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1. Semi-group estimates

a).

$$\begin{cases} \partial_t P + \partial_x V = 6(V-P) & ① \\ \partial_t V + \partial_x P = 6(P-V) & ② \end{cases}$$

$$\begin{array}{l} ①+② \\ ①-② \end{array} \Rightarrow \begin{cases} \partial_t(P+V) + \partial_x(P+V) = 0 \\ \partial_t(P-V) - \partial_x(P-V) = -26(P-V) \end{cases}$$

$$\begin{array}{l} U_1 = P+V \\ U_2 = P-V \end{array} \Rightarrow \begin{cases} \partial_t U_1 + \partial_x U_1 = 0, & U_{1,0} = P_0 + V_0 & ①' \\ \partial_t U_2 - \partial_x U_2 = -26 U_2, & U_{2,0} = P_0 - V_0 & ②' \end{cases}$$

We know the solution for ①' will be $U_1(x,t) = U_{1,0}(x-t)$

For part ②' we first solve $\partial_t U - \partial_x U = 0$, assume the solution has the form $U = U(t, x(t))$, ~~then~~ $\cancel{\partial_t U \; \partial_x U = U' = U' \frac{\partial x(t)}{\partial t} = 0}$

$\cancel{\text{So } \frac{\partial x(t)}{\partial (t)} = 1 \Rightarrow x(t) = t + c, \; x(0) = x_0}$

~~i.e.~~ $\cancel{U = U(t, t+x_0) \text{ is the solution for } \partial_t U - \partial_x U = 0}$ $\cancel{= U(t, x(t))}$ $\cancel{\Rightarrow x(t) = t + x_0 \Rightarrow x_0 = x(t) - t}$

$\cancel{\text{and } \frac{\partial U}{\partial t} = U' + U' \frac{\partial x(t)}{\partial t}}$

assume U is constant w.r.t. time t.

i.e. $\frac{\partial U}{\partial t} = 0 = \partial_t U(t, x(t)) + \partial_x U(t, x(t)) \frac{\partial x(t)}{\partial t} = 0$

$$\left.\begin{array}{l} \Rightarrow \partial_t U + \partial_x U \, \partial_t x = 0 \\ \text{and we have } \cancel{\partial_t U + \partial_x U} \; \partial_t U - \partial_x U = 0 \end{array}\right\} \Rightarrow \partial_t x = -1$$

$\Rightarrow x(t) = -t + c = c - t, \quad x(0) = x_0 \Rightarrow x(t) = x_0 - t$

$\Rightarrow x_0 = x(t) + t$

and since $\frac{\partial U}{\partial t} = 0 \Rightarrow U(t, x(t)) = U(0, x(0)) = U(x(0))$

$= U(x(t) + t)$

Now we try to solve $\partial_t U_2 - \partial_x U_2 = -26 U_2$

~~assume~~ $\frac{d\, U_2(t, x(t))}{dt} = -26 U_2$

$\Rightarrow U_2 = C_0 e^{-26t}$

i.e. $U_2(t, x(t)) = U_{2,0}(0, x(0)) e^{-26t} = U_{2,0}(x(t)+t) e^{-26t}$

Now we have $\begin{cases} u_1(x,t) = u_{1,0}(x-t) & \text{solution of } ①' \\ u_2(t, x(t)) = u_{2,0}(x(t)+t)\, e^{-2\sigma t} & \text{solution of } ②' \end{cases}$

$u_1 = p+v$, $u_2 = p-v$

$\Rightarrow p = \frac{u_1+u_2}{2}$; $v = \frac{u_1-u_2}{2}$, For convinence, we write $u_1 = z$, $u_2 = W$ and $u_{1,0} = z_0$, $u_{2,0} = W_0$

$\Rightarrow p = \frac{u_{1,0}(x-t) + u_{2,0}(x(t)+t)\, e^{-2\sigma t}}{2}$, with $z(t, x(t)) = z_0(x-t) = z_0(x_0^z)$

$= \frac{z_0(x-t) + W_0(x(t)+t)\, e^{-2\sigma t}}{2}$

$W(t, x(t)) = W_0(x(t)+t)\, e^{-2\sigma t} = W_0(x_0^w)\, e^{-2\sigma t}$

$v = \frac{z_0(x-t) - W_0(x(t)+t)\, e^{-2\sigma t}}{2}$

and $p_0 = \frac{z_0(x_0^z) + W_0(x_0^w)}{2}$, $v_0 = \frac{z_0(x_0^z) - W_0(x_0^w)}{2}$ $\Rightarrow$ $z_0(x_0^z) = p_0 + v_0$, $W_0(x_0^w) = p_0 - v_0$

$\Rightarrow \begin{cases} p = \frac{p_0+v_0 + (p_0-v_0)\, e^{-2\sigma t}}{2} = \frac{p_0(x-t)+v_0(x-t) + (p_0(x+t) - v_0(x+t))\, e^{-2\sigma t}}{2} \\ v = \frac{p_0+v_0 - (p_0-v_0)\, e^{-2\sigma t}}{2} = \frac{p_0(x-t)+v_0(x-t) - (p_0(x+t) - v_0(x+t))\, e^{-2\sigma t}}{2} \end{cases}$

$U(x,t) = (p(x,t), v(x,t))$ $\quad U_0 = U(x_0^u) = (p_0, v_0)$

$$U(t) = \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} = e^{At} U_0 = e^{At} \begin{bmatrix} p_0 \\ v_0 \end{bmatrix}$$

~~and we have~~ $\cancel{\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1+e^{-2\sigma t}}{2} & \frac{1-e^{-2\sigma t}}{2} \\ \frac{1-e^{-2\sigma t}}{2} & \frac{1+e^{-2\sigma t}}{2} \end{bmatrix}}_{M} \begin{bmatrix} p_0 \\ v_0 \end{bmatrix}}$

i.e. ~~we must have~~

$\cancel{e^{At} = M}$

~~Semi-group estimates~~. We could rewrite the system as:

$$U = \binom{P}{V}, \quad \partial_t U + M \partial_x U = 6 N U$$

with where M, N is derived from the system

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Then

$$\partial_t U = \underbrace{(6N - M\partial_x)}_{\text{independent of } t} U, \quad U_0 = (P_0, V_0)$$

$$\Rightarrow \quad U = e^{(6N - M\partial_x)t} U_0$$

i.e. we have found $A = 6N - M\partial_x$.

b). Now we show $\|e^{tA}\|_{L(Y_p)} \leq (1 + e^{-26t})$ for $t \geq 0$

$$U = \binom{P}{V} \Rightarrow \|U\|_1 = \|P\|_{L^1(\mathbb{R})} + \|V\|_{L^1(\mathbb{R})}$$

and from part a), we have

$$P = \frac{P_0(x-t) + V_0(x-t) + (P_0(x+t) - V_0(x+t))\, e^{-26t}}{2}$$

$$V = \frac{P_0(x-t) + V_0(x-t) - (P_0(x+t) - V_0(x+t))\, e^{-26t}}{2}$$

write $P_0(x-t) = \hat{P}_0$, $P_0(x+t) = \overset{*}{P}_0$, $V_0(x-t) = \hat{V}_0$, $V_0(x+t) = \overset{*}{V}_0$

$$\|P\|_{L^1(\mathbb{R})} = \int \left| \tfrac{1}{2} \left(\hat{P}_0 + \hat{V}_0 + (\overset{*}{P}_0 - \overset{*}{V}_0) e^{-26t} \right) \right| dx$$

$$\leq \frac{1}{2} \int \left(|\hat{P}_0| + |\hat{V}_0| + (|\overset{*}{P}_0| + |\overset{*}{V}_0|) e^{-26t} \right) dx$$

$$= \frac{1}{2} \left(\|\hat{P}_0\|_{L^1(\mathbb{R})} + \|\hat{V}_0\|_{L^1(\mathbb{R})} + (\|\overset{*}{P}_0\|_{L^1(\mathbb{R})} + \|\overset{*}{V}_0\|_{L^1(\mathbb{R})}) e^{-26t} \right)$$

with $t = 0$

$$= \frac{1}{2} \left[(1 + e^{-26t}) \left(\|P_0\|_{L^1(\mathbb{R})} + \|V_0\|_{L^1(\mathbb{R})} \right) \right]$$

$$= \frac{1}{2} (1 + e^{-26t}) \|U_0\|_1$$

i.e. we have $\|P\|_{L^1(\mathbb{R})} \leq \frac{1}{2} (1 + e^{-26t}) \|U_0\|_1$

Doing the same calculation for V, we have

$$\|V\|_{L^1(\mathbb{R})} \leq \frac{1}{2} (1 + e^{-26t}) \|U_0\|_1$$

then
$$\|U\|_1 = \|P\|_{L^1(R)} + \|V\|_{L^1(R)} \leq \tfrac{1}{2}(1+e^{-26t})\|U_0\|_1 + \tfrac{1}{2}(1+e^{-26t})\|U_0\|_1$$
$$= (1+e^{-26t})\|U_0\|_1$$

We have from part a) that $U = e^{At}U_0$

$$\Rightarrow \|e^{At}U_0\|_1 \leq (1+e^{-26t})\|U_0\|_1$$
$$\Rightarrow \|e^{At}\|_{L(Y_1)} \leq 1+e^{-26t}$$

c).

The prove is quite similar to part b).

$$\|U\|_2 = \sqrt{\|P\|^2_{L^2(R)} + \|V\|^2_{L^2(R)}}$$

$p^* = p_0(x+t)$

$\hat{p} = p_0(x-t)$

$$\|P\|^2_{L^2(R)} = \int \left(\tfrac{1}{2}(\hat{p}_0 + \hat{v}_0 + (p_0^* - v_0^*)e^{-26t})\right)^2 dx$$
$$= \frac{1}{4}\int [(p_0^* - v_0^*)^2 e^{-46t} + \hat{p}_0^2 + \hat{v}_0^2 + 2\hat{p}_0\hat{v}_0 + 2(\hat{p}_0+\hat{v}_0)(p_0^*-v_0^*)e^{-26t}]dx$$

Similarly

$$\|V\|^2_{L^2(R)} = \frac{1}{4}\int [(p_0^*-v_0^*)^2 e^{-46t} + \hat{p}_0^2 + \hat{v}_0^2 + 2\hat{p}_0\hat{v}_0 - 2(\hat{p}_0+\hat{v}_0)(p_0^*-v_0^*)e^{-26t}]dx$$

$$\|P\|^2_{L^2(R)} + \|V\|^2_{L^2(R)} = \frac{1}{4}\int [2(p_0^*-v_0^*)^2 e^{-46t} + 2(\hat{p}_0+\hat{v}_0)^2]dx$$
$$= \frac{e^{-46t}}{2}\int (p_0^*-v_0^*)^2 dx + \frac{1}{2}\int (\hat{p}_0+\hat{v}_0)^2 dx$$

$m = \dfrac{\max(1, e^{-46t})}{2}$,
$$\leq m\int (p_0^*-v_0^*)^2 dx + m\int(\hat{p}_0+\hat{v}_0)^2 dx$$
$$\overset{t=0}{=} m\left[\int (p_0^2 - 2p_0v_0 + v_0^2 + p_0^2 + 2p_0v_0 + v_0^2)dx\right.$$
$$= 2m\int (p_0^2+v_0^2)dx = 2m\left[\|P_0\|^2_{L^2(R)} + \|V_0\|^2_{L^2(R)}\right] = 2m\|U_0\|_2^2$$

$$\|U\|_2^2 = \|P\|^2_{L^2(R)} + \|V\|^2_{L^2(R)} \leq 2m\|U_0\|_2^2$$
$$= \max(1, e^{-46t})\|U_0\|_2^2$$

$U = e^{At}U_0$

$$\Rightarrow \|U\|_2 \leq \max(1, e^{-26t})\|U_0\|_2$$
$$\Rightarrow \|e^{At}U_0\|_2 \leq \max(1, e^{-26t})\|U_0\|_2$$
$$\Rightarrow \|e^{At}\|_{L(Y_2)} \leq \max(1, e^{-26t})$$

d).

The prove ~~⇔~~ is similar to part b) and c).

$$\|P\|_{L^\infty(\mathbb{R})} = \sup_x \left|\tfrac{1}{2}(\hat{P}_0+\hat{V}_0+(\hat{P}_0^*-\hat{V}_0^*)e^{-2\sigma t})\right|$$
$$\le \tfrac{1}{2}\left[\sup_x|\hat{P}_0|+\sup_x|\hat{V}_0|+(\sup_x|P_0^*|+\sup_x|V_0^*|)e^{-2\sigma t}\right]$$

set: $t=0$

$$= \tfrac{1}{2}(1+e^{-2\sigma t})(\|P_0\|_{L^\infty(\mathbb{R})}+\|V_0\|_{L^\infty(\mathbb{R})})$$
$$\le (1+e^{-2\sigma t})\max(\|P_0\|_{L^\infty(\mathbb{R})},\|V_0\|_{L^\infty(\mathbb{R})})$$

Similarly,

$$\|V\|_{L^\infty(\mathbb{R})} \le (1+e^{-2\sigma t})\max(\|P_0\|_{L^\infty(\mathbb{R})},\|V_0\|_{L^\infty(\mathbb{R})})$$

Then
$$\|U(t)\|_\infty = \max(\|P\|_{L^\infty(\mathbb{R})},\|V\|_{L^\infty(\mathbb{R})})$$
$$= (1+e^{-2\sigma t})\max(\|P_0\|_{L^\infty(\mathbb{R})},\|V_0\|_{L^\infty(\mathbb{R})})$$
$$= (1+e^{-2\sigma t})\|U_0\|_\infty$$

$$\Rightarrow \|e^{At}U_0\|_\infty \le (1+e^{-2\sigma t})\|U_0\|_\infty$$

$$\Rightarrow \|e^{At}\|_{L(Y_\infty)} \le 1+e^{-2\sigma t}$$

Take $\sigma=0$ and $t\ge 0$, we have $\|e^{At}\|_{L(Y_\infty)} \le 2$

With $\sigma=0$ ~~$U(t)=\begin{pmatrix}P\\V\end{pmatrix}=\begin{pmatrix}\hat{P}_0\\\hat{V}_0\end{pmatrix}=\begin{pmatrix}P_0(x-t)\\V_0(x-t)\end{pmatrix}\xrightarrow{\text{let } V_0=P_0}\begin{pmatrix}P_0(x-t)\\P_0(x-t)\end{pmatrix}$~~

~~Then $\|U\|_\infty=$~~ $U(t)=\begin{pmatrix}P\\V\end{pmatrix}=\begin{pmatrix}\frac{\hat{P}_0+\hat{V}_0+(\hat{P}_0^*-\hat{V}_0^*)}{2}\\\frac{\hat{P}_0+\hat{V}_0-(\hat{P}_0^*-\hat{V}_0^*)}{2}\end{pmatrix}$, ~~$U_0$~~ when $t=0$ $\hat{P}_0=P_0^*$, $\hat{V}_0=V_0^*$, $U_0=\begin{pmatrix}P_0\\V_0\end{pmatrix}$

Let $V_0=P_0$, then $U(t)=\begin{pmatrix}\hat{P}_0\\\hat{P}_0\end{pmatrix}=\begin{pmatrix}P_0(x-t)\\P_0(x-t)\end{pmatrix}$, $U_0=\begin{pmatrix}P_0\\V_0\end{pmatrix}=\begin{pmatrix}\hat{P}_0\\\hat{P}_0\end{pmatrix}=\begin{pmatrix}P_0(x-0)\\V_0(x-0)\end{pmatrix}=\begin{pmatrix}P_0(x)\\V_0(x)\end{pmatrix}$

i.e we have $\|U\|_\infty = \|P_0(x-t)\|_\infty$, $\|U_0\|=\|P_0(x)\|_\infty$

~~we want $\|e^{At}U_0\|=$~~

Let $P_0(x)=\frac{1}{x}+1$, $x\in[-2,-1]$, then $P_0(x-t)=\frac{1}{x-t}+1$, $x\in[-2,-1]$

i.e. $\|U\|_\infty=\|\frac{1}{x-t}+1\|_\infty=1$, $t\ge0$ $\|U_0\|_\infty=\|\frac{1}{x}+1\|_\infty=\frac{1}{2}$, $x\in[-2,-1]$

$\Rightarrow \|U(t)\|_\infty=\|e^{At}U_0\|_\infty=2\|U_0\|_\infty \Rightarrow \|e^{At}\|_{L(Y_\infty)}=2$

2 Numerical Methods:

a). Determine the symbol of the scheme.

$$\frac{U_j^{n+1}-U_j^n}{\Delta t}-\frac{4}{3}\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{\Delta x^2}+\frac{U_{j+2}^n-2U_j^n+U_{j-2}^n}{12\Delta x^2}$$
$$-\frac{\Delta t}{2}\frac{U_{j+2}^n-4U_{j+1}^n+6U_j^n-4U_{j-1}^n+U_{j-2}^n}{\Delta x^4}=0$$

$$\Rightarrow U_j^{n+1}=U_j^n+\frac{4}{3}\frac{\Delta t}{\Delta x^2}\left(U_{j+1}^n-2U_j^n+U_{j-1}^n\right)$$
$$+\frac{1}{12}\frac{\Delta t}{\Delta x^2}\left(U_{j+2}^n-2U_j^n+U_{j-2}^n\right)\cancel{-\frac{1}{2}\left(\frac{\Delta t}{\Delta x^2}\right)^2}$$
$$-\frac{1}{2}\left(\frac{\Delta t}{\Delta x^2}\right)^2\left(U_{j+2}^n-4U_{j+1}^n+6U_j^n-4U_{j-1}^n+U_{j-2}^n\right)$$

Denote $V=\frac{\Delta t}{\Delta x^2}$, then $\left(\frac{\Delta t}{\Delta x^2}\right)^2=V^2$

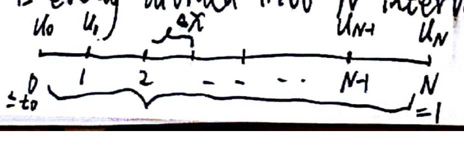
$$\Rightarrow U_j^{n+1}=\left(\frac{V^2}{2}-\frac{V}{12}\right)U_{j+2}^n+\left(\frac{4}{3}V-2V^2\right)U_{j+1}^n+\left(3V^2-\frac{5}{2}V+1\right)U_j^n$$
$$+\left(\frac{4}{3}V-2V^2\right)U_{j-1}^n+\left(\frac{V^2}{2}-\frac{V}{12}\right)U_{j-2}^n$$

Denote $P_1=\frac{V^2}{2}-\frac{V}{12}$, $P_2=\frac{4}{3}V-2V^2$, $P_3=3V^2-\frac{5}{2}V+1$, we have $2P_1+2P_2=-3V^2+\frac{5}{2}V=1-P_3$

$$\Rightarrow U_j^{n+1}=P_1U_{j+2}^n+P_2U_{j+1}^n+P_3U_j^n+P_2U_{j-1}^n+P_1U_{j-2}^n$$
$$=\left[P_1,P_2,P_3,P_2,P_1\right]\begin{bmatrix}U_{j-2}^n\\U_{j-1}^n\\U_j^n\\U_{j+1}^n\\U_{j+2}^n\end{bmatrix}$$

In the following, I write ~~Uj~~ $\vec{U}^{n+1}=J_{h,\Delta t}\vec{U}^n$ for the formula of the programing question f).

The solution is 1-periodic, that is $u(t,x+1)=u(t,x)$, $x_j=j\Delta x$, assuming 1 is evenly divided into N interval.



, with $\Delta x=\frac{1}{N}$

Then $x_0 = 0 \cdot \Delta x = 0$, $x_1 = 1 \cdot \Delta x = \Delta x$, $x_N = N \cdot \Delta x = N \cdot \frac{1}{N} = 1$

So $U_0 = U_N$, $U_1 = U_{N+1}$, $U_{-1} = U_{N-1}$, $U_{-2} = U_{N-2}$

Therefore

$$U_0^{n+1} = \begin{bmatrix} P_1 & P_2 & P_3 & P_2 & P_1 \end{bmatrix} \begin{bmatrix} U_{N-2}^n \\ U_{N-1}^n \\ U_0^n \\ U_1^n \\ U_2^n \end{bmatrix}$$

$$U_1^{n+1} = \begin{bmatrix} P_1 & P_2 & P_3 & P_2 & P_1 \end{bmatrix} \begin{bmatrix} U_{N-1}^n \\ U_0^n \\ U_1^n \\ U_2^n \\ U_3^n \end{bmatrix}$$

$$\vdots$$

i.e. we have

$$U^{n+1} = \underbrace{\begin{array}{c} \begin{array}{cccccccccccccc} U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & \cdots & U_{j-2} & U_{j-1} & U_j & U_{j+1} & U_{j+2} & \cdots & U_{N-2} \; U_{N-1} \end{array} \\ \begin{bmatrix} P_3 & P_2 & P_1 & 0 & - & - & - & - & \cdots & & & 0 & P_1 & P_2 \\ P_2 & P_3 & P_2 & P_1 & 0 & - & - & - & - & & \cdots & & 0 & P_1 \\ P_1 & P_2 & P_3 & P_2 & P_1 & 0 & - & - & - & - & & - & 0 & 0 \\ 0 & P_1 & P_2 & P_3 & P_2 & P_1 & 0 & - & - & - & - & - & 0 & 0 \\ \vdots & & & & & & & & & & & & & \\ 0 & - & - & - & 0 & P_1 & P_2 & P_3 & P_2 & P_1 & 0 & - & - & - \\ & & & & & & & & & & & & & \\ P_1 & 0 & & & & & & & & & & & P_3 & P_2 \\ P_2 & P_1 & 0 & - & - & - & - & - & - & \cdots & 0 & P_1 & P_2 & P_3 \end{bmatrix} \end{array}}_{= J_{h,\Delta t}} \begin{bmatrix} U_0^n \\ U_1^n \\ U_2^n \\ U_3^n \\ \vdots \\ U_j^n \\ \vdots \\ U_{N-2}^n \\ U_{N-1}^n \end{bmatrix}$$

~~i.e~~

i.e. $\vec{U}^{n+1} = J_{h,\Delta t} \vec{U}^n$, $J_{h,\Delta t}$ is above matrix , with

$$\begin{cases} P_1 = \frac{V^2}{2} - \frac{V}{12} \\ P_2 = \frac{4}{3}V - 2V^2 \\ P_3 = 3V^2 - \frac{5}{2}V + 1 \\ V = \frac{\Delta t}{\Delta x^2} \end{cases}$$

b).

$$u_j^{n+1} = a_{-2}u_{j-2}^n + a_{-1}u_{j-1}^n + a_0 u_j^n + a_1 u_{j+1}^n + a_2 u_{j+2}^n$$

$$a_{-2} = a_2 = P_1, \quad a_{-1} = a_1 = P_2, \quad a_3 = P_3, \quad (\text{and we have } 2P_1 + 2P_2 = 1 - P_3)$$

$$\lambda_h(\theta) = \sum_{r=-2}^{2} a_r e^{-i\theta r} = P_1 e^{2i\theta} + P_2 e^{i\theta} + P_3 + P_2 e^{-i\theta} + P_1 e^{-2i\theta}$$

$$\partial_t u - \partial_{xx} u = 0$$

$$\partial_t u = \partial_{xx} u = Au, \quad A = \partial_{xx}$$

$$A e^{i\theta x} = \partial_{xx} e^{i\theta x} = -\theta^2 e^{i\theta x} = u(\theta) e^{i\theta x}, \text{ with } u(\theta) = -\theta^2$$

$$\left| e^{u(\theta)\Delta t} - \lambda_h(\theta\Delta x) \right| \xlongequal[]{\text{Denote } \hat\theta = \theta\Delta x} \left| e^{u\left(\frac{\hat\theta}{\Delta x}\right)\Delta t} - \lambda_h(\hat\theta) \right|$$

$$\hat\theta = \theta\Delta x$$

$$\hat\theta^2 V = \hat\theta^2 \cdot \frac{\Delta t}{\Delta x^2} = \theta^2 \Delta x^2 \frac{\Delta t}{\Delta x^2} = \theta^2 \Delta t$$

$$= \left| e^{-\hat\theta^2 \frac{\Delta t}{\Delta x^2}} - P_1 e^{2i\hat\theta} + P_2 e^{i\hat\theta} + P_3 + P_2 e^{-i\hat\theta} + P_1 e^{-2i\hat\theta} \right|$$

$$= \left| e^{-\hat\theta^2 V} - [\cancel{P_1(\cos 2\hat\theta +} P_1 \cos 2\hat\theta + P_2 \cos\hat\theta + P_2 \cos(-\hat\theta) + P_1 \cos(-2\hat\theta) + i(P_1 \sin 2\hat\theta + P_2 \sin\hat\theta + P_2 \sin(-\hat\theta) + P_1 \sin(-2\hat\theta)) + P_3 \right|$$

$$= \left| e^{-\hat\theta^2 V} - (2P_1 \cos 2\hat\theta + 2P_2 \cos\hat\theta + P_3) \right|$$

with $\hat\theta^2 V = \theta^2 \Delta t$, $\hat\theta = \theta\Delta x$

$$= \Bigg| 1 + (-\hat\theta^2 V) + \frac{(-\hat\theta^2 V)^2}{2} + \frac{(-\hat\theta^2 V)^3}{6} + O(\Delta t^4) - \Big[2P_1\Big(1 - \frac{(2\hat\theta)^2}{2} + \frac{(2\hat\theta)^4}{4!} - \frac{(2\hat\theta)^6}{6!} + O(\Delta x^8)\Big) + 2P_2\Big(1 - \frac{\hat\theta^2}{2} + \frac{\hat\theta^4}{4!} - \frac{\hat\theta^6}{6!} + O(\Delta x^8)\Big) + P_3 \Bigg|$$

$$= \Bigg| \underbrace{1 - (\underbrace{2P_1 + 2P_2}_{=1-P_3} + P_3)}_{=0} + \hat\theta^2 \underbrace{(-V + 4P_1 + P_2)}_{=0} + \hat\theta^4 \underbrace{\Big(\frac{V^2}{2} - \underbrace{\frac{2\times 2^4}{4!} P_1 - \frac{2P_2}{4!}}\Big)}_{=0}$$

$$P_1 = \frac{V^2}{2} - \frac{V}{12}, \quad P_2 = \frac{4V}{3} - 2V^2, \quad 4P_1 + P_2 = V$$

$$= \frac{V^2}{2} - \frac{16P_1 + P_2}{12} = 0$$

$$+ \hat\theta^6 \underbrace{\Big(-\frac{V^3}{6} + \frac{2P_1 \cdot 2^6 + 2P_2}{6!}\Big)}_{= \frac{30V^3 + 15V^2 - 2V}{\cancel{90}\ 180}} + O(\Delta t^4) - \cancel{2P_1 O(\Delta} (2P_1 + 2P_2) O(\Delta x^8) \Bigg|$$

$$= \left| (\theta \Delta x)^6 \frac{-\cancel{30}v^3+5v^2-v}{\cancel{90}\ 180} + O(\Delta t^4) - \underbrace{(2P_1+2P_2)}_{=-3v^2+\frac{5}{2}v} O(\Delta x^8) \right|$$

$$= \left| \left(-\frac{15}{90}\right) \cdot \theta^6 \Delta t^3 + \frac{51}{\cancel{90}\,36} \theta^6 \Delta t^2 \Delta x^2 - \frac{1}{180} \Delta t \Delta x^4 + O(\Delta t^4) \right.$$
$$\left. + (-3) \cdot \Delta t^2 O(\Delta x^4) + \frac{5}{2} \Delta t \cdot O(\Delta x^6) \right|$$

$$\leq C_1 |\theta^6| \, |\Delta t^3 + \Delta t^2 \Delta x^2 + \Delta t \Delta x^4|$$

$$ab \leq \frac{a^2+b^2}{2} \quad C_1 |\theta^6| \Delta t \, |\Delta t^2 + \Delta t^2 + \Delta x^4 + \Delta x^4|$$

$$\leq \cancel{C_2 |\theta^6| \, |\Delta t^3 + \Delta t^4 + \Delta x^4 + \Delta t^2 + \Delta x^8|}$$

$$\leq C_3 |\theta^6| \, |\Delta t^2 + \Delta x^4| \Delta t$$

i.e. we have proved that the scheme is consistent at order 2 in time and 4 in space.

c)

We need to show that under a CFL condition, the scheme is stable in the quadratic norm.

I.e. we need to prove $\sup_\theta |\lambda_h(\theta)| \leq 1$

$$\lambda_h(\theta) = \sum_{r=-2}^{2} \alpha_r e^{-i\theta r} = 2p_1 \cos 2\theta + 2p_2 \cos\theta + p_3$$

$$\sup_\theta |\lambda_h(\theta)| = \sup_\theta |2p_1 \cos 2\theta + 2p_2\cos\theta + p_3|$$
$$= \sup_\theta |2p(2\cos^2\theta - 1) + 2p_2\cos\theta + p_3|$$
$$= \sup_{t\in[-1,1]} |2p(2t^2-1) + 2pt + p_3|$$

Denote $g(t) = 2p_1(2t^2-1) + 2p_2 t + p_3$, $t \in [-1,1]$
$= 4p_1 t^2 + 2p_2 t + p_3 - 2p_1$, $t \in [-1,1]$

$g(t)$ is a qudratic function of t, and only reach min/max value at boundary point $-1, 1$ or critical point t^* where $g'(t^*)=0$.

At upper bound
$$g(1) = 4p_1 + 2p_2 + p_3 - 2p_1 = 2p_1 + 2p_2 + p_3 = 1, \quad |g(1)| \leq 1$$

At lower bound $g(-1) = 4p_1 + 2p_2(-1) + p_3 - 2p_1 = 2p_1 - 2p_2 + p_3$
$$= 2\left(\frac{v^2}{2} - \frac{v}{12}\right) - 2\left(\frac{4}{3}v - 2v^2\right) + 3v^2 - \frac{5}{2}v + 1$$
$$= 8v^2 - 5v + 1, \quad v = \frac{\Delta t}{\Delta x^2}, \quad 0 \leq v.$$
$$= 8\left(v - \frac{5}{16}\right)^2 + \frac{7}{32} \geq 0$$

$$|g(-1)| = g(-1) = 8\left(v - \frac{5}{16}\right)^2 + \frac{7}{32} \leq 1 \Rightarrow \left(v - \frac{5}{16}\right)^2 \leq \frac{5^2}{16^2}$$
$$\Rightarrow -\frac{5}{16} \leq v - \frac{5}{16} \leq \frac{5}{16}$$
$$\Rightarrow 0 \leq v \leq \frac{10}{16} = \frac{5}{8}, \quad \text{i.e. when } 0 \leq v \leq \frac{5}{8}, |g(-1)| \leq 1$$

At critical point t^*, where $g'(t^*)=0$
$$g'(t) = 8p_1 t + 2p_2 = 0 \Rightarrow t^* = -\frac{p_2}{4p_1}$$
$$g(t^*) = 4p_1 \cdot \left(-\frac{p_2}{4p_1}\right)^2 + 2p_2\left(-\frac{p_2}{4p_1}\right) + p_3 - 2p_1$$
$$= -\frac{p_2^2}{4p_1} + 1 - 2p_1 - 2p_2 - 2p_1$$
$$= 1 - \frac{(4p_1+p_2)^2}{4p_1}$$

$p_1 = \frac{v^2}{2} - \frac{v}{12}$

$p_2 = \frac{4}{3}v - 2v^2$

$4p_1 + p_2 = v$

$$= 1 - \frac{v^2}{2v^2 - \frac{v}{3}} = 1 - \frac{3v}{6v-1} = \frac{1}{2} - \frac{1}{12v-2}$$

$$|g(t^*)| \leq 1 \Rightarrow \left|\frac{1}{2} - \frac{1}{12v-2}\right| \leq 1, v > 0 \Rightarrow v \geq \frac{2}{9}$$ (if $g(t)$ could reach the critical point)

We also have the condition that if $g(t)$ could reach the minimum t^*, t^* should be in $[-1,1]$.

$$t^* = -\frac{p_2}{4p_1} = \frac{2v-\frac{4}{3}}{2v-\frac{1}{3}} \in [-1,1], \quad v \geq 0$$

$\Rightarrow v \geq \frac{5}{12}$, and since $g(t)$ is qudratic function and could only reach max/min value at boundary and stationary points

Then i.e. $\forall \; v \geq \max(\frac{2}{9}, \frac{5}{12})$ and $v \leq \frac{5}{8}$

i.e. $\forall \; \frac{5}{12} \leq v \leq \frac{5}{8}$, we have $\forall t \in [-1,1]$, $|g(t)| < 1$

Notice $g(t)$ may not be able to reach the critial point if t^* is outside $[-1,1]$, 非 then the part $v \geq \frac{5}{12}$ may be dropped. ~~then the upperbound $v \geq \frac{5}{12}$ condition may not be needed. Need to be checked.~~

Then under CFL condition

We are sure that $\forall v$, $\frac{5}{12} \leq v \leq \frac{5}{8}$, i.e. $\frac{5}{12} \leq \frac{\Delta t}{\Delta x^2} \leq \frac{5}{8}$, the scheme is L^2-stable and since this scheme is both consistent and stable, it could also converge. ~~Notice~~ The numerical test show the scheme is stable even for $v \leq \frac{5}{12}$, so it may be the case t^* is ~~often~~ out of ~~the reg~~ $[-1,1]$, then the constraint on t^* does not play a role. The stable region becomes $0 \leq v \leq \frac{5}{8}$

e).

$$U_j^{n+1} = U_j^n + \Delta t \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

Denote $v = \frac{\Delta t}{\Delta x^2}$

$$= \underbrace{(1-2v)}_{a_0} U_j^n + \underbrace{v}_{a_1} U_{j+1}^n + \underbrace{v}_{a_{-1}} U_{j-1}^n$$

$$\partial_t u = \partial_{xx} u, \quad A = \partial_{xx}, \quad A e^{i\theta x} = -\theta^2 e^{i\theta x}, \quad u(\theta) = -\theta^2$$

$$\lambda_h(\theta) = \sum_{r=-1}^{1} a_r e^{-i\theta r} = a_{-1} e^{i\theta} + a_0 + a_1 e^{-i\theta}$$

$$= v e^{i\theta} + (1-2v) + v e^{-i\theta}$$

$$\left| e^{u(\theta)\Delta t} - \lambda_h(\theta \Delta x) \right| \underset{\theta = \frac{\hat\theta}{\Delta x}}{\overset{\text{denote } \hat\theta = \theta\Delta x}{=\!=\!=}} \left| e^{u(\frac{\hat\theta}{\Delta x})\Delta t} - \lambda_h(\hat\theta) \right|$$

$$= \left| e^{-\hat\theta^2 \frac{\Delta t}{\Delta x^2}} - (v e^{i\hat\theta} + (1-2v) + v e^{-i\hat\theta}) \right|$$

$$= \left| e^{-\hat\theta^2 v} - \left(v(\cos\hat\theta + \cos(-\hat\theta)) + (1-2v) + vi(\sin\hat\theta + \sin(-\hat\theta)) \right) \right|$$

$\hat\theta^2 v = \theta^2 \Delta t$

$\Rightarrow (\hat\theta^2 v)^3 = \theta^6 (\Delta t)^3$

$\hat\theta = \theta \Delta x$

$(\hat\theta)^6 = \theta^6 (\Delta x)^6$

$$= \left| e^{-\hat\theta^2 v} - 2v\cos\hat\theta - 1 + 2v \right|$$

$$= \left| 1 + (-\hat\theta^2 v) + \frac{(-\hat\theta^2 v)^2}{2} + O(\Delta t^3) - 2v\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + O(\Delta x^6)\right) - 1 + 2v \right|$$

$$= \left| \underbrace{1-2v-1+2v}_{=0} + \underbrace{\hat\theta^2(-v + 2v \cdot \tfrac{1}{2})}_{=0} + \theta^4\left(\frac{v^2}{2} - \frac{2v}{4!}\right) + O(\Delta t^3) - 2v\, O(\Delta x^6) \right|$$

when $v = \frac{\Delta t}{\Delta x^2} = \frac{1}{6}$, $\frac{v^2}{2} - \frac{2v}{4!} = 0$

Then above equation will be

$$= \left| 0 + 0 + 0 + O(\Delta t^3) - \cancel{2\frac{\Delta t}{\Delta x^2} * O(\Delta x^6)} \; 2\frac{\Delta t}{\Delta x^2} O(\Delta x^6) \right|$$

$$= \left| O(\Delta t^3) + O(\Delta t) O(\Delta x^4) \right|$$

$$\leq 1 \cdot \left| O(\Delta t^3) + O(\Delta x^4) \right|$$

i.e. when $\frac{\Delta t}{\Delta x^2}$, the 3-points scheme is fourth order in space.

(f). See the python file

g). See the python file

g). From both the numerical result in the group as well as from the formula $U_j^{n+1} = \sum_{k=-2}^{2} d_k U_k^n$, with $\sum_{k=-2}^{2} d_k \leq 1$

we can have that

$$\max_j |U_j^n| \leq \max_j |U_j^0|$$

Then we could conclude that the maximum principle is satisfied.

NA_pde_hw_b004_Yu_Xiang

November 16, 2018

0.0.1 This is for the assignment of Numerical methods for unsteady PDEs: finite differences and finite volumes.

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In [313]:
```
import os
import matplotlib.pyplot as plt
from matplotlib import cm
from matplotlib.lines import Line2D
import numpy as np
import random
from scipy.linalg import norm
```

In [314]:
```
# choose a large font size by default and use tex for math
fontsize = 10
params = {'axes.labelsize': fontsize + 2,
      'font.size': fontsize + 2,
      'legend.fontsize': fontsize + 2,
      'xtick.labelsize': fontsize,
      'ytick.labelsize': fontsize}
plt.rcParams.update(params)
```

In [315]:
```
# hyper_parameters and initialization
pi = 3.14159265358979323846

NrX = 100 + 1
LenX = 1
NrT = 100

# CFL
v = 0.5 # v = CFL  = dt / (dx ** 2)
U_data = np.zeros((NrX,NrT))
U_old = np.zeros((NrX,1))
U_new = np.zeros(NrX)
xx_grid = np.zeros((NrX, 1))
dx = LenX / (NrX - 1)
```

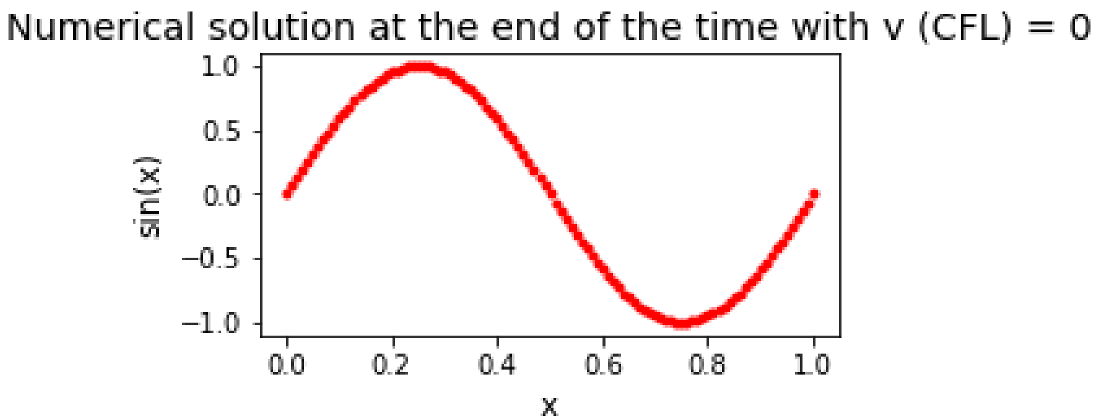
In [316]:
```
# create the xx grid
for i in np.arange(0,NrX):
    xx_grid[i,0] = i * dx

# initial as a sin function
U_old[:,0] = np.sin(2 * pi * xx_grid[:,0])  # time 0 value is sin(2 * pi * x)
```

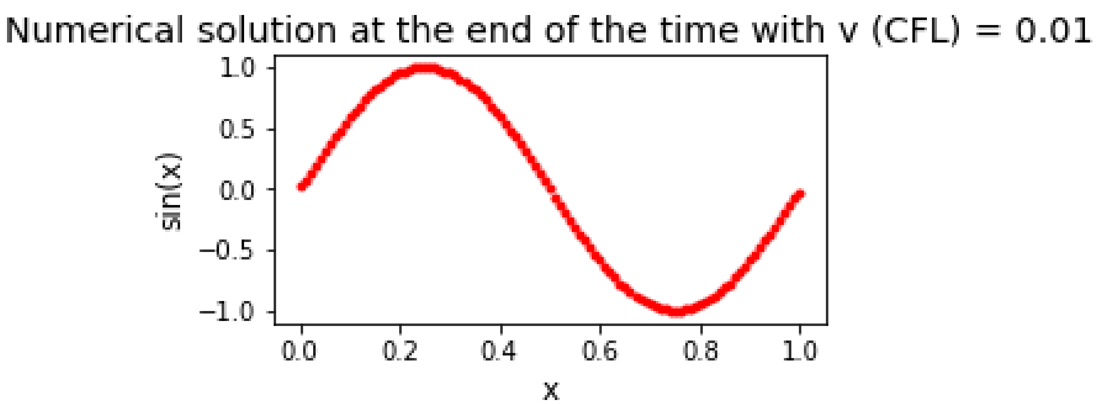
In [317]:
```
def cal_U (v, U_old):

    p1 = v ** 2 / 2. - v / 12.
    p2 = 4 / 3. * v  - 2. * v ** 2
    p3 = 1 - 2 * (p1 + p2)

    dt = v * (dx ** 2)

    M = np.zeros((NrX, NrX)) # maxitrix of Jh_dt = M, : U(n+1) = M * U(n)
    U_data = np.zeros((NrX,NrT))

    for i in np.arange(NrX):
        pre1c = (NrX + i - 1) % NrX
        pre2c = (NrX + i - 2) % NrX
        aft1c = (NrX + i + 1) % NrX
        aft2c = (NrX + i + 2) % NrX

        M[i, pre2c] = p1
        M[i, aft2c] = p1
        M[i, pre1c] = p2
        M[i, aft1c] = p2
        M[i, i] = p3

    time = 0.
    U_data[:,0] = U_old[:,0]
    for t in np.arange(1,NrT):
        time = time + dt
        U_data[:,t] = M.dot(U_data[:,t-1])

    return U_data
```

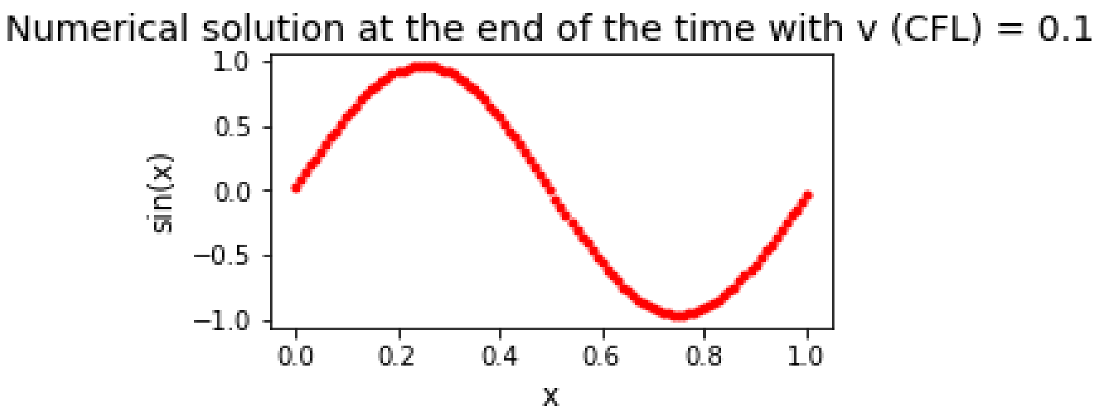
In [318]:
```
# test with different CFL
vs = [0, 0.01, 0.1, 0.35, 0.5, 0.65, 0.75, 0.85]

Nr_v = len(vs)
U_news = np.zeros((NrX, Nr_v))
for vi in np.arange(Nr_v):
    U_data = cal_U(vs[vi], U_old)
    U_news[:,vi] = U_data[:,-1]
```

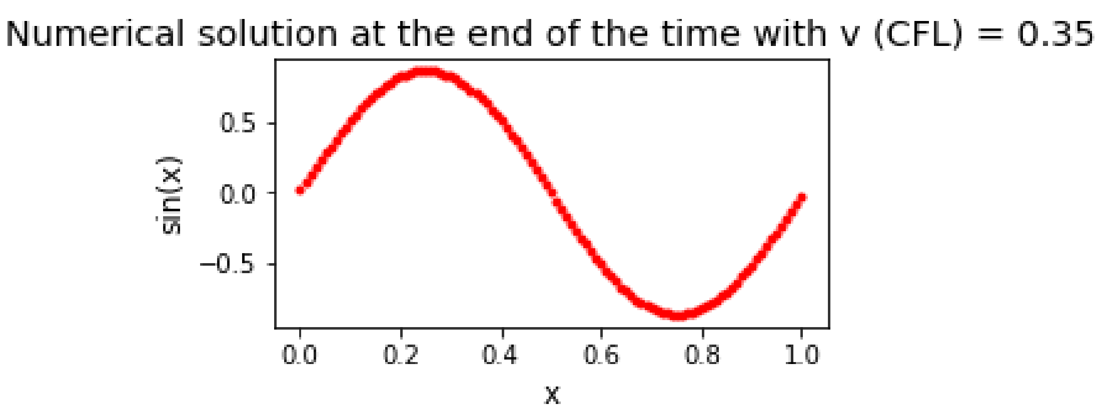
In [319]:
```
# plot the result with differnt CFL: check stablility
```

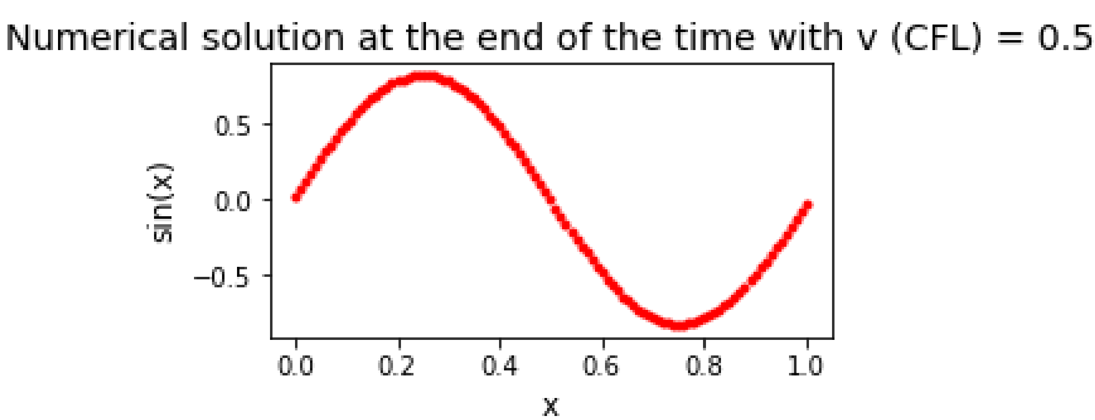
```
for vi in np.arange(Nr_v):
    fig, ax = plt.subplots(figsize=(4, 2))
    ax.set_xlabel('x')
    ax.set_ylabel('sin(x)')
    ax.set_title('Numerical solution at the end of the time with v (CFL) = '
                 + str(vs[vi]))
    plt.plot(xx_grid, U_news[:,vi],  'r.')

plt.show()
```

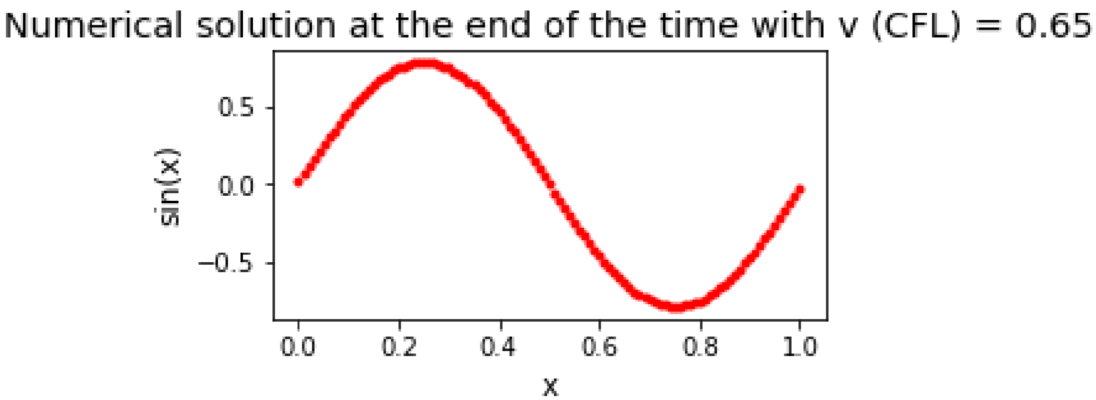


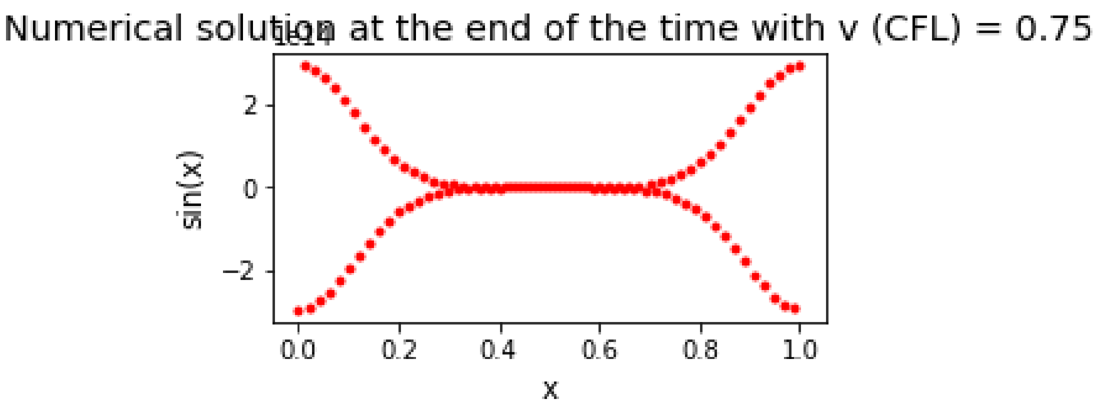


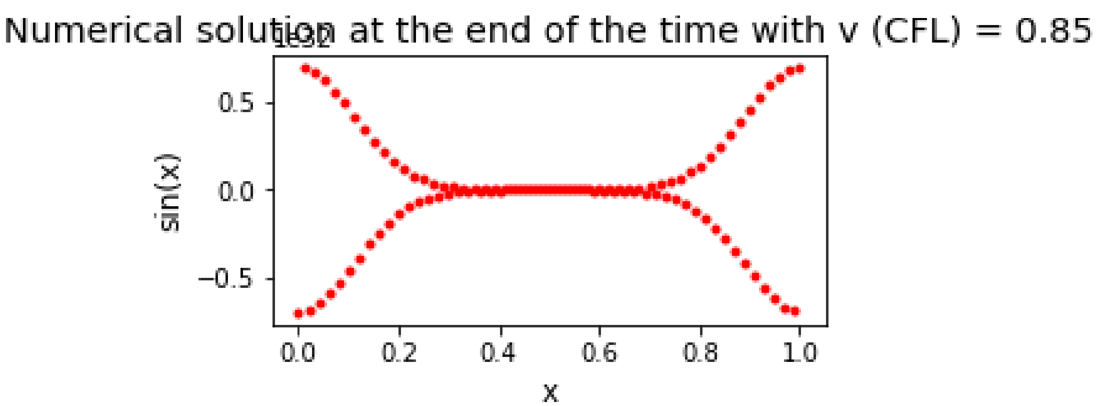

Numerical solution at the end of the time with v (CFL) = 0.1
1.0
0.5
0.0
−0.5
−1.0
sin(x)
0.0
0.2
0.4
0.6
0.8
1.0
x


Numerical solution at the end of the time with v (CFL) = 0.35
0.5
0.0
−0.5
sin(x)
0.0
0.2
0.4
0.6
0.8
1.0
x


Numerical solution at the end of the time with v (CFL) = 0.5
0.5
0.0
−0.5
sin(x)
0.0
0.2
0.4
0.6
0.8
1.0
x


Numerical solution at the end of the time with v (CFL) = 0.65
sin(x)
x

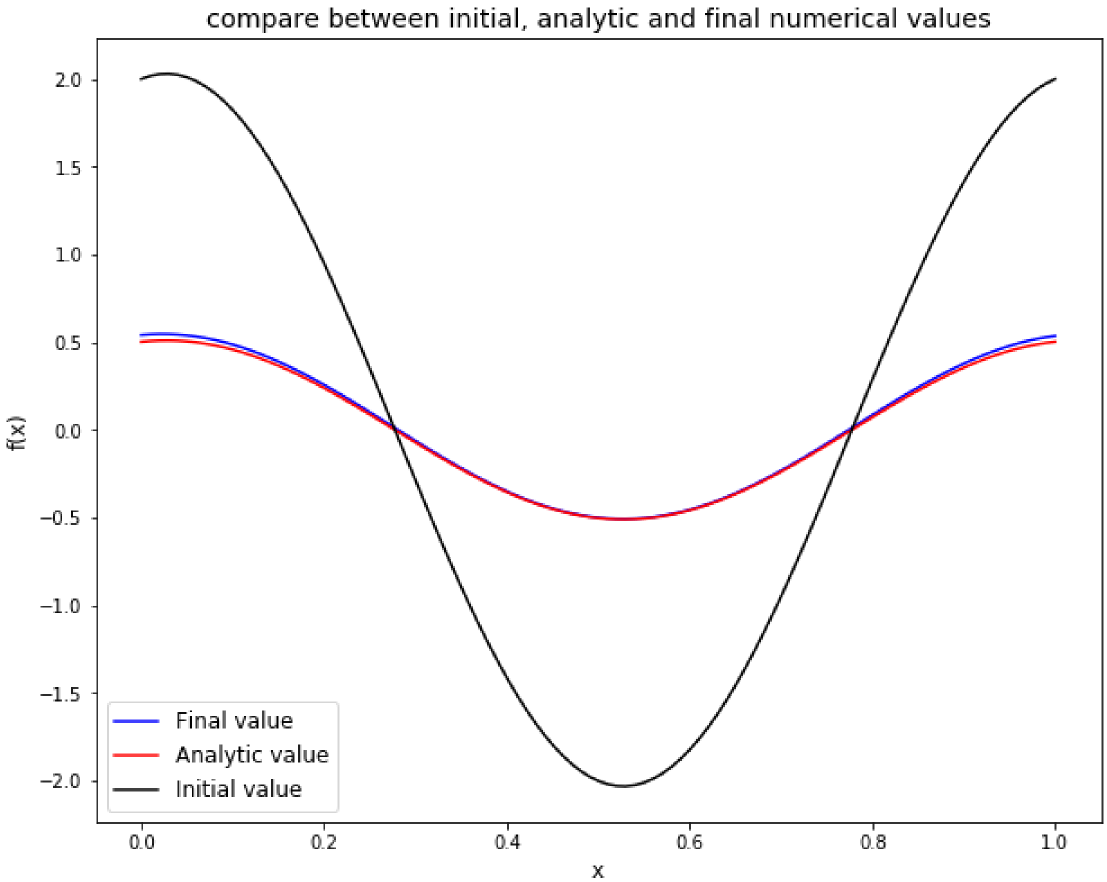

Numerical solution at the end of the time with v (CFL) = 0.75
1e14
sin(x)
x


Numerical solution at the end of the time with v (CFL) = 0.85
1e32
sin(x)
x

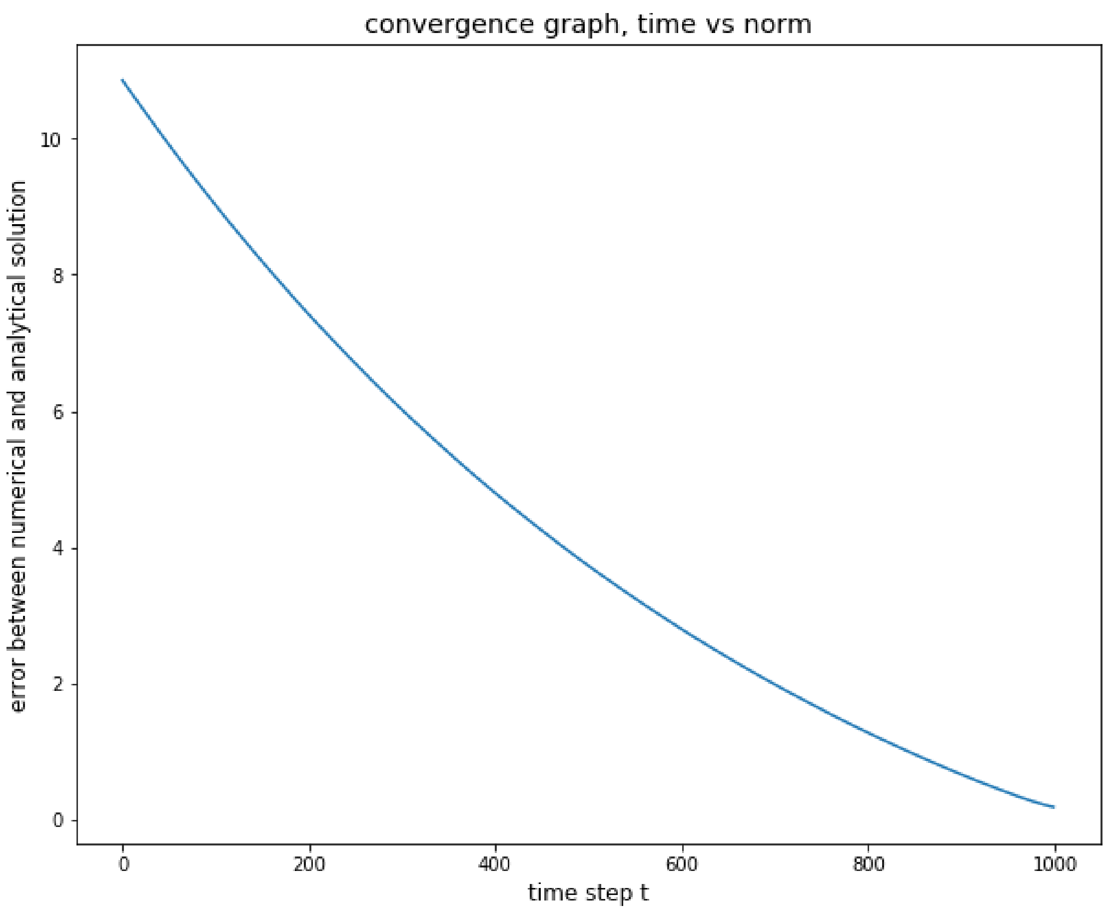
0.0.2 Clearly the scheme is stable when v(CFL) is no larger than around 0.65

In [320]:
```
# test with another function
alpha = 0.35
beta = 2

# CFL
v = 0.35
NrT = 1000
dt = v * (dx ** 2)
T = dt * NrT
U_data = np.zeros((NrX,NrT))

for i in np.arange(0,NrX):
    xx_grid[i,0] = i * dx


U_old = alpha * np.sin(2 * pi * xx_grid) + beta * np.cos(2 * pi * xx_grid)
U_analytic = (alpha * np.sin(2 * pi * xx_grid)
              + beta * np.cos(2 * pi * xx_grid))
              * np.exp(-4 * pi ** 2 * T)
U_data = cal_U(v, U_old)
U_new = U_data[:,-1]
```

In [321]:
```
# plot the result
plt.figure(figsize=(10, 8))
plt.plot(xx_grid, U_new,"b",label="Final value")
plt.plot(xx_grid, U_analytic,"r",label="Analytic value")
plt.plot(xx_grid, U_old,"k",label="Initial value")
plt.title('compare between initial, analytic and final numerical values')
plt.xlabel('x')
plt.ylabel('f(x)')
plt.legend()
plt.show()
```




```
In [322]: u_diff = U_data - U_analytic
          norm_u_diff = norm(u_diff, ord=2, axis=0)

In [323]: plt.figure(figsize=(10, 8))
          plt.plot(np.arange(NrT), norm_u_diff)
          plt.xlabel('time step t')
          plt.ylabel('error between numerical and analytical solution')
          plt.title('convergence graph, time vs norm')
          plt.show()
```



0.0.3 the convergence rate, as shown from above graph looks like linear