Department of Mathematics and Computer Science

Heidelberg University

Master thesis

in Scientific Computing

submitted by

Yu Xiang

born in November 10, 1989

2022

Gradient method of solving parameterized optimal control problems, with a case study in state constrained rocket car

This master thesis has been carried out by Yu Xiang

at the

Heidelberg University

under the supervision of

Professor Dr. Ekaterina A. Kostina

(Titel der Masterarbeit - deutsch):

(Abstract in Deutsch, max. 200 Worte. Beispiel: lorem ipsum)
in dieser Diplomarbeit,

(Title of Master thesis - english):

In this thesis, we focus on the mathematical models and numerical methods for constrained parameterized the optimal control problem, with a case study in state constrained rocket car.

1 Introduction

Many real life problems, can be modeled as parameterized optimization problems, such as the therapy design of Cerebral Palsy problem described in Schlöder [2022]. In this paper, we focus on the using gradient method to solve parameterized optimization problems, with a case study in state constrained rocket car.

Without giving a rigorous condition and definition[1], a general optimization problem is typically of the form

$$\begin{aligned} & min\ f(x) \\ s.t.\quad & g(x) = 0, \\ & h(x) \geq 0 \end{aligned} \tag{1.1}$$

where $f(x)$ is the objective or cost function, $g(x) = 0$ and $h(x) \geq 0$ are the constraints. Some optimization problems may have uncertain parameters whose value are priori unknown, and the optimal objective value depends on the parameter value. This kind of problem is called the parameterized optimization problems and is of the form

$$\begin{aligned} & min\ f(x,p) \\ s.t.\quad & g(x,p) = 0, \\ & h(x,p) \geq 0 \\ & x = x(p) \\ & x = x(p^0)\ if\ p = p^0 \\ & p \in \mathbb{P} \end{aligned} \tag{1.2}$$

where p^0 is a fixed value in the feasible uncertainty set $\mathbb{P}$, where the parameter p can take value from.

In the paper Schlöder [2022], multiple methods of solving the parameterized optimization problem have been discussed, and the main focus (of solving the Cerebral Palsy problem) in that paper is the "worst-case treatment planning by bilevel optimal control". Assuming that $\tilde{p}$ lies in an uncertainty set $\mathbb{P}$, the objective is to identify a worst possible solution with respect to $\tilde{p}$, i.e. solving a lower level problem. Based on the result of lower level, the author continues to find the best solution with respect to x, i.e. solving a upper level problem. The "worst-case treatment planning by bilevel optimal control", i.e. a bilevel optimization problem, is an optimization problem in which another optimization problem enters the constraints. Mathematically, the problem can be formulated as following

$$\begin{aligned} & \underset{x}{min}\ \tilde{f}(x) \\ where\quad & \tilde{f}(x) = \begin{cases} \underset{p\in\mathbb{P}}{max} & f(x,p) \\ s.t. & g(x,p) = 0,\ h(x,p) \geq 0 \end{cases} \end{aligned} \tag{1.3}$$

[1]We do not give a rigorous definition on purpose so that the problem we have described here can be applied to more general cases when such condition and definition are more clearly defined.

In a simplified notation, the problem can be written as

$$\begin{aligned} &\underset{x}{min}\ \underset{p\in\mathbb{P}}{max}\ f(x,p) \\ &\qquad\quad s.t.\ g(x,p)=0,\ \ h(x,p)\geq 0 \end{aligned} \tag{1.4}$$

Due to the *min max* notation, this approach of solving the bilevel problem can also be called *minmax* approach.

Please notice

As stated in Schlöder [2022], many different methods can be used to solve a bilevel problem, mainly transformation of the bilevel problem to a single level problem, a classical approach and a training approach. A common approach is to transfer the bilevel problem into a single level problem, however, in general the resulting single level problem is not equivalent to the original bilevel problem and this approach is also out of the focus of the paper Schlöder [2022] as well as this paper at hand. A classical approach is consistent with the *minmax* appoach.

The paper Schlöder [2022] introduces the "Training approach". It is based on the idea that in the real world, during the training period, an intervention is introduce and a certain, but a priori unknown, parameter $p \in \mathbb{P}$ is realized. What follows the training period (during which the parameter p is realized), the patient is able to react to it in an optimal manner, i.e. an optimal value $f(x,p)$ will be obtained given the realized parameter p. The paper Schlöder [2022] call this approach "worst case modelling Training Approach", and it can be written as

$$\begin{aligned} &\underset{p\in\mathbb{P}}{max}\ \underset{x}{min}\ f(x,p) \\ &\qquad\quad s.t.\ g(x,p)=0,\ \ h(x,p)\geq 0 \end{aligned} \tag{1.5}$$

Due to the *max mix* notation, this approach of solving the bilevel problem can also be called *maxmin* approach.

The paper Schlöder [2022] use a derivative free method in the Training Approach. This paper at hand will focus on a gradient method to solve the *maxmin* problem. In particular, we are interested in how to compute the derivatives theoretically and numerically. We would like to apply the quasi-Newton and multiple shooting method when solving the problem numerically. The whole approach will be demonstrated with a case study in state constrained rocket car.

We choose this case for two reasons: firstly, the case is relatively easy to understand yet is quite representaive of the general usage in real life; secondly, the case has theoretical solution and we can compare the numerical results with the theoretical value so that we can check whether our gradient method can find the optimal solution and how fast it converges.

The structure of this paper is as follows: in Chapter 1, i.e. this chapter, we give an introduction on what problems this paper intends to address. In the Chapter 2, we introduce the case of the state constrained rocket car. In the Chapter 3, we show the theoretical value of the chosen case. In Chapter 4, we give the mathematical background of the quasi-Newton and multiple shooting method. In Chapter 5, we show how we can solve the case numerically using the methods described in Chapter 4. In Chapter 6, we compare our theoretical and numerical results and conclude the paper.

Contents

Part I

Appendix

A Lists

A.1 List of Figures

A.2 List of Tables

Bibliography

lorem ipsum. URL http://la.wikisource.org/wiki/Lorem_ipsum.

Matthias Schlöder. Numerical methods for optimal control of constrained biomechanical multi-body systems appearing in therapy design of cerebral palsy. 2022.

Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den (Datum) ....

Declaration:

I hereby confirm that I wrote this work independently and did not use any sources other than those indicated.

Heidelberg, (Date) ....