

Department of Mathematics and Computer Science

Heidelberg University

Master thesis

in Scientific Computing

submitted by

Yu Xiang

born in November 10, 1989

2022

Gradient method of solving parameterized optimal  
control problems, with a case study in state  
constrained rocket car

This master thesis has been carried out by Yu Xiang

at the

Heidelberg University

under the supervision of

Professor Dr. Ekaterina A. Kostina

# Contents

|          |                                                   |           |
|----------|---------------------------------------------------|-----------|
| <b>1</b> | <b>Introduction</b>                               | <b>4</b>  |
| <b>2</b> | <b>Rocket car case</b>                            | <b>6</b>  |
| 2.1      | Theoretical solution to rocket car case . . . . . | 7         |
| <b>3</b> | <b>The Classical and Training Approach</b>        | <b>8</b>  |
| 3.1      | The Classical Approach . . . . .                  | 8         |
| 3.2      | The Training Approach . . . . .                   | 9         |
| <b>I</b> | <b>Appendix</b>                                   | <b>12</b> |

# 1 Introduction

Many real life problems, can be modeled as parameterized optimization problems, such as the therapy design of Cerebral Palsy (CP) problem described in [Schlöder \[2022\]](#). In this paper, we focus on using gradient method to solve parameterized optimization problems, with a case study in state constrained rocket car.

Without giving a rigorous condition and definition<sup>1</sup>, a general optimization problem is typically of the form

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) = 0, \\ & h(x) \geq 0 \end{aligned} \tag{1.1}$$

where  $f(x)$  is the objective or cost function,  $g(x) = 0$  and  $h(x) \geq 0$  are the constraints. Some optimization problems may have uncertain parameters whose value are priori unknown, and the optimal objective value depends on the parameter value. This kind of problem is called the parameterized optimization problems and is of the form

$$\begin{aligned} \min \quad & f(x, p) \\ \text{s.t.} \quad & g(x, p) = 0, \\ & h(x, p) \geq 0 \\ & x = x(p) \\ & x = x(p^0) \text{ if } p = p^0 \\ & p \in \Omega_P \end{aligned} \tag{1.2}$$

where  $p^0$  is a fixed value in the feasible uncertainty set  $\Omega_P$ , where the parameter  $p$  can take value from.

Parameterized optimization problems are very difficult to solve due to the uncertainty in the parameter  $p$ . In the paper [Schlöder \[2022\]](#), multiple methods of solving the parameterized optimization problem have been discussed. The main focus (of solving the Cerebral Palsy problem) of the paper [Schlöder \[2022\]](#), is the "worst-case treatment planning by bilevel optimal control", i.e. a bilevel optimization problem. The bilevel optimisation method in paper [Schlöder \[2022\]](#) solves the parameterized optimization problems, e.g. the Cerebral Palsy problem, in a conservative way.

One method of solving the original CP problem in a conservative way is to transform the problem 1.2 into another form. Assuming that the parameter  $\tilde{p}$  lies in an uncertainty set  $\Omega_P$ , we can firstly reach one objective, i.e. identifying a worst possible solution with respect to  $\tilde{p}$ , i.e. solving a lower level problem. Based on the result of lower level, we can continue to find the best solution with respect to  $x$ , i.e. solving a upper level problem. The "worst-case treatment planning by bilevel optimal control", i.e. a bilevel optimization

---

<sup>1</sup>We do not give a rigorous definition on purpose so that the problem we have described here can be applied to more general cases when such condition and definition are more clearly defined.

problem, is an optimization problem in which another optimization problem enters the constraints. Mathematically, the problem 1.2 is transformed into another form, and can be formulated in a simplified notation, as following

$$\begin{aligned} \min_x \max_{p \in \Omega_P} f(x, p) \\ \text{s.t. } g(x, p) = 0, \quad h(x, p) \geq 0 \end{aligned} \tag{1.3}$$

Due to the *min max* notation, this classical approach of solving the bilevel problem can also be called *minmax* approach.

As stated in Schlöder [2022], many different methods can be used to solve a bilevel problem, three approaches have been discussed in detail, i.e. a transformation of the bilevel problem to a single level problem, a classical approach and a training approach. A intuitive approach is to transfer the bilevel problem into a single level problem, however, in general the resulting single level problem is not equivalent to the original bilevel problem and this approach is also out of the focus of the paper Schlöder [2022] as well as this paper at hand. A classical approach, aka a robust optimization approach, is consistent with the *minmax* approach, which will be discussed in more detail in Chapter 2.

The paper Schlöder [2022] introduces the "Training Approach". It is based on the idea that in the real world, during the training period, an intervention is introduced and a certain, but a priori unknown, parameter  $p \in \Omega_P$  is realized. What follows the training period (during which the parameter  $p$  is realized), the patient is able to react to it in an optimal manner, i.e. an optimal value  $f(x, p)$  will be obtained given the realized parameter  $p$ . The paper Schlöder [2022] call this approach "worst case modeling Training Approach", and it can be written as

$$\begin{aligned} \max_{p \in \Omega_P} \min_x f(x, p) \\ \text{s.t. } g(x, p) = 0, \quad h(x, p) \geq 0 \end{aligned} \tag{1.4}$$

Due to the *max min* notation, this approach of solving the bilevel problem can also be called *maxmin* approach.

The paper Schlöder [2022] use a derivative free method in the Training Approach. This paper at hand will focus on a gradient method to solve the *maxmin* problem. In particular, we are interested in how to compute the derivatives theoretically and numerically. We would like to apply the quasi-Newton and multiple shooting method when solving the problem numerically. The approaches discussed in this paper at hand will be demonstrated with a case study in state constrained rocket car.

We choose this rocket car case for two reasons: firstly, the case is relatively easy to understand and is quite representative of the general usage in real life; secondly, the case has theoretical solution and we can compare the numerical results with the theoretical value so that we can check whether our gradient method can find the optimal solution and how fast it converges.

The structure of this paper is as follows: in Chapter 1, i.e. this chapter, we give an introduction on what problems this paper intends to address. In the Chapter 2, we introduce the case of the state constrained rocket car. In the Chapter 3, we discuss the classical approach and training approach, and show the theoretical value of the chosen case. In Chapter 4, we give the mathematical background of the quasi-Newton and

multiple shooting method. In Chapter 5, we show how we can solve the case numerically using the methods described in Chapter 4. In Chapter 6, we compare our theoretical and numerical results and conclude the paper.

## 2 Rocket car case

Since the approaches we are going to use in this paper will be demonstrated with the case of rocket car, we decide to describe the rocket car case first. So that, when we are discussing our approaches, we can directly describe how they can be used in solving the rocket car case. The description of the rocket car case is mostly coming from the paper [Schlöder \[2022\]](#), with content either verbatim or in a modified form.

We consider the rocket car case with state constraints, i.e. the one-dimensional movement of a mass point under the influence of some constant acceleration/deceleration, e.g. modeling head-wind or sliding friction, which can accelerate and decelerate in order to reach a desired position. The mass of the car is normalized to 1 unit<sup>1</sup> and the constant acceleration/deceleration enters the model in form of an unknown parameter  $p \in \Omega_P \subset \mathbb{R}$  suffering from uncertainty, with the uncertainty set  $\Omega_P$  convex and compact. We consider a problem in which the rocket car shall reach a final feasible position and velocity in a minimum time:

$$\min_{T, u(\cdot), x(\cdot; p)} T \tag{2.1a}$$

$$s.t. \quad x = (x_1, x_2) \tag{2.1b}$$

$$\dot{x} = T \begin{pmatrix} x_2(t; p) \\ u(t) - p \end{pmatrix}, \quad t \in [0, 1], \tag{2.1c}$$

$$x(0, p) = 0, \tag{2.1d}$$

$$x_1(1; p) \geq 10, \tag{2.1e}$$

$$x_2(t; p) \leq 4, \quad t \in [0, 1], \tag{2.1f}$$

$$x_2(1; p) \leq 0, \tag{2.1g}$$

$$T \geq 0, \tag{2.1h}$$

$$u(t) \in [-10, 10], \quad t \in [0, 1]. \tag{2.1i}$$

where  $x$  represents the variables of the rocket car, and it has two components  $x = (x_1, x_2)$ . The first component  $x_1$  is the (time-transformed) position of the rocket car. The second component  $x_2$  is (time-transformed) velocity of the rocket car. The condition 2.1d, i.e.  $x(0, p) = 0$ , indicates that at  $t = 0$ , both the position and velocity of the car is 0. The condition 2.1e, i.e.  $x_1(1; p) \geq 10$ , indicates that the position of the car at  $t = 1$  must be greater or equal to 10. The condition 2.1f, i.e.  $x_2(t; p) \leq 4$ , indicates that the velocity of the car is always smaller or equal to 4 across the whole period. The condition 2.1g, i.e.  $x_2(1; p) \leq 0$ , indicates that the velocity of the car at  $t = 1$  is always smaller or

---

<sup>1</sup>We do not specify the unit on purpose since the actual unit, either one kilogram or meter, does not play a role in the modeling. We are more concerned about the scale.

equal to 0. Here, a negative velocity means that the car is moving in a direction that decreases the position. To make the rocket car case even simpler, we can limit the size of the uncertainty set, as following

$$p \in \Omega_P = [p_l, p_u] \subset [0, 9], \quad (2.2)$$

where  $p_l < p_u$ , with  $p_l$  and  $p_u$  the lower and upper boundary of the parameter  $p$ .

The decision variable in the problem 2.1 is the controllable parameter  $T$ , which encodes the process duration of the corresponding problem with free end time. The control function  $u : [0, 1] \rightarrow \mathbb{R}$  represents the acceleration/deceleration value, and is dependent on the unknown parameter  $p$ , as shown in the condition 2.1c. The second component of the condition 2.1c, i.e.  $\dot{x}_2 = T(u(t) - p)$ , indicates the change in the velocity of the car at time  $t$  is subject to the value of  $T, u(t)$  and  $p$ . The first component of the condition 2.1c, i.e.  $\dot{x}_1 = T x_2(t; p)$ , indicates the position of the car at time  $t$  is subject to the value of  $T$  and the velocity  $x_2(t; p)$  at time  $t$ . The variable  $x(t : p)$  is a dependent variable, and is uniquely determined by  $T, u(\cdot)$  and  $p$ . The goal is to minimize  $T$  such that the variable  $x(t : p)$  satisfies all the conditions in 2.1.

## 2.1 Theoretical solution to rocket car case

As explained in Chapter 1, we choose the rocket car case for two main reasons, i.e. the easyness of understanding and the existence of theoretical solution, which will be shown in this section.

The optimization problem 2.1 has a unique global solution, and no further local solution exists. The optimal controllable parameter is given by

$$T^* = T^*(p) = 2.5 + \frac{40}{100 - p^2}, \quad (2.3)$$

and the optimal control function  $u^*(\cdot)(= u^*(\cdot; p))$  by

$$u^*(\cdot) = \begin{cases} 10, & \text{for } 0 \leq t < \frac{4}{(10-p)T^*} \\ p & \text{for } \frac{4}{(10-p)T^*} \leq t < 1 - \frac{4}{(10+p)T^*} \\ -10 & \text{for } 1 - \frac{4}{(10+p)T^*} \leq t \leq 1 \end{cases} \quad (2.4)$$

In words, we accelerate as strongly as possible (the acceleration value  $u^*(t) = 10$ ) until the velocity  $x_2^*(t; p) = 4$ , and then keep the velocity  $x_2^*(t; p)$  constant for a certain period of time<sup>2</sup>, and eventually decelerate as strongly as possible until the velocity is  $x_2(1; p) \leq 0$ . The moment  $x_2(1; p)$  reaching the value of 0 is the moment that we find the optimal/smallest  $T$  that all the conditions are satisfied.

The proof of the theoretical solution is given in Appendix B of [Schlöder \[2022\]](#). Because of the simplicity nature of the rocket car case, we can find the theoretical solution of the nominal/original problem for our case 2.1. But for many real life problems, it is very difficult to find a direct solution to the original problem, and for some cases not feasible, due to the uncertainty in the parameter  $p$ . That is why in the paper [Schlöder \[2022\]](#), a classical (in the form *minmax*) approach and a training (in the form *maxmin*) approach

---

<sup>2</sup>The acceleration value cancels out with a inherent deceleration value so that the velocity can stay constant. The inherent deceleration value can be result of a friction or head wind.

have been discussed, and both approaches will lead to a conservative solution to the original problem. A conservative solution to the CP problem is a acceptable (or desired) result since less risk should be taken regarding the therapy design of CP problem. In the next chapter, we discuss, in details, the classical approaches and training approach for the rocket car problem. After that, we focus on the quasi-Newton and multi shooting approach to the same problem.

## 3 The Classical and Training Approach

The paper on hand focuses on using quasi-Newton and multi-shooting method for the Training Approach. In this chapter, we shortly introduce the Classical Approach first and then we discuss the Training Approach in greater detail. In the next chapter, we can introduce the quasi-Newton and multi-shooting method, and elaborate in detail how they can be used for the Training Approach.

### 3.1 The Classical Approach

As stated in the introduction part, the classical approach is consistent with the *minmax* approach, during which, two level optimization problems are solved.

In the lower level, we solve an optimization problem ( $\max f(x, p)$ ) with respect to  $p$ , and in the upper level, we continue to find the best solution with respect to  $x$ , as shown in 1.3. In the case of the rocket car, the classical approach will be expressed in the following form

$$\min_{T, u(\cdot)} \max_{p \in \Omega_P, x(\cdot, p)} T \quad (3.1a)$$

$$s.t. \quad x = (x_1, x_2) \quad (3.1b)$$

$$\dot{x} = T \begin{pmatrix} x_2(t; p) \\ u(t) - p \end{pmatrix}, \quad t \in [0, 1], \quad (3.1c)$$

$$x(0, p) = 0, \quad (3.1d)$$

$$x_1(1; p) \geq 10, \quad \text{for all } p \in \Omega_P, \quad (3.1e)$$

$$x_2(t; p) \leq 4, \quad t \in [0, 1], \text{ for all } p \in \Omega_P, \quad (3.1f)$$

$$x_2(1; p) \leq 0, \quad \text{for all } p \in \Omega_P, \quad (3.1g)$$

$$T \geq 0, \quad (3.1h)$$

$$u(t) \in [-10, 10], \quad t \in [0, 1]. \quad (3.1i)$$

In the Classical Approach, the set of feasible controllable parameters and control functions are given by those  $T$  and  $u(\cdot)$ , which yield feasible trajectories  $x(\cdot, p)$  for all  $p \in \Omega_P$ . The value of the objective function in the lower level does not depend on  $p$  and  $x(\cdot, p)$ . In other words, in this approach, the driver has no prior knowledge about the value of the parameter  $p$  and gets no feedback during the process and has to set up the driving strategy in advance.



## 3.2 The Training Approach

Contrast to the Classical Approach, in the Training Approach it is assumed that the driver of the rocket car is able to perform optimally for every  $p$  because of a preceding training period. Thus the worst possible optimal performance is given by a solution of the problem

$$\max_{p \in \Omega_P, T, u(\cdot), x(\cdot, p)} \min T \quad (3.2a)$$

$$s.t. \quad x = (x_1, x_2) \quad (3.2b)$$

$$\dot{x} = T \begin{pmatrix} x_2(t; p) \\ u(t) - p \end{pmatrix}, \quad t \in [0, 1], \quad (3.2c)$$

$$x(0, p) = 0, \quad (3.2d)$$

$$x_1(1; p) \geq 10, \quad (3.2e)$$

$$x_2(t; p) \leq 4, \quad t \in [0, 1], \quad (3.2f)$$

$$x_2(1; p) \leq 0, \quad (3.2g)$$

$$T \geq 0, \quad (3.2h)$$

$$u(t) \in [-10, 10], \quad t \in [0, 1]. \quad (3.2i)$$

The solution of the Training Approach in paper [Schlöder \[2022\]](#) is given by a gradient-free method, more precisely, a so-called model-based DFO approach for box-constrained optimization problems is used. The BOBYQA algorithm is chosen for such approach to solve problems of the form

$$\begin{aligned} \min_{x \in \mathcal{R}^n} F(x) \\ s.t. \quad a_i \leq x_i \leq b_i, i = 1, \dots, n \end{aligned} \quad (3.3)$$

The name BOBYQA is an acronym for "Bound Optimization BY Quadratic Approximation", and is used to solve lower level problem of 3.2. In the general DFO method, the objective function  $F(\cdot)$  is considered a black box. For a given  $p$ , the parametric lower level OCP of the Training Approach 3.2 is solved with a direct approach and the resulting (finite dimensional) solution is viewed as dependent variable. Furthermore, the uncertainty set  $\Omega_P$  is box-shaped, and hence the BOBYQA algorithm is applicable to the problem in the Training Approach. The BOBYQA algorithm has been introduced in details in the paper [Powell \[2009\]](#), and we reiterate the main idea in the text that follows.

The method of BOBYQA is iterative,  $k$  and  $n$  being reserved for the iteration number and the number of variables, respectively. Further,  $m$  is reserved for the number of interpolation conditions that are imposed on a quadratic approximation  $Q_k(x), x \in \mathcal{R}$ , to  $F(x), \mathcal{R}$ , with  $m$  is a chosen constant integer from the interval  $[n + 2, \frac{1}{2}(n + 1)(n + 2)]$ . The approximation is available at the beginning of the  $k$ -th iteration, the interpolation equations have the form

$$Q_k(y_j) = F(y_j), \quad j = 1, 2, \dots, m, \quad (3.4)$$

We let  $x_k$  be the point in the set  $\{y_j : j = 1, 2, \dots, m\}$  that has the property

$$F(x_k) = \min \{F(y_j), \quad j = 1, 2, \dots, m\}, \quad (3.5)$$

with any ties being broken by giving priority to an earlier evaluation of the least function value  $F(x_k)$ . A positive number  $\Delta_k$ , called the “trust region radius”, is also available at the beginning of the  $k$ -th iteration. If a termination condition<sup>1</sup> is satisfied, then the iteration stops. Otherwise, a step  $d_k$  from  $x_k$  is constructed such that  $\|d_k\| \leq \Delta_k$  holds, such that  $x = x_k + d_k$  is within the bounds 3.3, and such that  $x_k + d_k$  is not one of the interpolation points  $y_j : j = 1, 2, \dots, m$ . Then the new function value  $F(x_k + d_k)$  is calculated, and one of the interpolation points,  $y_t$  say, is replaced by  $x_k + d_k$ , where  $y_t$  is different from  $x_k$ . It follows that  $x_{k+1}$  is defined by the formula

$$x_{k+1} \begin{cases} x_k, & F(x_k + d_k) \geq F(x_k) \\ x_k + d_k, & F(x_k + d_k) < F(x_k) \end{cases} \quad (3.6)$$

Further,  $\Delta_{k+1}$  and  $Q_{k+1}$  are generated for the next iteration,  $Q_{k+1}$  being subject to the constraints

$$Q_{k+1}(\hat{y}_j) = F(\hat{y}_j), \quad j = 1, 2, \dots, m, \quad (3.7)$$

at the new interpolation points

$$\hat{y}_j = \begin{cases} y_j, & j \neq t, \\ x_k + d_k, & j = t \end{cases}, \quad j = 1, 2, \dots, m. \quad (3.8)$$

The operations of BOBYQA algorithm requires the user to provide an initial vector of variables  $x_0 \in \mathcal{R}^n$ , the initial trust region  $\Delta_1$ , in addition to the bounds of problem 3.3, and the number  $m$  of interpolation conditions where  $n + 2 \leq m \leq \frac{1}{2}(n + 1)(n + 2)$ . Two different ways have been proposed for constructing the step  $d_k$  from  $x_k$  and updating procedures from the  $k$ -th iteration to the  $k + 1$ -th iteration in the paper [Powell \[2009\]](#), with both methods having utilized the "quadratic" nature of the approximation function  $Q(\cdot)$ .

The lower level OCP of the Training Approach 3.2 can be solved with the BOBYQA algorithm for a given  $p$ , since the lower level problem can be re-written into the form of 3.3 and the constraints are box-shaped. With the BOBYQA algorithm computing local extrema, the upper level problem still needs to be solved globally. In our rocket car case, this is straight-forward, i.e. maximizing over all  $p$ .

Nevertheless, the BOBYQA algorithm has limitations with several strong assumptions being made. Firstly, it has been assumed the uncertainty set is of moderate size and is box-shaped. Secondly, it has been assumed that there is only one local extrema, i.e. the lower level problem has only one solution for each  $p \in \Omega_P$ . In general, we cannot expect the second assumption to be valid. Though the BOBYQA algorithm is a gradient-free method with respect to the objective function  $F(\cdot)$ , it still utilizes the gradient of the approximation function  $Q(\cdot)$  while updating the iterations. Therefore, this BOBYQA algorithm, or a general DFO approach, is still subject to the numerical errors and computational costs while calculating the gradients of the approximation function  $Q(\cdot)$ .

The paper at hand, instead, is utilizing the gradient of the objective function  $F(\cdot)$  directly, and we have used the multiple shooting and quasi-Newton method for solving

---

<sup>1</sup>Typically, a termination condition is satisfied when the objective value can not be improved further after some iterations, for the case of BOBYQA algorithm, please refer the the paper [Powell \[2009\]](#) for more details.

the lower level problem of the Training Approach 3.2. In the chapter that follows, we give the introduction to the multiple shooting and quasi-Newton method.

## 4 The Introduction to Multiple Shooting and Quasi-Newton Method

In this chapter, we first introduce the basics of solutions to nonlinear problems and the focus on the quasi-Newton and multiple shooting method.

As stated in the introduction chapter 1, a general optimization problem is typically of the form

$$\begin{aligned} \min f(x) \\ \text{s.t. } x \in \Omega \end{aligned} \tag{4.1}$$

Here  $x \in \Omega$  represents the constraints for which  $x$  must satisfy, it may be in the form of  $g(x) = 0, h(x) \geq 0$  as in the problem 1.1 in the Introduction chapter, i.e. the feasible set  $\Omega = \arg \{g(x) = 0, h(x) \geq 0\}$ .

Such problem can be solved via Newton's method, which attempts to solve this problem by constructing a sequence  $\{x_K\}$  from an initial guess (starting point)  $x_0 \in \Omega$  that converges towards a minimizer  $x^*$  of  $f(x)$  by using a sequence of first and/or second order Taylor approximations of  $f(x)$  around the iterates. The second-order Taylor expansion of  $f(x)$  around  $x_k$  is

$$f(x_k + \delta_x) \approx f(x_k) + f'(x_k)\delta_x + \frac{1}{2}f''(x_k)\delta_x^2$$

where  $\delta_x$  represents a small change (with respect to  $x$ ), and  $f', f''$  are the first and second order derivatives of the original function  $f(x)$ .

# Part I

## Appendix

# A Lists

## Bibliography

Michael J. D. Powell. The bobyqa algorithm for bound constrained optimization without derivatives. 2009.

Matthias Schlöder. Numerical methods for optimal control of constrained biomechanical multi-body systems appearing in therapy design of cerebral palsy. 2022.

Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den (Datum) .....

Declaration:

I hereby confirm that I wrote this work independently and did not use any sources other than those indicated.