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Master thesis

in Scientific Computing

submitted by

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born in November 10, 1989

2022

Gradient method of solving parameterized optimal control problems, with a case study in state constrained rocket car

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at the

Heidelberg University

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Contents

1 Introduction

Many real life problems, can be modeled as parameterized optimization problems, such as the therapy design of Cerebral Palsy (CP) problem described in Schlöder [2022]. In this paper, we focus on using gradient method to solve parameterized optimization problems, with a case study in state constrained rocket car.

Without giving a rigorous condition and definition[1], a general optimization problem is typically of the form

$$\begin{aligned} & min\ f(x) \\ s.t.\quad & g(x) = 0, \\ & h(x) \geq 0 \end{aligned} \tag{1.1}$$

where $f(x)$ is the objective or cost function, $g(x) = 0$ and $h(x) \geq 0$ are the constraints. Some optimization problems may have uncertain parameters whose value are priori unknown, and the optimal objective value depends on the parameter value. This kind of problem is called the parameterized optimization problems and is of the form

$$\begin{aligned} & min\ f(x,p) \\ s.t.\quad & g(x,p) = 0, \\ & h(x,p) \geq 0 \\ & x = x(p) \\ & x = x(p^0)\ if\ p = p^0 \\ & p \in \Omega_P \end{aligned} \tag{1.2}$$

where p^0 is a fixed value in the feasible uncertainty set Ω_P, where the parameter p can take value from.

Parameterized optimization problems are very difficult to solve due to the uncertainty in the parameter p. In the paper Schlöder [2022], multiple methods of solving the parameterized optimization problem have been discussed. The main focus (of solving the Cerebral Palsy problem) of the paper Schlöder [2022], is the "worst-case treatment planning by bilevel optimal control", i.e. a bilevel optimization problem. The bilevel optimisation method in paper Schlöder [2022] solves the parameterized optimization problems, e.g. the Cerebral Palsy problem, in a conservative way.

One method of solving the original CP problem in a conservative way is to transform the problem 1.2 into another form. Assuming that the parameter $\tilde{p}$ lies in an uncertainty set Ω_P, we can firstly reach one objective, i.e. identifying a worst possible solution with respect to $\tilde{p}$, i.e. solving a lower level problem. Based on the result of lower level, we can continue to find the best solution with respect to x, i.e. solving a upper level problem. The "worst-case treatment planning by bilevel optimal control", i.e. a bilevel optimization

[1] We do not give a rigorous definition on purpose so that the problem we have described here can be applied to more general cases when such condition and definition are more clearly defined.

problem, is an optimization problem in which another optimization problem enters the constraints. Mathematically, the problem 1.2 is transformed into another form, and can be formulated in a simplified notation, as following

$$\begin{aligned} &\underset{x}{min}\ \underset{p\in\Omega_P}{max}\ f(x,p) \\ &\qquad s.t.\ g(x,p)=0,\ \ h(x,p)\geq 0 \end{aligned} \tag{1.3}$$

Due to the *min max* notation, this classical approach of solving the bilevel problem can also be called *minmax* approach.

As stated in Schlöder [2022], many different methods can be used to solve a bilevel problem, three approaches have been discussed in detail, i.e. a transformation of the bilevel problem to a single level problem, a classical approach and a training approach. A intuitive approach is to transfer the bilevel problem into a single level problem, however, in general the resulting single level problem is not equivalent to the original bilevel problem and this approach is also out of the focus of the paper Schlöder [2022] as well as this paper at hand. A classical approach, aka a robust optimization appraoch, is consistent with the *minmax* appoach, which will be discussed in more detail in Chapter 2.

The paper Schlöder [2022] introduces the "Training Approach". It is based on the idea that in the real world, during the training period, an intervention is introduced and a certain, but a priori unknown, parameter $p \in \Omega_P$ is realized. What follows the training period (during which the parameter p is realized), the patient is able to react to it in an optimal manner, i.e. an optimal value $f(x,p)$ will be obtained given the realized parameter p. The paper Schlöder [2022] call this approach "worst case modeling Training Approach", and it can be written as

$$\begin{aligned} &\underset{p\in\Omega_P}{max}\ \underset{x}{min}\ f(x,p) \\ &\qquad s.t.\ g(x,p)=0,\ \ h(x,p)\geq 0 \end{aligned} \tag{1.4}$$

Due to the *max mix* notation, this approach of solving the bilevel problem can also be called *maxmin* approach.

The paper Schlöder [2022] use a derivative free method in the Training Approach. This paper at hand will focus on a gradient method to solve the *maxmin* problem. In particular, we are interested in how to compute the derivatives theoretically and numerically. We would like to apply the quasi-Newton and multiple shooting method when solving the problem numerically. The approaches discussed in this paper at hand will be demonstrated with a case study in state constrained rocket car.

We choose this rocket car case for two reasons: firstly, the case is relatively easy to understand and is quite representative of the general usage in real life; secondly, the case has theoretical solution and we can compare the numerical results with the theoretical value so that we can check whether our gradient method can find the optimal solution and how fast it converges.

The structure of this paper is as follows: in Chapter 1, i.e. this chapter, we give an introduction on what problems this paper intends to address. In the Chapter 2, we introduce the case of the state constrained rocket car. In the Chapter 3, we discuss the classical approach and training approach, and show the theoretical value of the chosen case. In Chapter 4, we give the mathematical background of the quasi-Newton and multiple shooting method. In Chapter 5, we show how we can solve the case numerically using the methods described in Chapter 4. In Chapter 6, we compare our theoretical and numerical results and conclude the paper.

2 Rocket car case

Since the approaches we are going to use in this paper will be demonstrated with the case of rocket car, we decide to describe the rocket car case first. So that, when we are discussing our approaches, we can directly describe how they can be used in solving the rocket car case. The description of the rocket car case is mostly coming from the paper Schlöder [2022], with content either verbatim or in a modified form.

We consider the rocket car case with state constraints, i.e. the one-dimensional movement of a mass point under the influence of some constant acceleration/deceleration, e.g. modeling head-wind or sliding friction, which can accelerate and decelerate in order to reach a desired position. The mass of the car is normalized to 1 unit[1] and the constant acceleration/deceleration enters the model in form of an unknown parameter $p \in \Omega_P \subset \mathbb{R}$ suffering from uncertainty, with the uncertainty set Ω_P convex and compact. We consider a problem in which the rocket car shall reach a final feasible position and velocity in a minimum time:

$$\begin{aligned}
\underset{T,u(\cdot),x(\cdot;p)}{min} \quad & T && && (2.1a)\\
s.t. \quad & x = (x_1, x_2) && && (2.1b)\\
& \dot{x} = T \begin{pmatrix} x_2(t;p) \\ u(t) - p \end{pmatrix}, && t \in [0,1], && (2.1c)\\
& x(0,p) = 0, && && (2.1d)\\
& x_1(1;p) \geq 10, && && (2.1e)\\
& x_2(t;p) \leq 4, && t \in [0,1], && (2.1f)\\
& x_2(1;p) \leq 0, && && (2.1g)\\
& T \geq 0, && && (2.1h)\\
& u(t) \in [-10, 10], && t \in [0,1]. && (2.1i)
\end{aligned}$$

where x represents the variables of the rocket car, and it has two components $x = (x_1, x_2)$. The first component x_1 is the (time-transformed) position of the rocket car. The second component x_2 is (time-transformed) velocity of the rocket car. The condition 2.1d, i.e. $x(0,p) = 0$, indicates that at $t = 0$, both the position and velocity of the car is 0. The condition 2.1e, i.e. $x_1(1;p) \geq 10$, indicates that the position of the car at $t = 1$ must be greater or equal to 10. The condition 2.1, i.e. $x_2(t;p) \leq 4$, indicates that the velocity of the car is always smaller or equal to 4 across the whole period. The condition 2.1, i.e. $x_2(1;p) \leq 0$, indicates that the velocity of the car at $t = 1$ is always smaller or

[1] We do not specify the unit on purpose since the actual unit, either one kilogram or meter, does not play a role in the modeling. We are more concerned about the scale.

equal to 0. Here, a negative velocity means that the car is moving in a direction that decreases the position. To make the rocket car case even simpler, we can limit the size of the uncertainty set, as following

$$p \in \Omega_P = [p_l, p_u] \subset [0, 9], \tag{2.2}$$

where $p_l < p_u$, with p_l and p_u the lower and upper boundary of the parameter p.

The decision variable in the problem 2.1 is the controllable parameter T, which encodes the process duration of the corresponding problem with free end time. The control function $u : [0, 1] \to \mathbb{R}$ represents the acceleration/deceleration value, and is dependent on the unknown parameter p, as shown in the condition 2.1c. The second component of the condition 2.1c, i.e. $\dot{x_2} = T(u(t) - p)$, indicates the change in the velocity of the car at time t is subject to the value of $T, u(t)$ and p. The first component of the condition 2.1c, i.e. $\dot{x_1} = Tx_2(t; p)$, indicates the position of the car at time t is subject to the value of T and the velocity $x_2(t; p)$ at time t. The variable $x(t : p)$ is a dependent variable, and is uniquely determined by $T, u(\cdot)$ and p. The goal is to minimize T such that the variable $x(t : p)$ satisfies all the conditions in 2.1.

2.1 Theoretical solution to rocket car case

As explained in Chapter 1, we choose the rocket car case for two main reasons, i.e. the easyness of understanding and the existence of theoretical solution, which will be shown in this section.

There are three optimization variables in the optimization problem 2.1, i.e. T, u and x, and they belong to the following normed space

$$(T, u(\cdot), x(:, p)) \in \mathbb{R} \times \mathbb{L}^{\infty}([0, 1], \mathbb{R}) \times \mathbb{W}^{1,\infty}([0, 1], \mathbb{R}^2) \tag{2.3}$$

And the optimization problem 2.1 has a unique global solution, and no further local solution exists. The optimal controllable parameter is given by

$$T^\star = T^\star(p) = 2.5 + \frac{40}{100 - p^2}, \tag{2.4}$$

and the optimal control function $u^\star(\cdot)(= u^\star(\cdot; p))$ by

$$u^\star(\cdot) = \begin{cases} 10, & for\ 0 \le t < \frac{4}{(10-p)T^\star} \\ p & for\ \frac{4}{(10-p)T^\star} \le t < 1 - \frac{4}{(10+p)T^\star} \\ -10 & for\ 1 - \frac{4}{(10+p)T^\star} \le t \le 1 \end{cases} \tag{2.5}$$

In words, we accelerate as strongly as possible (the acceleration value $u^\star(t) = 10$) until the velocity $x_2^\star(t; p) = 4$, and then keep the velocity $x_2^\star(t; p)$ constant for a certain periofd of time[2], and eventually decelerate as as strongly as possible until the velocity is $x_2(1; p) \le 0$,. The moment $x_2(1; p)$ reaching the value of 0 is the moment that we finds the optimal/smallest T that all the conditions are satisfied.

The proof of the theoretical solution is given in Appendix B of Schlöder [2022]. Because of the simplicity nature of the rocket car case, we can find the theoretical solution of the

[2] The acceleration value cancels out with a inherent deceleration value so that the velocity can stay constant. The inherent deceleration value can be result of a friction or head wind.

nominal/original problem for our case 2.1. But for many real life problems, it is very difficult to find a direct solution to the original problem, and for some cases not feasible, due to the uncertainty in the parameter p. That is why in the paper Schlöder [2022], a classical (in the form *minmax*) approach and a training (in the form *maxmin*) approach have been discussed, and both approaches will lead to a conservative solution to the original problem. A conservative solution to the CP problem is a acceptable (or desired) result since less risk should be taken regarding the therapy design of CP problem. In the next chapter, we discuss, in details, the classical approaches and training approach for the rocket car problem. After that, we focus on the quasi-Newton and multi shooting approach to the same problem.

3 The Classical and Training Approach

The paper on hand focuses on using quasi-Newton and multi-shooting method for the Training Approach. In this chapter, we shortly introduce the Classical Approach first and then we discuss the Training Approach in greater detail. In the next chapter, we can introduce the quasi-Newton and multi-shooting method, and elaborate in detail how they can be used for the Training Approach.

3.1 The Classical Approach

As stated in the introduction part, the classical approach is consistent with the *minmax* approach, during which, two level optimization problems are solved.

In the lower level, we solve a optimization problem ($max\ f(x,p)$) with respect to p, and in the upper level, we continue to find the best solution with respect to x, as shown in 1.3. In the case of the rocket car, the classical approach will be expressed in the following form

$$
\begin{aligned}
\underset{T,u(\cdot)}{min}\ \underset{p\in\Omega_P,x(\cdot,p)}{max}\quad & T && && (3.1a)\\
s.t.\quad & x=(x_1,x_2) && && (3.1b)\\
& \dot{x}=T\begin{pmatrix}x_2(t;p)\\ u(t)-p\end{pmatrix}, && t\in[0,1], && (3.1c)\\
& x(0,p)=0, && && (3.1d)\\
& x_1(1;p)\geq 10, && for\ all\ p\in\Omega_P, && (3.1e)\\
& x_2(t;p)\leq 4, && t\in[0,1],\ for\ all\ p\in\Omega_P, && (3.1f)\\
& x_2(1;p)\leq 0, && for\ all\ p\in\Omega_P, && (3.1g)\\
& T\geq 0, && && (3.1h)\\
& u(t)\in[-10,10], && t\in[0,1]. && (3.1i)
\end{aligned}
$$

In the Classical Approach, the set of feasible controllable parameters and control functions are given by those T and $u(\cdot)$, which yield feasible trajectories $x(\cdot,p)$ for all $p\in\Omega_P$.

The value of the objective function in the lower level does not depend on p and $x(\cdot, p)$. In other words, in this approach, the driver has no prior knowledge about the value of the parameter p and gets no feedback during the process and has to set up the driving strategy in advance.

3.2 The Training Approach

Contrast to the Classical Approach, in the Training Approach it is assumed that the driver of the rocket car is able to perform optmially for every p because of a preceding training period. Thus the worst possible optimal performance is given by a solution of the problem

$$
\begin{aligned}
\max_{p\in\Omega_P, T, u(\cdot), x(\cdot,p)} \quad \min \; & T && && (3.2a)\\
s.t. \; & x = (x_1, x_2) && && (3.2b)\\
& \dot{x} = T \begin{pmatrix} x_2(t;p) \\ u(t) - p \end{pmatrix}, && t \in [0,1], && (3.2c)\\
& x(0,p) = 0, && && (3.2d)\\
& x_1(1;p) \geq 10, && && (3.2e)\\
& x_2(t;p) \leq 4, && t \in [0,1], && (3.2f)\\
& x_2(1;p) \leq 0, && && (3.2g)\\
& T \geq 0, && && (3.2h)\\
& u(t) \in [-10, 10], && t \in [0,1]. && (3.2i)
\end{aligned}
$$

Part I

Appendix

A Lists

A.1 List of Figures

A.2 List of Tables

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Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den (Datum) . . . .

Declaration:

I hereby confirm that I wrote this work independently and did not use any sources other than those indicated.