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Gradient method of solving parameterized optimal control problems, with a case study in state constrained rocket car

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(Titel der Masterarbeit - deutsch):

(Abstract in Deutsch, max. 200 Worte. Beispiel: lorem ipsum)
in dieser Diplomarbeit,

(Title of Master thesis - english):

In this thesis, we focus on the mathematical models and numerical methods for constrained parameterized the optimal control problem, with a case study in state constrained rocket car.

1 Introduction

Many real life problems, can be modeled as parameterized optimization problems, such as the therapy design of Cerebral Palsy problem described in Schlöder [2022]. In this paper, we focus on using gradient method to solve parameterized optimization problems, with a case study in state constrained rocket car.

Without giving a rigorous condition and definition[1], a general optimization problem is typically of the form

$$\begin{aligned} & min\ f(x) \\ s.t.\quad & g(x) = 0, \\ & h(x) \geq 0 \end{aligned} \tag{1.1}$$

where $f(x)$ is the objective or cost function, $g(x) = 0$ and $h(x) \geq 0$ are the constraints. Some optimization problems may have uncertain parameters whose value are priori unknown, and the optimal objective value depends on the parameter value. This kind of problem is called the parameterized optimization problems and is of the form

$$\begin{aligned} & min\ f(x,p) \\ s.t.\quad & g(x,p) = 0, \\ & h(x,p) \geq 0 \\ & x = x(p) \\ & x = x(p^0)\ if\ p = p^0 \\ & p \in \Omega_P \end{aligned} \tag{1.2}$$

where p^0 is a fixed value in the feasible uncertainty set Ω_P, where the parameter p can take value from.

Parameterized optimization problems are very difficult to solve due to the uncertainty in the parameter p. In the paper Schlöder [2022], multiple methods of solving the parameterized optimization problem have been discussed. The main focus (of solving the Cerebral Palsy problem) of the paper Schlöder [2022], is the "worst-case treatment planning by bilevel optimal control", i.e. a bilevel optimization problem. The bilevel optimisation method in paper Schlöder [2022] solves the parameterized optimization problems, e.g. the Cerebral Palsy problem, in a conservative way.

Assuming that the parameter $\tilde{p}$ lies in an uncertainty set Ω_P, the objective is to identify a worst possible solution with respect to $\tilde{p}$, i.e. solving a lower level problem. Based on the result of lower level, the author continues to find the best solution with respect to x, i.e. solving a upper level problem. The "worst-case treatment planning by bilevel optimal control", i.e. a bilevel optimization problem, is an optimization problem in which another optimization problem enters the constraints. Mathematically, the problem can be formulated in a simplified form, as following

[1] We do not give a rigorous definition on purpose so that the problem we have described here can be applied to more general cases when such condition and definition are more clearly defined.

$$\begin{aligned} &\underset{x}{min} \underset{p\in\Omega_P}{max} \; f(x,p) \\ &\qquad\quad s.t. \; g(x,p)=0, \; h(x,p)\geq 0 \end{aligned} \tag{1.3}$$

Due to the *min max* notation, this approach of solving the bilevel problem can also be called *minmax* approach.

As stated in Schlöder [2022], many different methods can be used to solve a bilevel problem, three approaches have been discussed in detail, i.e. a transformation of the bilevel problem to a single level problem, a classical approach and a training approach. A intuitive approach is to transfer the bilevel problem into a single level problem, however, in general the resulting single level problem is not equivalent to the original bilevel problem and this approach is also out of the focus of the paper Schlöder [2022] as well as this paper at hand. A classical approach, aka a robust optimization appraoch, is consistent with the *minmax* appoach, which will be discussed in more detail in Chapter 2.

The paper Schlöder [2022] introduces the "Training Approach". It is based on the idea that in the real world, during the training period, an intervention is introduced and a certain, but a priori unknown, parameter $p \in \Omega_P$ is realized. What follows the training period (during which the parameter p is realized), the patient is able to react to it in an optimal manner, i.e. an optimal value $f(x,p)$ will be obtained given the realized parameter p. The paper Schlöder [2022] call this approach "worst case modeling Training Approach", and it can be written as

$$\begin{aligned} &\underset{p\in\Omega_P}{max} \underset{x}{min} \; f(x,p) \\ &\qquad\quad s.t. \; g(x,p)=0, \; h(x,p)\geq 0 \end{aligned} \tag{1.4}$$

Due to the *max mix* notation, this approach of solving the bilevel problem can also be called *maxmin* approach.

The paper Schlöder [2022] use a derivative free method in the Training Approach. This paper at hand will focus on a gradient method to solve the *maxmin* problem. In particular, we are interested in how to compute the derivatives theoretically and numerically. We would like to apply the quasi-Newton and multiple shooting method when solving the problem numerically. The approaches discussed in this paper at hand will be demonstrated with a case study in state constrained rocket car.

We choose this rocket car case for two reasons: firstly, the case is relatively easy to understand and is quite representative of the general usage in real life; secondly, the case has theoretical solution and we can compare the numerical results with the theoretical value so that we can check whether our gradient method can find the optimal solution and how fast it converges.

The structure of this paper is as follows: in Chapter 1, i.e. this chapter, we give an introduction on what problems this paper intends to address. In the Chapter 2, we introduce the case of the state constrained rocket car. In the Chapter 3, we discuss the classical approach and training approach, and show the theoretical value of the chosen case. In Chapter 4, we give the mathematical background of the quasi-Newton and multiple shooting method. In Chapter 5, we show how we can solve the case numerically using the methods described in Chapter 4. In Chapter 6, we compare our theoretical and numerical results and conclude the paper.

2 Rocket car case

Since the approaches we are going to use in this paper will be demonstrated with the case of rocket car, we decide to describe the rocket car case first. So that, when we are discussing our approaches, we can directly describe how they can be used in solving the rocket car case. The description of the rocket car case is mostly coming from the paper Schlöder [2022], with content either verbatim or in a modified form.

We consider the rocket car case with state constraints, i.e. the one-dimensional movement of a mass point under the influence of some constant acceleration/deceleration, e.g. modeling head-wind or sliding friction, which can accelerate and decelerate in order to reach a desired position. The mass of the car is normalized to 1 unit[1] and the constant acceleration/deceleration enters the model in form of an unknown parameter $p \in \Omega_P \subset \mathbb{R}$ suffering from uncertainty, with the uncertainty set Ω_P convex and compact. We consider a problem in which the rocket car shall reach a final feasible position and velocity in a minimum time:

$$
\begin{aligned}
\underset{T,u(\cdot),x(\cdot;p)}{min} \quad & T && && (2.1a)\\
s.t. \quad & x = (x_1, x_2) && && (2.1b)\\
& \dot{x} = T \begin{pmatrix} x_2(t;p) \\ u(t) - p \end{pmatrix}, && t \in [0,1], && (2.1c)\\
& x(0,p) = 0, && && (2.1d)\\
& x_1(1;p) \geq 10, && && (2.1e)\\
& x_2(t;p) \leq 4, && t \in [0,1], && (2.1f)\\
& x_2(1;p) \leq 0, && && (2.1g)\\
& T \geq 0, && && (2.1h)\\
& u(t) \in [-10, 10], && t \in [0,1]. && (2.1i)
\end{aligned}
$$

where x represents the variables of the rocket car, and it has two components $x = (x_1, x_2)$. The first component x_1 is the (time-transformed) position of the rocket car. The second component x_2 is (time-transformed) velocity of the rocket car. The condition 2.1d, i.e. $x(0,p) = 0$, indicates that at $t = 0$, both the position and velocity of the car is 0. The condition 2.1e, i.e. $x_1(1;p) \geq 10$, indicates that the position of the car at $t = 1$ must be greater or equal to 10. The condition 2.1g, i.e. $x_2(t;p) \leq 4$, indicates that the velocity of the car is always smaller or equal to 4 across the whole period. The condition 2.1g, i.e. $x_2(1;p) \leq 0$, indicates that the velocity of the car at $t = 1$ is always smaller or equal to 0. Here, a negative velocity means that the car is moving in a direction that decreases the position.

[1]We do not specify the unit on purpose since the actual unit, either one kilogram or meter, does not play a role in the modeling. We are more concerned about the scale.

The decision variable in the problem 2.1 is the controllable parameter T, which encodes the process duration of the corresponding problem with free end time. The control function $u : [0, 1] \to \mathbb{R}$ represents the acceleration/deceleration value, and is dependent on the unknown parameter p, as shown in the condition 2.1c. The second component of the condition 2.1c, i.e. $\dot{x_2} = T(u(t) - p)$, indicates the change in the velocity of the car at time t is subject to the value of $T, u(t)$ and p. The first component of the condition 2.1c, i.e. $\dot{x_1} = Tx_2(t; p)$, indicates the position of the car at time t is subject to the value of T and the velocity $x_2(t; p)$ at time t. $x(t : p)$ is a dependent variable, and is uniquely determined by $T, u(\cdot)$ and p. The goal is to minimize T such that the variable $x(t : p)$ satisfies all the conditions in 2.1.

2.1 Theoretical solution to rocket car case

As explained in Chapter 1, we choose the rocket car case for two main reasons, i.e. the easyness of understanding and the existence of theoretical solution, which will be shown in this section.

There are three optimization variables in the optimization problem 2.1, i.e. T, u and x, and they belong to the following normed space

$$(T, u(\cdot), x(:, p)) \in \mathbb{R} \times \mathbb{L}^{\infty}([0, 1], \mathbb{R}) \times \mathbb{W}^{1,\infty}([0, 1], \mathbb{R}^2) \tag{2.2}$$

And the optimization problem 2.1 has a unique global solution, and no further local solution exists. The optimal controllable parameter is given by

$$T^{\star} = T^{\star}(p) = 2.5 + \frac{40}{100 - p^2}, \tag{2.3}$$

and the optimal control function $u^{\star}(\cdot)(= u^{\star}(\cdot; p))$ by

$$u^{\star}(\cdot) = \begin{cases} 10, & for\ 0 \le t < \frac{4}{(10-p)T^{\star}} \\ p & for\ \frac{4}{(10-p)T^{\star}} \le< 1 - \frac{4}{(10+p)T^{\star}} \\ -10 & for\ 1 - \frac{4}{(10+p)T^{\star}} \le t \le 1 \end{cases} \tag{2.4}$$

In words, we accelerate as strongly as possible (the acceleration value $u^{\star}(t) = 10$) until $x_2^{\star}(t; p) = 4$, and then keep $x_2^{\star}(t; p)$ constant for a certain periof of time, and eventually decelerate as as strongly as possible.

Contents

Part I

Appendix

A Lists

A.1 List of Figures

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Bibliography

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den (Datum) ....

Declaration:

I hereby confirm that I wrote this work independently and did not use any sources other than those indicated.

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