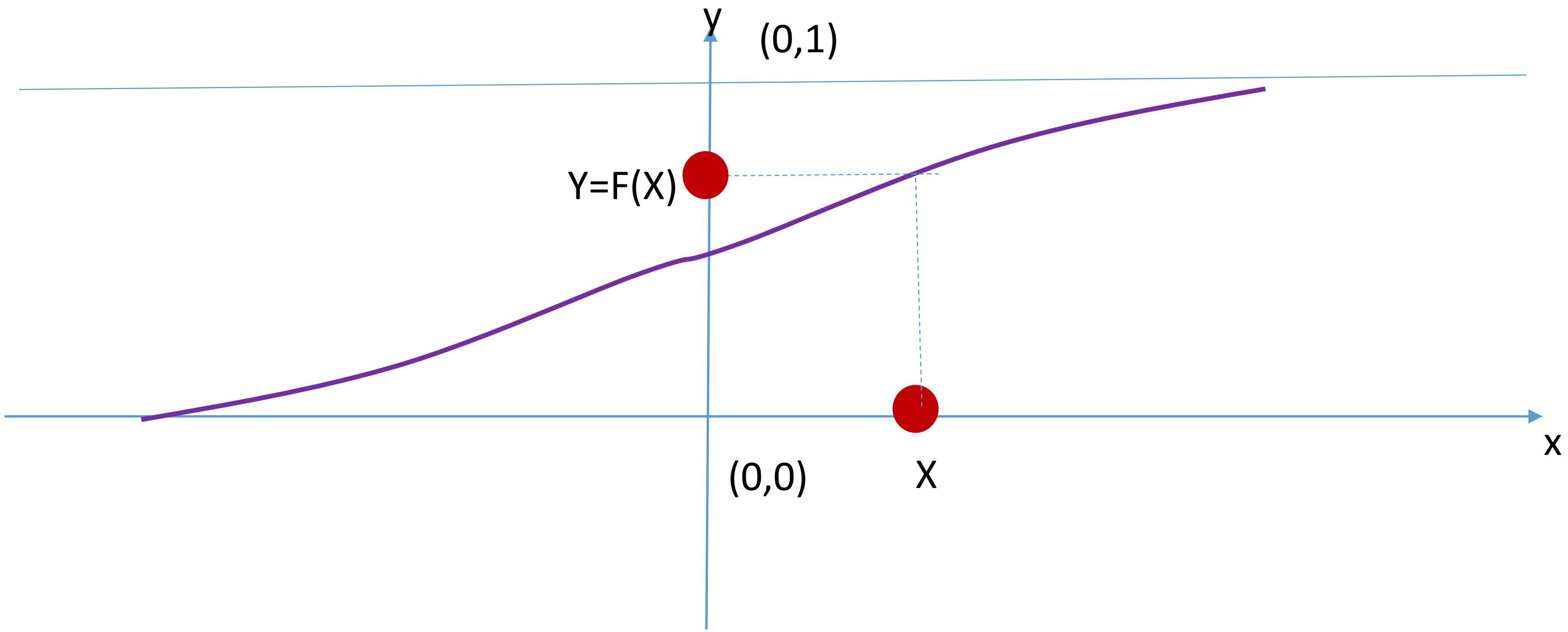
Anderson-Darling

Risk and Portfolio Management with Econometrics

The pull-back

Let X be a random variable with distribution F(x). Then Y=F(X) is <u>uniformly distributed</u>.



Empirical Distribution

Let $(X_1, X_2, \dots X_N)$ be a sample drawn from the distribution F(x).

The empirical distribution is defined as

$$F_N(x) = \frac{1}{N} \times \#\{i : X_i \leq x\}$$

Kolmogorov-Smirnoff Theorem: the empirical distribution function converges uniformly to the true distribution.

$$\lim_{N\to\infty} \sup_x |F_N(x) - F(x)| = 0$$

Kolmogorov-Smirnoff Test

$$KS = \max_{1 \le i \le N} \left| \frac{i}{N} - F\left(X^{(i)}\right) \right|$$

Here $X^{(i)}$ is the ith ranked statistic.

The KS test is based on the statistical distribution of the KS for large values on N.

It is a very popular test, but does not focus on the tails of the PDF – all observations are equally weighted.

Available as a black-box in most statistical packages.

Anderson-Darling Test

Based on the integral of the error, weighted.

$$A_N{}^2 = N \int_{-\infty}^{+\infty} \left(F_N(x) - F(x)\right)^2 w(x) dF(x)$$

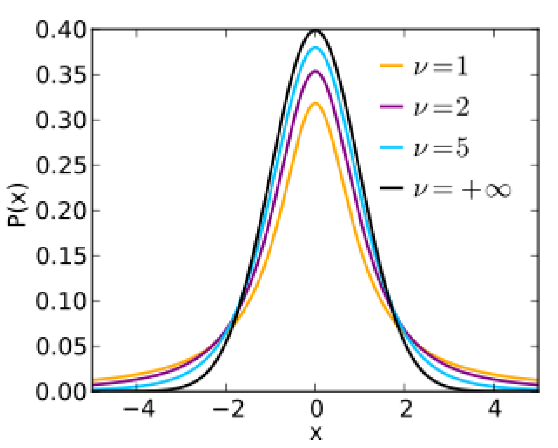
$$w(x) = \frac{1}{F(x)(1 - F(x))}$$

In terms of the sorted sample, this gives rise to the Anderson-Darling test statistic

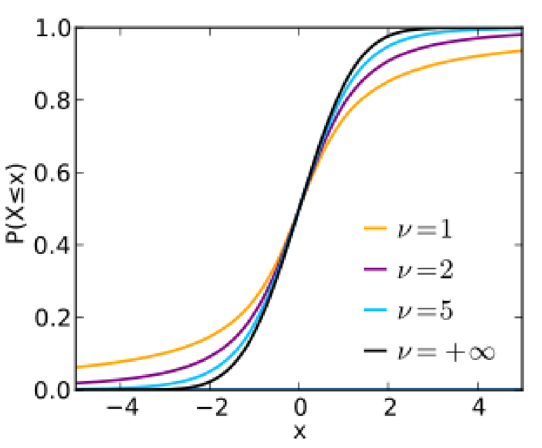
$$AD = -N - \sum_{i=1}^{N} \frac{2i - 1}{N} \left[\ln F(X^{(i)}) + \ln(1 - F(X^{(N-i+1)}))\right]$$

Student's t

Probability density function



Cumulative distribution function



Parameters	$\nu > 0$ degrees of freedom (real)
Support	$x \in (-\infty; +\infty)$
PDF	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$
CDF	$\frac{1}{2}+x\Gamma\left(\frac{\nu+1}{2}\right)\times \frac{{}_2F_1\left(\frac{1}{2},\frac{\nu+1}{2};\frac{3}{2};-\frac{x^2}{\nu}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}$ where ${}_2F_1$ is the hypergeometric function
Mean	0 for $\nu > 1$, otherwise undefined
Median	0
Mode	0
Variance	$\frac{\nu}{\nu-2}$ for $\nu > 2$, ∞ for $1 < \nu \le 2$, otherwise undefined
Skewness	0 for $\nu > 3$, otherwise undefined
Ex. kurtosis	$\frac{6}{\nu-4}$ for $\nu > 4$, ∞ for $2 < \nu \le 4$, otherwise undefined
Entropy	$\frac{\nu+1}{2}\left[\psi\left(\frac{1+\nu}{2}\right)-\psi\left(\frac{\nu}{2}\right)\right]$ $+\ln\left[\sqrt{\nu}B\left(\frac{\nu}{2},\frac{1}{2}\right)\right]$ (nats) • ψ: digamma function, • B: beta function
MGF	undefined
CF	$\frac{K_{\nu/2}(\sqrt{\nu}\lvert t\rvert)\cdot(\sqrt{\nu}\lvert t\rvert)^{\nu/2}}{\Gamma(\nu/2)2^{\nu/2-1}}$ for $\nu > 0$ • $K_\nu(x)$: Modified Bessel function of the second kind[1]

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Student-t distribution

-- Power-law tails
-- Symmetric

$$f(x) = \frac{C(\nu)}{\left(1+\frac{x^2}{\nu}\right)^{\frac{\nu+1}{2}}}$$