B.2 Proofs for Section 6.3

B.2.1 Proof of Remark 6.1

For the proof, we need the following

Lemma B.1
Let $\Omega_{\mathbf{x}} \subset \mathbb{R}^{n_x}$ and $\Omega_{\mathbf{p}} \subset \mathbb{R}^{n_p}$ be compact subsets, and $h : \Omega_{\mathbf{x}} \times \Omega_{\mathbf{p}} \to \mathbb{R}$ a continuous function. Then

$$g : \Omega_{\mathbf{x}} \to \mathbb{R}, \ \mathbf{x} \mapsto \max_{\mathbf{p} \in \Omega_{\mathbf{p}}} h(\mathbf{x}, \mathbf{p})$$

is continuous, and the same holds for $g' : \Omega_{\mathbf{x}} \to \mathbb{R}$, $\mathbf{x} \mapsto \min_{\mathbf{p} \in \Omega_{\mathbf{p}}} h(\mathbf{x}, \mathbf{p})$.

Proof As $h(\cdot)$ is continuous on a compact set, it is uniformly continuous (Heine-Cantor theorem). Let $\mathbf{x} \in \Omega_{\mathbf{x}}$ and $(\mathbf{x}^n)_{n \in \mathbb{N}}$ be a sequence in $\Omega_{\mathbf{x}}$ converging to $\mathbf{x}$. As $h(\cdot)$ is uniformly continuous we have

$$\forall \varepsilon > 0 \, \exists N(\varepsilon) \in \mathbb{N} \text{ s.t. } \left| h(\mathbf{x}^n, \mathbf{p}) - h(\mathbf{x}, \mathbf{p}) \right| < \varepsilon \ \forall n \geq N(\varepsilon) \ \forall \mathbf{p} \in \Omega_{\mathbf{p}} .$$

In particular, $N(\varepsilon)$ does not depend on $\mathbf{p}$. Let $\varepsilon > 0$. We have

$$h(\mathbf{x}, \mathbf{p}) - \varepsilon < h(\mathbf{x}^n, \mathbf{p}) < h(\mathbf{x}, \mathbf{p}) + \varepsilon \ \ \forall n \geq N(\varepsilon)$$

for all $\mathbf{p} \in \Omega_{\mathbf{p}}$, and therefore also

$$\max_{\mathbf{p} \in \Omega_{\mathbf{p}}} h(\mathbf{x}, \mathbf{p}) - \varepsilon < \max_{\mathbf{p} \in \Omega_{\mathbf{p}}} h(\mathbf{x}^n, \mathbf{p}) < \max_{\mathbf{p} \in \Omega_{\mathbf{p}}} h(\mathbf{x}, \mathbf{p}) + \varepsilon \ \ \forall n \geq N(\varepsilon)$$

(note that a continuous function takes its maximum on a compact set according to a generalization of the extreme value theorem). Altogether, we get

$$\left| g(\mathbf{x}^n) - g(\mathbf{x}) \right| = \left| \max_{\mathbf{p} \in \Omega_{\mathbf{p}}} h(\mathbf{x}^n, \mathbf{p}) - \max_{\mathbf{p} \in \Omega_{\mathbf{p}}} h(\mathbf{x}, \mathbf{p}) \right| < \varepsilon \ \ \forall n \geq N(\varepsilon) \, ,$$

which shows the continuity of $g(\cdot)$. The continuity of $g'(\cdot)$ can be proven in a similar manner. □

Now, we are able to prove Remark 6.1. We have

$$f(\mathbf{x}, \mathbf{p}) \leq \max_{\mathbf{p} \in \Omega_{\mathbf{p}}} f(\mathbf{x}, \mathbf{p})$$

for all $\mathbf{x} \in \Omega_\mathbf{x}$ and $\mathbf{p} \in \Omega_\mathbf{p}$, where $\max_{\mathbf{p}\in\Omega_\mathbf{p}} f(\mathbf{x},\mathbf{p})$ depends on $\mathbf{x}$ only. Consequently,

$$\min_{\mathbf{x}\in\Omega_\mathbf{x}} f(\mathbf{x},\mathbf{p}) \le \min_{\mathbf{x}\in\Omega_\mathbf{x}} \max_{\mathbf{p}\in\Omega_\mathbf{p}} f(\mathbf{x},\mathbf{p})$$

for all $\mathbf{p} \in \Omega_\mathbf{p}$, and thus

$$\max_{\mathbf{p}\in\Omega_\mathbf{p}} \min_{\mathbf{x}\in\Omega_\mathbf{x}} f(\mathbf{x},\mathbf{p}) \le \min_{\mathbf{x}\in\Omega_\mathbf{x}} \max_{\mathbf{p}\in\Omega_\mathbf{p}} f(\mathbf{x},\mathbf{p})\,.$$

The existence of all above maxima and minima follows from Lemma B.1 and a generalization of the extreme value theorem.

B.2.2 max min vs min max NLP – Objective Function Value

For $\Omega_x = [-5,5]$, $\Omega_p = [-1,1]$ we consider the function

$$f : \Omega_x \times \Omega_p \to \mathbb{R},\ \ (x,p) \mapsto (x-p)^2 + p\,.$$

Since $\Omega_p \subset \Omega_x$ we have $\min_{x\in\Omega_x} f(x,p) = f(p,p) = p$ and hence

$$\max_{p\in\Omega_p} \min_{x\in\Omega_x} f(x,p) = 1\,.$$

For the min max problem we consider

$$f(x,p) = (x-p)^2 + p = p^2 - p(2x-1) + x^2\,.$$

For a fixed $\tilde{x}$, the function $f(\tilde{x},\cdot)$ is a convex parabola with vertex $\bar{p}(\tilde{x}) = \frac{2\tilde{x}-1}{2}$. Since $\Omega_p = [-1,1]$, by the symmetry of the parabola we get

$$g(x) \overset{\text{def}}{=} \max_{p\in\Omega_p} f(x,p) = \begin{cases} f(x,1) & \text{if } \bar{p}(x) \le 0 \\ f(x,-1) & \text{if } \bar{p}(x) > 0 \end{cases} = \begin{cases} (x-1)^2+1 & \text{if } x \le 0.5 \\ (x+1)^2-1 & \text{if } x > 0.5 \end{cases}.$$

For the derivative of $g(\cdot)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}x} g(x) = \begin{cases} 2(x-1) & \text{if } x < 0.5 \\ \text{undefined} & \text{if } x = 0.5 \\ 2(x+1) & \text{if } x > 0.5 \end{cases}.$$

Hence, the continuous function $g(\cdot)$ is strictly monotonically decreasing for $x < 0.5$ and strictly monotonically increasing for $x > 0.5$, thus having a minimum at $x = 0.5$.

Altogether, we get

$$\max_{p\in\Omega_p}\min_{x\in\Omega_x} f(x,p) = 1 < 1.25 = f(0.5,1) = \min_{x\in\Omega_x}\max_{p\in\Omega_p} f(x,p)\,.$$

B.2.3 Proof of Proposition 6.2

The proof works similar to the one of Remark 6.1. We have

$$\Phi(\mathbf{x}(1;\mathbf{p})) \le \max_{\substack{\mathbf{p}\in\Omega_{\mathbf{p}},\\ \mathbf{x}(\cdot;\mathbf{p})}} \Phi(\mathbf{x}(1;\mathbf{p}))$$

for all $\mathbf{p}\in\Omega_{\mathbf{p}}$ and $(\mathbf{u},\mathbf{u}(\cdot))\in\tilde{\mathcal{C}}(\Omega_{\mathbf{p}})$. Thus,

$$\min_{\substack{(\mathbf{u},\mathbf{u}(\cdot))\in\tilde{\mathcal{C}}(\Omega_{\mathbf{p}}),\\ \mathbf{x}(\cdot;\mathbf{p})}} \Phi(\mathbf{x}(1;\mathbf{p})) \le \min_{(\mathbf{u},\mathbf{u}(\cdot))\in\tilde{\mathcal{C}}(\Omega_{\mathbf{p}})} \max_{\substack{\mathbf{p}\in\Omega_{\mathbf{p}},\\ \mathbf{x}(\cdot;\mathbf{p})}} \Phi(\mathbf{x}(1;\mathbf{p}))$$

for all $\mathbf{p}\in\Omega_{\mathbf{p}}$. As $\tilde{\mathcal{C}}(\Omega_{\mathbf{p}})\subseteq\mathcal{C}(\mathbf{p})$, we get

$$\min_{\substack{(\mathbf{u},\mathbf{u}(\cdot))\in\mathcal{C}(\mathbf{p}),\\ \mathbf{x}(\cdot;\mathbf{p})}} \Phi(\mathbf{x}(1;\mathbf{p})) \le \min_{(\mathbf{u},\mathbf{u}(\cdot))\in\tilde{\mathcal{C}}(\Omega_{\mathbf{p}})} \max_{\substack{\mathbf{p}\in\Omega_{\mathbf{p}},\\ \mathbf{x}(\cdot;\mathbf{p})}} \Phi(\mathbf{x}(1;\mathbf{p}))$$

for all $\mathbf{p}\in\Omega_{\mathbf{p}}$, and consequently

$$\max_{\mathbf{p}\in\Omega_{\mathbf{p}}} \min_{\substack{(\mathbf{u},\mathbf{u}(\cdot))\in\mathcal{C}(\mathbf{p}),\\ \mathbf{x}(\cdot;\mathbf{p})}} \Phi(\mathbf{x}(1;\mathbf{p})) \le \min_{(\mathbf{u},\mathbf{u}(\cdot))\in\tilde{\mathcal{C}}(\Omega_{\mathbf{p}})} \max_{\substack{\mathbf{p}\in\Omega_{\mathbf{p}},\\ \mathbf{x}(\cdot;\mathbf{p})}} \Phi(\mathbf{x}(1;\mathbf{p}))\,.$$

All maxima and minima above exist per assumption.

B.2.4 Solution of Problem (6.9)

Let $p\in[0,9]$. We consider Problem (6.9) with variables

$$\left(T,u(\cdot),\mathbf{x}(\cdot;p)\right)\in\mathbb{R}\times L^{\infty}([0,1],\mathbb{R})\times W^{1,\infty}\left([0,1],\mathbb{R}^2\right),$$

see Section 2.1 for the spaces and corresponding norms, where $\mathbf{x}(\cdot;p)$ denotes the (unique) solution of Initial Value Problem (IVP) (6.9b-6.9c) for given T, $u(\cdot)$, and p. The product space $\mathbb{R}\times L^{\infty}([0,1],\mathbb{R})\times W^{1,\infty}\left([0,1],\mathbb{R}^2\right)$, equipped with the norm

$$\left\|\left(T,u(\cdot),\mathbf{x}(\cdot;p)\right)\right\| = \max\left[|T|,\|u(\cdot)\|_{\infty},\|\mathbf{x}(\cdot;p)\|_{1,\infty}\right],$$

is a Banach space. In this section, we state the unique globally optimal solution of Problem (6.9), prove its optimality, and show that no different local optima exist. The main results of this section can be found in Corollary B.4 and Proposition B.9.

The Globally Optimal Solution of Problem (6.9)

We first consider the global optimum and start with

Lemma B.2
Let $\left(T, u(\cdot), \mathbf{x}(\cdot\,; p)\right)$ be feasible for Problem (6.9). Then $T > \frac{4}{10+p} + \frac{4}{10-p}$.

Proof Due to the Boundary Conditions (6.9c) and (6.9e) we have $T > 0$. Furthermore,

$$\mathbf{x}_2(t; p) = \mathbf{x}_2(t_0; p) + \int_{t_0}^{t} T(u(\tau) - p)\,\mathrm{d}\tau$$

for all $0 \le t_0 \le t \le 1$. Since $\mathbf{x}_2(0; p) = 0$ and $u(t) \le 10$ we get

$$\mathbf{x}_2(t; p) \le (10 - p)\,T\,t\,.$$

On the other hand, because of $\mathbf{x}_2(1; p) \le 0$ and $u(t) \ge -10$ we have

$$\mathbf{x}_2(t; p) \le (10 + p)\,T - (10 + p)\,T\,t$$

for all $t \in [0, 1]$. Otherwise, there exists a t_0 with $\mathbf{x}_2(t_0; p) > (10 + p)\,T - (10 + p)\,T\,t_0$ and we get

$$\begin{aligned}\mathbf{x}_2(1; p) &> (10 + p)\,T - (10 + p)\,T\,t_0 + \int_{t_0}^{1} T(u(\tau) - p)\,\mathrm{d}\tau \\ &\ge (10 + p)\,T - (10 + p)\,T\,t_0 - (10 + p)\,T(1 - t_0) = 0\end{aligned}$$

which contradicts the Terminal Condition (6.9f). We conclude

$$\begin{aligned}\mathbf{x}_2(t; p) &\le \min\left[(10 - p)\,T\,t,\ (10 + p)\,T - (10 + p)\,T\,t\right] \\ &= \begin{cases} (10 - p)\,T\,t & \text{if } 0 \le t \le \frac{10+p}{20}, \\ (10 + p)\,T - (10 + p)\,T\,t & \text{else,} \end{cases}\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_1(1;p) &= T\int_0^1 \mathbf{x}_2(t;p)\,\mathrm{d}t \le T\int_0^{\frac{10+p}{20}} (10-p)T\,t\,\mathrm{d}t + T\int_{\frac{10+p}{20}}^1 (10+p)T-(10+p)T\,t\,\mathrm{d}t\\ &= \frac{1}{2}(10-p)T^2\left(\frac{10+p}{20}\right)^2 + \frac{1}{2}(10-p)T^2\left(\frac{10+p}{20}\right)\left(\frac{10-p}{20}\right).\end{aligned}$$

Now, if we had $T \le \frac{4}{10+p} + \frac{4}{10-p} = \frac{80}{(10-p)(10+p)}$, because of $0 \le p \le 9$ we would obtain

$$\mathbf{x}_1(1;p) \le \frac{8}{10-p} + \frac{8}{10+p} \le 8 + \frac{8}{10} < 10$$

which contradicts the Terminal Condition (6.9e). Hence, $T > \frac{4}{10+p} + \frac{4}{10-p}$. ☐

To state the global optimum and to prove its optimality, we consider a certain class of control functions and the corresponding differential states. Let $T > \frac{4}{10+p} + \frac{4}{10-p}$. We set

$$t_1 = t_1(T,p) = \frac{4}{T(10-p)} \quad\text{and}\quad t_2 = t_2(T,p) = 1 - \frac{4}{T(10+p)}.$$

Then $0 < t_1 < t_2 < 1$ and we define

$$u_T(t;p) \stackrel{\text{def}}{=} \begin{cases} 10 & \text{for } 0 \le t < t_1(T,p), \\ p & \text{for } t_1(T,p) \le t < t_2(T,p), \\ -10 & \text{for } t_2(T,p) \le t \le 1. \end{cases} \tag{B.5}$$

Let $\mathbf{x}_T(\cdot\,;p)$ denote the differential states which are determined by p, T, and $u_T(\cdot\,;p)$. We have

$$\mathbf{x}_{T,2}(t;p) = \begin{cases} T(10-p)\,t & \text{for } 0 \le t < t_1(T,p), \\ 4 & \text{for } t_1(T,p) \le t < t_2(T,p), \\ 4 - T(10+p)\big(t - t_2(T,p)\big) & \text{for } t_2(T,p) \le t \le 1. \end{cases}$$

In particular, $\mathbf{x}_{T,2}(t;p) \le 4$ for all $t \in [0,1]$ and $\mathbf{x}_{T,2}(1;p) = 0$. Furthermore, we get

$$\begin{aligned}\mathbf{x}_{T,1}(t_1;p) - \mathbf{x}_{T,1}(0;p) &= \frac{1}{2}(10-p)T^2 t_1^2 \;= \frac{8}{10-p}\\ \mathbf{x}_{T,1}(t_2;p) - \mathbf{x}_{T,1}(t_1;p) &= 4T(t_2-t_1) = 4T - \frac{16}{10-p} - \frac{16}{10+p},\\ \mathbf{x}_{T,1}(1;p) - \mathbf{x}_{T,1}(t_2;p) &= \frac{1}{2}4T\,(1-t_2) = \frac{8}{10+p}\end{aligned}$$

(e. g., by geometrical considerations) and therefore

$$\mathbf{x}_{T,1}(1;p) = \mathbf{x}_{T,1}(1;p) - \mathbf{x}_{T,1}(0;p) = 4T - \frac{8}{10-p} - \frac{8}{10+p} \tag{B.6}$$

since $\mathbf{x}_{T,1}(0;p) = 0$. In particular,

$$\mathbf{x}_{T,1}(1;p) = 10 \iff T = T^*(p) \stackrel{\text{def}}{=} 2.5 + \frac{40}{100-p^2}, \tag{B.7}$$

$\mathbf{x}_{T,1}(1;p) < 10$ for $\frac{4}{10+p} + \frac{4}{10-p} < T < T^*(p)$ and $\mathbf{x}_{T,1}(1;p) > 10$ for $T > T^*(p)$. Hence, the tuple $\big(T, u_T(\cdot;p), \mathbf{x}_T(\cdot;p)\big)$ is feasible if and only if $T \geq T^*(p)$. We define

$$T^* \stackrel{\text{def}}{=} T^*(p), \quad u^*(\cdot) = u^*(\cdot;p) \stackrel{\text{def}}{=} u_{T^*}(\cdot;p), \quad \text{and} \quad \mathbf{x}^*(\cdot;p) \stackrel{\text{def}}{=} \mathbf{x}_{T^*}(\cdot;p).$$

Proposition B.3
Let $\big(T, u(\cdot), \mathbf{x}(\cdot;p)\big)$ be feasible for Problem (6.9) with $u(\cdot) \neq u_T(\cdot;p)$ (in $L^\infty([0,1],\mathbb{R})$), where $u_T(\cdot;p)$ is given by (B.5). Then $\big(T, u_T(\cdot;p), \mathbf{x}_T(\cdot;p)\big)$ is feasible as well and we have

$$10 \leq \mathbf{x}_1(1;p) < \mathbf{x}_{T,1}(1;p).$$

Proof From Lemma B.2 we get $T > \frac{4}{10+p} + \frac{4}{10-p}$, and $u_T(\cdot;p)$ is well-defined. First, we show that $\mathbf{x}_2(t;p) \leq \mathbf{x}_{T,2}(t;p)$ for all $t \in [0,1]$. Let $t_1 = \frac{4}{T(10-p)}$ and $t_2 = 1 - \frac{4}{T(10+p)}$. For $t \in [0, t_1]$, we have

$$\mathbf{x}_2(t;p) = T\int_0^t u(t) - p\,\mathrm{d}\tau \leq T\int_0^t 10 - p\,\mathrm{d}\tau = \mathbf{x}_{T,2}(t;p),$$

and due to feasibility $\mathbf{x}_2(t;p) \leq 4 = \mathbf{x}_{T,2}(t;p)$ for $t \in [t_1, t_2]$. If there was a $t' \in [t_2, 1]$ with $\mathbf{x}_2(t';p) > \mathbf{x}_{T,2}(t';p)$, we would get

$$\mathbf{x}_2(1;p) = \mathbf{x}_2(t';p) + T\int_{t'}^1 u(t) - p\,\mathrm{d}t > \mathbf{x}_{T,2}(t';p) + T\int_{t'}^1 -10 - p\,\mathrm{d}t = \mathbf{x}_{T,2}(1;p) = 0$$

which contradicts the feasibility of $\big(T, u(\cdot), \mathbf{x}(\cdot;p)\big)$. Thus, we have $\mathbf{x}_2(t;p) \leq \mathbf{x}_{T,2}(t;p)$ for all $t \in [0,1]$.

Next, we show that there is a $t' \in [0,1]$ with $\mathbf{x}_2(t';p) < \mathbf{x}_{T,2}(t';p)$. By assumption, we have $u(\cdot) \neq u_T(\cdot;p)$. We distinct three cases:

Case 1): $u(\cdot) \not\equiv u_T(\cdot\,;p)$ in $[0,t_1]$ (almost surely). Then there is an $\varepsilon > 0$ and a subset $\mathcal{A} \subseteq [0,t_1]$ with non-zero measure such that $u(t) < 10-\varepsilon = u_T(t;p)-\varepsilon$ for $t \in \mathcal{A}$ (almost surely). Thus, we get $\mathbf{x}_2(t_1;p) < \mathbf{x}_{T,2}(t_1;p)$.

Case 2): $u(\cdot) \equiv u_T(\cdot\,;p)$ in $[0,t_1]$ and $u(\cdot) \not\equiv u_T(\cdot\,;p)$ in $[t_1,t_2]$ (almost surely, respectively). Then $\mathbf{x}_2(t_1;p) = 4$. We claim that there is an $\varepsilon > 0$ and a subset $\mathcal{A} \subseteq [t_1,t_2]$ with non-zero measure such that $u(t) < p-\varepsilon = u_T(t;p)-\varepsilon$ for $t \in \mathcal{A}$. Indeed, if such a subset does not exist, we have $u(t) \geq u_T(t;p)$ in $[t_1,t_2]$ (almost surely), and since the control functions differ on a set with non-zero measure we get $\mathbf{x}_2(t;p) > 4$ for some $t \in [t_1,t_2]$. This contradicts the feasibility of $\big(T,u(\cdot),\mathbf{x}(\cdot\,;p)\big)$. Hence such a subset exists and consequently $\mathbf{x}_2(t;p) < \mathbf{x}_{T,2}(t;p)$ for some t' in $[t_1,t_2]$.

Case 3): $u(\cdot) \equiv u_T(\cdot\,;p)$ in $[0,t_2]$ and $u(\cdot) \not\equiv u_T(\cdot\,;p)$ in $[t_2,1]$ (almost surely, respectively). Then we have $\mathbf{x}_2(t_2;p) = \mathbf{x}_{T,2}(t_2;p)$ and there is an $\varepsilon > 0$ and a subset $\mathcal{A} \subseteq [t_2,1]$ with non-zero measure such that $u(t) > u_T(t;p)+\varepsilon = -10+\varepsilon$ for $t \in \mathcal{A}$ (almost surely). Consequently, $\mathbf{x}_2(1;p) > \mathbf{x}_{T,2}(1;p) = 0$ which contradicts the feasibility of $\big(T,u(\cdot),\mathbf{x}(\cdot\,;p)\big)$. Thus, Case 3) does not occur.

Altogether, we have seen that $\mathbf{x}_2(t;p) \leq \mathbf{x}_{T,2}(t;p)$ for all $t \in [0,1]$ and there is a t' with $\mathbf{x}_2(t';p) < \mathbf{x}_{T,2}(t';p)$. By the continuity of $\mathbf{x}_2(\cdot\,;p)$ and $\mathbf{x}_{T,2}(\cdot\,;p)$ we conclude

$$\mathbf{x}_{T,1}(1;p) > \mathbf{x}_1(1;p) \geq 10\,,$$

which shows the feasibility of $\big(T,u_T(\cdot\,;p),\mathbf{x}_T(\cdot\,;p)\big)$ and completes the proof. □

Corollary B.4
The tuple $\big(T^*,u^*(\cdot),\mathbf{x}^*(\cdot\,;p)\big)$ is the unique global optimum of Problem (6.9).

Proof By construction, $\big(T^*,u^*(\cdot),\mathbf{x}^*(\cdot\,;p)\big)$ is feasible with $\mathbf{x}_1^*(1;p) = 10$ and we have $\mathbf{x}_{T,1}(1;p) < 10$ for all $\frac{4}{10+p}+\frac{4}{10-p} < T < T^*$, see (B.6) and (B.7). Let $\big(T,u(\cdot),\mathbf{x}(\cdot\,;p)\big)$ be any feasible tuple for Problem (6.9). If $T < T^*$, from Proposition B.3 we get

$$10 \leq \mathbf{x}_1(1;p) \leq \mathbf{x}_{T,1}(1;p) < 10\,,$$

which is a contradiction. Hence, $T \geq T^*$ and $\big(T^*,u^*(\cdot),\mathbf{x}^*(\cdot\,;p)\big)$ is globally optimal. Furthermore, let $\big(T,u(\cdot),\mathbf{x}(\cdot\,;p)\big) \neq \big(T^*,u^*(\cdot),\mathbf{x}^*(\cdot\,;p)\big)$ be feasible for Problem (6.9) with $T = T^*$ and $u(\cdot) \neq u^*(\cdot\,;p) = u_{T^*}(\cdot\,;p)$. Again we apply Proposition B.3 and find the contradiction

$$10 \le \mathbf{x}_1(1;p) < \mathbf{x}_{T,1}(1;p) = \mathbf{x}_{T^*,1}(1;p) = 10\,,$$

which shows the uniqueness of the global optimum $\left(T^*, u^*(\cdot), \mathbf{x}^*(\cdot\,;p)\right)$. $\square$

The Unique Solvability of Problem (6.9)

We show that $\left(T^*, u^*(\cdot), \mathbf{x}^*(\cdot\,;p)\right)$ is the only local optimum of Problem (6.9) in the considered normed space. For a proof we need several auxiliary results.

Lemma B.5
Let $T > 0$, $u(\cdot) \in L^\infty([0,1],\mathbb{R})$, and $\mathbf{x}(\cdot\,;p) \in W^{1,\infty}\left([0,1],\mathbb{R}^2\right)$ be the differential states which are determined by T, $u(\cdot)$, and p. Let $\varepsilon > 0$. Then there exist $\delta_T, \delta_u > 0$ such that

$$\|\mathbf{x}'(\cdot\,;p) - \mathbf{x}(\cdot\,;p)\|_{1,\infty} < \varepsilon$$

for all $T', u'(\cdot)$ with $|T' - T| < \delta_T$ and $\|u'(\cdot) - u(\cdot)\|_\infty < \delta_u$, where $\mathbf{x}'(\cdot\,;p)$ denotes the differential states which are determined by T', $u'(\cdot)$, and p.

Proof Let $\bar{\delta}_T, \bar{\delta}_u > 0$, $|T' - T| < \bar{\delta}_T$, and $\|u'(\cdot) - u(\cdot)\|_\infty < \bar{\delta}_u$. Then

$$\begin{aligned}
\left\|\dot{\mathbf{x}}_2'(\cdot\,;p) - \dot{\mathbf{x}}_2(\cdot\,;p)\right\|_\infty &= \left\|T'\left(u'(\cdot) - p\right) - T\left(u(\cdot) - p\right)\right\|_\infty \\
&= \left\|(T' - T + T)\left(u'(\cdot) - p\right) - T\left(u(\cdot) - p\right)\right\|_\infty \\
&\le |T' - T|\left\|u'(\cdot) - p\right\|_\infty + |T|\left\|u'(\cdot) - u(\cdot)\right\|_\infty \\
&= |T' - T|\left\|u'(\cdot) - u(\cdot) + u(\cdot) - p\right\|_\infty + |T|\left\|u'(\cdot) - u(\cdot)\right\|_\infty \\
&< \left(\bar{\delta}_T + |T|\right)\bar{\delta}_u + \bar{\delta}_T\left\|u(t) - p\right\|_\infty \\
&\le \left(\bar{\delta}_T + |T|\right)\bar{\delta}_u + \bar{\delta}_T\left(\|u(\cdot)\|_\infty + 9\right), \\
\left\|\mathbf{x}_2'(\cdot\,;p) - \mathbf{x}_2(\cdot\,;p)\right\|_\infty &= \sup_{t\in[0,1]}\left|T'\int_0^t \dot{\mathbf{x}}_2'(\tau;p)\,\mathrm{d}\tau - T\int_0^t \dot{\mathbf{x}}_2(\tau;p)\,\mathrm{d}\tau\right| \\
&\le \sup_{t\in[0,1]}\int_0^t \left\|T'\dot{\mathbf{x}}_2'(\cdot\,;p) - T\dot{\mathbf{x}}_2(\cdot\,;p)\right\|_\infty \mathrm{d}\tau \\
&= \left\|T'\dot{\mathbf{x}}_2'(\cdot\,;p) - T\dot{\mathbf{x}}_2(\cdot\,;p)\right\|_\infty \\
&< \left(\bar{\delta}_T + |T|\right)\left\|\dot{\mathbf{x}}_2'(\cdot\,;p) - \dot{\mathbf{x}}_2(\cdot\,;p)\right\|_\infty + \bar{\delta}_T\left\|\dot{\mathbf{x}}_2(\cdot\,;p)\right\|_\infty.
\end{aligned}$$

Furthermore, similar to what we have seen before we get

$$
\begin{aligned}
\left\|\dot{\mathbf{x}}_1'(\cdot\,;p)-\dot{\mathbf{x}}_1(\cdot\,;p)\right\|_\infty &= \left\|T'\mathbf{x}_2'(\cdot\,;p)-T\mathbf{x}_2(\cdot\,;p)\right\|_\infty \\
&< \left(\bar{\delta}_T+|T|\right)\left\|\mathbf{x}_2'(\cdot\,;p)-\mathbf{x}_2(\cdot\,;p)\right\|_\infty+\bar{\delta}_T\left\|\mathbf{x}_2(\cdot\,;p)\right\|_\infty, \\
\left\|\mathbf{x}_1'(\cdot\,;p)-\mathbf{x}_1(\cdot\,;p)\right\|_\infty &= \sup_{t\in[0,1]}\left|\int_0^t T'\dot{\mathbf{x}}_1'(\tau;p)\,\mathrm{d}\tau-\int_0^t T\dot{\mathbf{x}}_1(\tau;p)\,\mathrm{d}\tau\right| \\
&< \left(\bar{\delta}_T+|T|\right)\left\|\dot{\mathbf{x}}_1'(\cdot\,;p)-\dot{\mathbf{x}}_1(\cdot\,;p)\right\|_\infty+\bar{\delta}_T\left\|\dot{\mathbf{x}}_1(\cdot\,;p)\right\|_\infty.
\end{aligned}
$$

We see that $\left\|\dot{\mathbf{x}}_2'(\cdot\,;p)-\dot{\mathbf{x}}_2(\cdot\,;p)\right\|_\infty \to 0$ for $\bar{\delta}_T,\bar{\delta}_u\to 0$. Consequently, $\bar{\delta}_T,\bar{\delta}_u\to 0$ also successively implies

$$
\left\|\mathbf{x}_2'(\cdot\,;p)-\mathbf{x}_2(\cdot\,;p)\right\|_\infty \to 0,\quad \left\|\dot{\mathbf{x}}_1'(\cdot\,;p)-\dot{\mathbf{x}}_1(\cdot\,;p)\right\|_\infty \to 0,\quad \text{and } \left\|\mathbf{x}_1'(\cdot\,;p)-\mathbf{x}_1(\cdot\,;p)\right\|_\infty \to 0.
$$

Altogether, we get

$$
\left\|\mathbf{x}'(\cdot\,;p)-\mathbf{x}(\cdot\,;p)\right\|_{1,\infty}\to 0 \quad\text{for}\quad \bar{\delta}_T,\bar{\delta}_u\to 0,
$$

and the statement of the lemma follows. □

Due to Lemma B.5, it is sufficient to show that for each feasible $\left(T,u(\cdot),\mathbf{x}(\cdot\,;p)\right)\neq\left(T^*,u^*(\cdot),\mathbf{x}^*(\cdot\,;p)\right)$ and each $\delta_T,\delta_u>0$ there is a feasible tuple $\left(T',u'(\cdot),\mathbf{x}'(\cdot\,;p)\right)$ with $|T'-T|<\delta_T$, $\|u'(\cdot)-u(\cdot)\|_\infty<\delta_u$, and $T'<T$. In the following, we prove this claim. We start by investigating three different cases.

Lemma B.6
Let $\delta_u>0$ and $\left(T,u(\cdot),\mathbf{x}(\cdot\,;p)\right)$ be feasible for Problem (6.9) with

$$
\mathbf{x}_2(1;p)<0.
$$

Then there is a feasible $\left(T',u'(\cdot),\mathbf{x}'(\cdot\,;p)\right)$ with $T'=T$, $\|u'(\cdot)-u(\cdot)\|_\infty<\delta_u$, and $\mathbf{x}_1'(1;p)>10$.

Proof Due to $\mathbf{x}_2(1;p)<0$, by the continuity of $\mathbf{x}_2(\cdot;p)$ there is a $\varepsilon>0$ such that $\mathbf{x}_2(t;p)<0$ for all $t\in[1-\varepsilon,1]$ and $\mathbf{x}_2(1;p)<\mathbf{x}_2(1-\varepsilon;p)<0$. If we had $u(t)\geq p$ for $t\in[1-\varepsilon,1]$ (almost surely), then $\mathbf{x}_2(1;p)\geq\mathbf{x}_2(1-\varepsilon;p)$ which is a contradiction. Hence, there is a $\delta>0$ with $\delta<\min\left(\delta_u,4,|\mathbf{x}_2(1;p)|\right)$ and a subset $\mathcal{A}\subseteq[1-\varepsilon,1]$ with non-zero (Lebesgue-)measure $\lambda(\mathcal{A})$ such that $u(t)<p-\delta<p$ for $t\in\mathcal{A}$ (almost surely).

Let $\chi_{\mathcal{A}}(\cdot)$ be the characteristic function on the set $\mathcal{A}$. As $T^* = 2.5 + \frac{40}{100-p^2}$ is the global optimum of Problem (6.9) according to Corollary B.4, we have $T \geq T^* > 1$. We set

$$u'(t) = u(t) + \frac{\delta}{T}\chi_{\mathcal{A}}(t).$$

Then by $T > 1$ and the choice of δ we get $u'(t) \in [-10, 10]$ and $\|u'(\cdot) - u(\cdot)\|_\infty = \delta < \delta_u$. Let $\mathbf{x}'(\cdot\,; p)$ denote the differential states which are determined by T, $u'(\cdot)$, and p. We have $\mathbf{x}'_2(t;p) = \mathbf{x}_2(t;p)$ for all $t \in [0, 1-\varepsilon]$, and for $t \in [1-\varepsilon, 1]$ we get

$$\begin{aligned}
\mathbf{x}'_2(t;p) &= \mathbf{x}'_2(1-\varepsilon;p) + T\int_{1-\varepsilon}^{t} u'(\tau) - p \,\mathrm{d}\tau \\
&= \mathbf{x}_2(1-\varepsilon;p) + T\int_{1-\varepsilon}^{t} u(\tau) - p + \frac{\delta}{T}\chi_{\mathcal{A}}(\tau)\,\mathrm{d}\tau \\
&= \mathbf{x}_2(t;p) + \delta\lambda\left(\mathcal{A} \cap [1-\varepsilon, t]\right).
\end{aligned}$$

We conclude $\mathbf{x}'_2(t;p) \geq \mathbf{x}_2(t;p)$ for all $t \in [0,1]$ and $\mathbf{x}'_2(t;p) \leq \mathbf{x}_2(t;p) + \delta < \delta$ for all $t \in [1-\varepsilon, 1]$. Thus, by the choice of δ we get $\mathbf{x}'_2(t;p) \leq 4$ for all $t \in [0,1]$, and furthermore

$$\mathbf{x}_2(1;p) < \mathbf{x}'_2(1;p) = \mathbf{x}_2(1;p) + \delta\lambda(\mathcal{A}) \leq \mathbf{x}_2(1;p) + \delta < 0.$$

As $\mathbf{x}_2(\cdot\,;p)$ and $\mathbf{x}'_2(\cdot\,;p)$ are continuous functions, we have $10 \leq \mathbf{x}_1(1;p) < \mathbf{x}'_1(1;p)$. In particular, $\left(T, u'(\cdot), \mathbf{x}'(\cdot\,;p)\right)$ is feasible and has the desired properties. □

Lemma B.7
Let $\delta_T > 0$ and $\left(T, u(\cdot), \mathbf{x}(\cdot\,;p)\right)$ be feasible for Problem (6.9) with

$$\mathbf{x}_1(1;p) > 10.$$

Then there is a feasible $\left(T', u'(\cdot), \mathbf{x}'(\cdot\,;p)\right)$ with $u'(\cdot) = u(\cdot)$, $|T' - T| < \delta_T$, and $T' < T$.

Proof Since $\mathbf{x}_1(1;p) > 10$ there is a $0 < T' < T$ with $\left(\frac{T'}{T}\right)^2 \mathbf{x}_1(1;p) \geq 10$ and $|T' - T| < \delta_T$. Let $\mathbf{x}'(\cdot\,;p)$ denote the differential states which are determined by T', $u(\cdot)$, and p. We have

$$\mathbf{x}'_2(t;p) = T'\int_0^t u(\tau) - p\,\mathrm{d}\tau = \frac{T'}{T}\mathbf{x}_2(t;p) \leq \mathbf{x}_2(t;p)$$

for all $t \in [0,1]$, and

$$\mathbf{x}'_1(1;p) = T'\int_0^1 \mathbf{x}'_2(t;p)\,\mathrm{d}t = \left(\frac{T'}{T}\right)^2 T\int_0^1 \mathbf{x}_2(t;p)\,\mathrm{d}t = \left(\frac{T'}{T}\right)^2 \mathbf{x}_1(1;p) \geq 10.$$

Thus, $\big(T', u(\cdot), \mathbf{x}'(\cdot\,;p)\big)$ is feasible. $\square$

Proposition B.8
Let $\delta_u > 0$ and $\big(T, u(\cdot), \mathbf{x}(\cdot\,;p)\big) \neq \big(T^*, u^*(\cdot), \mathbf{x}^*(\cdot\,;p)\big)$ be feasible for Problem (6.9) with

$$\mathbf{x}_1(1\,;p) = 10 \quad \text{and} \quad \mathbf{x}_2(1\,;p) = 0\,.$$

Then there is a feasible tuple $\big(T', u'(\cdot), \mathbf{x}'(\cdot\,;p)\big)$ with $T' = T$, $\|u'(\cdot) - u(\cdot)\|_\infty < \delta_u$, and $\mathbf{x}_1'(1;p) > 10$.

Proof We have $T > T^*$ as the global optimum of Problem (6.9) is unique, see Corollary B.4. Let $t_1 = \frac{4}{T(10-p)}$ and $t_2 = 1 - \frac{4}{T(10+p)}$. From Lemma B.2 we get $0 < t_1 < t_2 < 1$. We distinct two cases:

Case 1): $u(\cdot) \not\equiv 10$ on $[0, t_1]$ (almost surely). Similar to the proof of Proposition B.3 we show $\mathbf{x}_2(t;p) < 4$ for all $t \in [0, t_1]$. In particular, there is $\delta > 0$ with $\mathbf{x}_2(t;p) \leq 4 - \delta$ for all $t \in [0, t_1]$. Furthermore, there is a $0 < \bar{\delta}_1 < \frac{\delta}{2}$ and a subset $\mathcal{A}_1 \subseteq [0, t_1]$ with non-zero (Lebesgue-)measure $\lambda(\mathcal{A}_1)$ (< 1) such that $u(t) < 10 - \bar{\delta}_1$ for $t \in \mathcal{A}_1$ (almost surely). Let $\chi_{\mathcal{A}_1}(\cdot)$ denote the characteristic function on the set $\mathcal{A}_1$. Then, for all $t \in [0, t_1]$ and $0 < \delta' < \bar{\delta}_1$ we get

$$\mathbf{x}_2(t;p) \leq T \int_0^t \left(u(\tau) - p + \frac{1}{T}\delta' \chi_{\mathcal{A}_1}(\tau) \right) \mathrm{d}\tau \leq \mathbf{x}_2(t;p) + \delta' \lambda(\mathcal{A}_1) < \mathbf{x}_2(t;p) + \frac{\delta}{2} < 4\,, \tag{B.8}$$

which we will need later in the course of the proof. Again, we distinct two cases:

Case i): $\mathbf{x}_2(t;p) + \frac{\delta}{2} \leq 4$ for all $t \geq t_1$. If we had $u(\cdot) \equiv -10$ on $[t_1, 1]$ (almost surely), due to $t_2 > t_1$ we would get

$$\begin{aligned}
\mathbf{x}_2(1;p) &< 4 + T \int_{t_1}^{t_2} -10 - p \,\mathrm{d}t + T \int_{t_2}^{1} -10 - p \,\mathrm{d}t \\
&= 4 - T(10+p)(1-t_2) + T \int_{t_1}^{t_2} -10 - p \,\mathrm{d}t = -T \int_{t_1}^{t_2} 10 + p \,\mathrm{d}t < 0
\end{aligned}$$

which contradicts the prerequisites of the proposition. Thus, there is a $\bar{\delta}_2 > 0$ and a subset $\mathcal{A}_2 \subseteq [t_1, 1]$ with non-zero measure $\lambda(\mathcal{A}_2)$ (< 1) such that $u(t) > -10 + \bar{\delta}_2$ on $\mathcal{A}_2$ (almost surely). We set $\delta' = \frac{1}{2}\min\big(\bar{\delta}_1, \bar{\delta}_2, \delta_u\big)\big(< \frac{\delta}{2}\big)$. Then

$$u(t) < 10 - \delta' \text{ for } t \in \mathcal{A}_1 \subseteq [0, t_1] \quad \text{and} \quad u(t) > -10 + \delta' \text{ for } t \in \mathcal{A}_2 \subseteq [t_1, 1]$$

(almost surely, respectively). We can choose $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\lambda(\mathcal{A}_1) = \lambda(\mathcal{A}_2)$ and define

$$u'(t) = u(t) + \frac{1}{T}\delta' \chi_{\mathcal{A}_1}(t) - \frac{1}{T}\delta' \chi_{\mathcal{A}_2}(t).$$

Since $T > T^* > 1$, we have $u'(t) \in [-10, 10]$ and $\|u(\cdot) - u'(\cdot)\|_\infty < \delta' < \delta_u$. Let $\mathbf{x}'(\cdot\,; p)$ denote the differential states which are determined by T, $u'(\cdot)$, and p. Due to (B.8), for all $t \in [0, t_1]$ we have

$$\mathbf{x}_2(t;p) \le T\int_0^t \left(u(\tau) - p + \frac{1}{T}\delta' \chi_{\mathcal{A}_1}(\tau) \right) \mathrm{d}\tau = \mathbf{x}_2'(t;p) < \mathbf{x}_2(t;p) + \delta' < \mathbf{x}_2(t;p) + \frac{\delta}{2} < 4$$

and $\mathbf{x}_2'(t_1;p) > \mathbf{x}_2(t_1;p)$. Furthermore, for $t \in [t_1, 1]$ we get

$$\begin{aligned}
\mathbf{x}_2'(t;p) &= \mathbf{x}_2(t;p) + \delta'\lambda(\mathcal{A}_1) - T\int_{t_1}^t \frac{1}{T}\delta'\chi_{\mathcal{A}_2}(\tau)\,\mathrm{d}\tau \\
&= \mathbf{x}_2(t;p) + \delta'\lambda(\mathcal{A}_1) - \delta'\lambda(\mathcal{A}_2 \cap [t_1, t]) \ge \mathbf{x}_2(t;p).
\end{aligned}$$

Altogether, we see $\mathbf{x}_2'(1;p) = \mathbf{x}_2(1;p) = 0$ and

$$\mathbf{x}_2(t;p) \le \mathbf{x}_2'(t;p) < \mathbf{x}_2(t;p) + \delta' < \mathbf{x}_2(t;p) + \frac{\delta}{2} \le 4$$

for all $t \in [0,1]$ due to the assumption in the beginning of Case i). Since $\mathbf{x}_2(t_1;p) < \mathbf{x}_2'(t_1;p)$, by the continuity of $\mathbf{x}_2(\cdot\,;p)$ and $\mathbf{x}_2'(\cdot\,;p)$ we get $\mathbf{x}_1'(1;p) > \mathbf{x}_1(1;p) = 10$, and $\big(T, u'(\cdot), \mathbf{x}'(\cdot\,;p)\big)$ is feasible.

Case ii): There is a $t \in [t_1, 1]$ with $\mathbf{x}_2(t;p) > 4 - \frac{\delta}{2}$. Since $\mathbf{x}_2(t_1;p) < 4 - \frac{\delta}{2}$ according to (B.8) and $\mathbf{x}_2(\cdot\,;p)$ is continuous, the minimum

$$\bar{t} = \min_{t\in[t_1,1]} \left\{ t \,\middle|\, \mathbf{x}_2(t;p) = 4 - \frac{\delta}{2} \right\}$$

exists and we have $t_1 < \bar{t}$. In particular, $\mathbf{x}_2(t;p) < 4 - \frac{\delta}{2}$ for all $t \in [t_1, \bar{t})$ and $\mathbf{x}_2(\bar{t};p) = 4 - \frac{\delta}{2}$. Hence, there exists a subset $\mathcal{A}_2 \subseteq [t_1, \bar{t}] \subseteq [t_1, 1]$ with non-zero measure $\lambda(\mathcal{A}_2)$ such that $u(t) > p$ on $\mathcal{A}_2$ (almost surely). We set $\delta' = \frac{1}{2}\min\big(\bar{\delta}_1, \delta_u\big)\big(< \frac{\delta}{2}\big)$. Then – as in Case i) – we have

$$u(t) < 10 - \delta' \text{ for } t \in \mathcal{A}_1 \subseteq [0, t_1] \quad \text{and} \quad u(t) > -10 + \delta' \text{ for } t \in \mathcal{A}_2 \subseteq [t_1, 1].$$

We can choose $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\lambda(\mathcal{A}_1) = \lambda(\mathcal{A}_2)$ and define

$$u'(t) = u(t) + \frac{1}{T}\delta' \chi_{\mathcal{A}_1}(t) - \frac{1}{T}\delta' \chi_{\mathcal{A}_2}(t).$$

As in Case i), we have $u'(t) \in [-10, 10]$ (almost surely) and $\|u(\cdot) - u'(\cdot)\|_\infty < \delta' < \delta_u$. Let $\mathbf{x}'(\cdot; p)$ denote the differential states which are determined by T, $u'(\cdot)$, and p. Then similar to Case i) we have $\mathbf{x}_2'(t; p) \leq \mathbf{x}_2(t; p) + \frac{\delta}{2} < 4$ for all $t \in [0, t_1]$ and $\mathbf{x}_2'(t_1; p) > \mathbf{x}_2(t_1; p)$. Furthermore, for $t \in [t_1, \bar{t}]$ we get

$$\mathbf{x}_2'(t; p) = \mathbf{x}_2(t; p) + \delta' \lambda(\mathcal{A}_1) - \delta' \lambda(\mathcal{A}_2 \cap [t_1, \bar{t}]) \geq \mathbf{x}_2(t; p).$$

In particular, $\mathbf{x}_2'(t; p) = \mathbf{x}_2(t; p)$ for all $t \geq \bar{t}$. As in Case i), we conclude that the tuple $\big(T, u'(\cdot), \mathbf{x}'(\cdot; p)\big)$ is feasible with $\mathbf{x}_1'(1; p) > \mathbf{x}_1(1; p) = 10$.

Case 2): $u(\cdot) \equiv 10$ in $[0, t_1]$ (almost surely). Then we have $\mathbf{x}_2(t_1; p) = 4$. If $\mathbf{x}_2(t; p) = 4$ for all $t \in [t_1, t_2]$, then $u(\cdot) \equiv -10$ in $[t_2, 1]$ (almost surely) due to the terminal constraint $\mathbf{x}_2(1; p) \leq 0$. Thus $u(\cdot) = u_T(\cdot; p)$ and $\mathbf{x}_1(1; p) > 10$ by construction since $T > T^*$. This contradicts the prerequisites of the proposition. Hence, there is a $t \in (t_1, t_2]$ with $\mathbf{x}_2(t; p) < 4$. Let

$$\bar{t} = \inf_{t \in [t_1, t_2]} \left\{ t \,\middle|\, \mathbf{x}_2(t; p) < 4 \right\}.$$

By the continuity of $\mathbf{x}_2(t; p)$ we get $t_1 \leq \bar{t} < t_2$ and $\mathbf{x}_2(t; p) = 4$ for all $t \in [t_1, \bar{t}]$. There is a $\varepsilon > 0$ with $\bar{t} + \varepsilon < t_2 - \varepsilon$. Again by the continuity of $\mathbf{x}_2(\cdot; p)$, there is a

$$t_l \in \arg\min_{t \in [\bar{t}, \bar{t}+\varepsilon]} \mathbf{x}_2(t; p),$$

and by the choice of $\bar{t}$ we have $t_l > \bar{t}$ and $\mathbf{x}_2(t_l; p) < 4$. Furthermore, there are $\varepsilon', \delta > 0$ such that

$$\mathbf{x}_2(t; p) < 4 - \delta \quad \text{for all } t \in [t_l - \varepsilon', t_l + \varepsilon'] \cap [\bar{t}, \bar{t} + \varepsilon] = [\tau_1, \tau_2].$$

We have $\tau_1 < t_l \leq \tau_2 \leq \bar{t} + \varepsilon < t_2 - \varepsilon$. Since $\mathbf{x}_2(t_l; p) \leq \mathbf{x}_2(t; p)$ for all $t \in [\tau_1, \tau_2] \subseteq [\bar{t}, \bar{t} + \varepsilon]$, there is a subset $\mathcal{A}_1 \subseteq [\tau_1, \tau_2]$ with non-zero measure such that $u(t) \leq p$ on $\mathcal{A}_1$. Otherwise, due to $\tau_1 < t_l$ we would get $\mathbf{x}_2(\tau_1; p) < \mathbf{x}_2(t_l; p)$. In particular, there is a $\bar{\delta}_1$ with $0 < \bar{\delta}_1 < \frac{\delta}{2}$ such that $u(t) < 10 - \bar{\delta}_1$ on $\mathcal{A}_1$ (almost surely). Similar to Case 1), for all $t \in [\tau_1, \tau_2]$ and $0 < \delta' < \bar{\delta}_1$ we get

$$\mathbf{x}_2(\tau_1; p) + T \int_{\tau_1}^{t} \left(u(\tau) - p + \frac{1}{T} \delta' \chi_{\mathcal{A}_1}(\tau) \right) \mathrm{d}\tau \leq \mathbf{x}_2(t; p) + \delta' \lambda(\mathcal{A}_1) < \mathbf{x}_2(t; p) + \frac{\delta}{2} < 4\,.$$

Again we distinct two cases: either $\mathbf{x}_2(t;p) + \frac{\delta}{2} \leq 4$ for all $t \geq \tau_2$, or there is a $t \in [\tau_2, 1]$ with $\mathbf{x}_2(t;p) + \frac{\delta}{2} > 4$. Since $\tau_2 \leq t_2 - \varepsilon$, we can argue as in Case 1), subcases i)-ii), and construct a feasible tuple a feasible $\big(T', u'(\cdot), \mathbf{x}'(\cdot\,; p)\big)$ with $T' = T$, $\|u'(\cdot) - u(\cdot)\|_\infty < \delta_u$, and $\mathbf{x}'_1(1;p) > 10$. □

We are now able to prove that Problem (6.9) is uniquely solvable:

Proposition B.9
The tuple $\big(T^*, u^*(\cdot), \mathbf{x}^*(\cdot\,; p)\big)$ is the only local optimum of Problem (6.9) in the considered normed space.

Proof We know that $\big(T^*, u^*(\cdot), \mathbf{x}^*(\cdot\,; p)\big)$ is the global minumum of Problem (6.9), see Proposition B.4. Now let $\big(T, u(\cdot), \mathbf{x}(\cdot\,; p)\big) \neq \big(T^*, u^*(\cdot), \mathbf{x}^*(\cdot\,; p)\big)$ be feasible for Problem (6.9). Since the global minimum is unique, we have $T > T^*$. Let $\delta_T, \delta_u > 0$. We distinct two cases:

Case 1): $\mathbf{x}_1(1;p) = 10$. We can apply Lemma B.6 or Proposition B.8 to construct a feasible $\big(T', u'(\cdot), \mathbf{x}'(\cdot\,; p)\big)$ with $T' = T$, $\|u'(\cdot) - u(\cdot)\|_\infty < \delta_u$, and $\mathbf{x}'_1(1;p) > 10$. Subsequently, we apply Lemma B.7 and find a feasible tuple $\big(T'', u''(\cdot), \mathbf{x}''(\cdot\,; p)\big)$ with $\|u''(\cdot) - u(\cdot)\|_\infty = \|u'(\cdot) - u(\cdot)\|_\infty < \delta_u$, $|T'' - T| = |T'' - T'| < \delta_T$, and $T'' < T$.

Case 2): $\mathbf{x}_1(1;p) > 10$. We can apply Lemma B.7 and find a feasible $\big(T', u'(\cdot), \mathbf{x}'(\cdot\,; p)\big)$ with $u'(\cdot) = u(\cdot)$, $|T' - T| < \delta_T$, and $T' < T$.

Using Lemma B.5 we conclude that $\big(T, u(\cdot), \mathbf{x}(\cdot\,; p)\big)$ is not a local minimum. □

B.2.5 Sketch of Proof of Remark 6.3

Let $p \geq 0$, $T = T^*(p) = 2.5 + \frac{40}{100-p^2}$ and $t_1 = \frac{4}{T(10-p)}$ and $t_2 = 1 - \frac{4}{T(10+p)}$ as in Appendix B.2.4. Then we get

$$
\begin{aligned}
& & t_2 &> t_1 \\
&\iff & T &> \tfrac{4}{10-p} + \tfrac{4}{10+p} \\
&\iff & 2.5 + \tfrac{40}{100-p^2} &> \tfrac{4}{10-p} + \tfrac{4}{10+p} = \tfrac{80}{100-p^2} \\
&\iff & 2.5 &> \tfrac{40}{100-p^2} \\
&\iff & 100 - p^2 &> 16 \\
&\iff & 84 &> p^2 \\
&\iff & 2\sqrt{21} &> p,
\end{aligned}
$$

as $p \geq 0$. Hence, for $p \geq 2\sqrt{21}$ the function $u_T(\cdot\,;p)$, see (B.5), is not well-defined anymore and the optimal strategy needs to be adapted. If $2\sqrt{21} \leq p < 10$, for the optimal controllable parameter we get $T^*(p) = \frac{20}{\sqrt{100-p^2}}$ and the optimal control function $u^*(\cdot)$ is given by

$$
u^*(t) = \begin{cases} 10 & \text{for } 0 \leq t < \frac{10+p}{20}, \\ -10 & \text{for } \frac{10+p}{20} \leq t \leq 1. \end{cases}
$$

The proof of this statement works similarly as the one given in Appendix B.2.4.

B.2.6 Solution of Problem (6.10)

Let $\Omega_p = [p_l, p_u] \subseteq [0, 9]$ with $p_l < p_u$. In this section, we consider Problem (6.10) with variables

$$
\left(T, u(\cdot), p, \mathbf{x}(\cdot\,;p)\right) \in \mathbb{R} \times L^\infty([0,1],\mathbb{R}) \times \mathbb{R} \times W^{1,\infty}\left([0,1],\mathbb{R}^2\right),
$$

see Section 2.1 for the spaces and corresponding norms, where $\mathbf{x}(\cdot\,;p)$ denotes the (unique) solution of IVP (6.10b-6.10c) for given T, $u(\cdot)$, and $p \in \Omega_p$. The product space $\mathbb{R} \times L^\infty([0,1],\mathbb{R}) \times \mathbb{R} \times W^{1,\infty}\left([0,1],\mathbb{R}^2\right)$, equipped with the norm

$$
\left\|\left(T, u(\cdot), p, \mathbf{x}(\cdot\,;p)\right)\right\| = \max\left[|T|, \|u(\cdot)\|_\infty, |p|, \|\mathbf{x}(\cdot\,;p)\|_{1,\infty}\right],
$$

is a Banach space. In this section, we derive a condition for the non-emptiness of the feasible set of Problem (6.10) and compute the set of globally optimal solutions for the case that the feasible set is non-empty. Furthermore, if $\left(T, u(\cdot), p, \mathbf{x}(\cdot\,;p)\right)$ is a (locally) optimal solution, then $T = T^*$ and $u(\cdot) = u^*(\cdot)$ for a certain parameter T^*

and a certain control function $u^*(\cdot)$. The main results of this section can be found in Proposition B.14 and Corollary B.22.

A Reformulation of Problem (6.10)

We state a problem reformulation which will be useful for the subsequent investigations. Let $T \geq 0$ and $u : [0,1] \to [-10,10]$. For all $p \in \Omega_p = [p_l, p_u]$ we have

$$u(t) - p_u \leq u(t) - p \leq u(t) - p_l\,, \quad t \in [0,1]\,.$$

By the monotonicity of the integral, we get

$$\mathbf{x}_2(t; p_u) \leq \mathbf{x}_2(t; p) \leq \mathbf{x}_2(t; p_l)\,, \quad t \in [0,1]\,,$$

and thus also

$$\mathbf{x}_1(t; p_u) \leq \mathbf{x}_1(t; p) \leq \mathbf{x}_1(t; p_l)\,, \quad t \in [0,1]\,.$$

In particular

$$\begin{bmatrix} \mathbf{x}_2(t;p) \leq 4\,, & t \in [0,1]\,, \text{ for all } p \in \Omega_p\,, \\ \mathbf{x}_1(1;p) \geq 10\,, & \text{for all } p \in \Omega_p\,, \\ \mathbf{x}_2(1;p) \leq 0\,, & \text{for all } p \in \Omega_p \end{bmatrix} \iff \begin{bmatrix} \mathbf{x}_2(t;p_l) \leq 4\,, & t \in [0,1]\,, \\ \mathbf{x}_1(1;p_u) \geq 10\,, & \\ \mathbf{x}_2(1;p_l) \leq 0 & \end{bmatrix}. \tag{B.9}$$

We consider the problem

$$\min_{T, u(\cdot), \mathbf{x}(\cdot;\Omega_p)} \quad T \tag{B.10a}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}\left(t;\Omega_p\right) = T \begin{pmatrix} \mathbf{x}_2\left(t;\Omega_p\right) \\ u(t) - p_l \\ \mathbf{x}_4\left(t;\Omega_p\right) \\ u(t) - p_u \end{pmatrix}, \quad t \in [0,1]\,, \tag{B.10b}$$

$$\mathbf{x}\left(0;\Omega_p\right) = \mathbf{0}\,, \tag{B.10c}$$

$$\mathbf{x}_2\left(t;\Omega_p\right) \leq 4\,, \quad t \in [0,1]\,, \tag{B.10d}$$

$$\mathbf{x}_3\left(1;\Omega_p\right) \geq 10\,, \tag{B.10e}$$

$$\mathbf{x}_2\left(1;\Omega_p\right) \leq 0\,, \tag{B.10f}$$

$$T \geq 0\,, \tag{B.10g}$$

$$u(t) \in [-10,10]\,, \quad t \in [0,1]\,, \tag{B.10h}$$

with variables

$$\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right) \in \mathbb{R} \times L^{\infty}([0,1],\mathbb{R}) \times W^{1,\infty}\left([0,1],\mathbb{R}^4\right)$$

where $\mathbf{x}\left(\cdot;\Omega_p\right)$ denotes the (unique) solution of the IVP (B.10b-B.10c) for given T, $u(\cdot)$, and Ω_p. If we equip the above space with the norm

$$\left\|\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)\right\| = \max\left[|T|, \|u(\cdot)\|_{\infty}, \left\|\mathbf{x}\left(\cdot;\Omega_p\right)\right\|_{1,\infty}\right],$$

it is a Banach space. We get

Lemma B.10
Let $T > 0$, $u(\cdot) \in L^{\infty}([0,1],\mathbb{R})$, and $\mathbf{x}\left(\cdot;\Omega_p\right) \in W^{1,\infty}\left([0,1],\mathbb{R}^4\right)$ the differential states which are determined by T, $u(\cdot)$, and Ω_p. Let $\varepsilon > 0$. Then there exist $\delta_T, \delta_u > 0$ such that

$$\left\|\mathbf{x}'\left(\cdot;\Omega_p\right) - \mathbf{x}\left(\cdot;\Omega_p\right)\right\|_{1,\infty} < \varepsilon$$

for all T', $u'(\cdot)$ with $|T' - T| < \delta_T$ and $\|u'(\cdot) - u(\cdot)\|_{\infty} < \delta_u$, where $\mathbf{x}'\left(\cdot;\Omega_p\right)$ denotes the differential states which are determined by T', $u'(\cdot)$, and Ω_p.

Proof Similar to proof of Lemma B.5. □

We find the following equivalence of the Problems (B.10) and (6.10):

Lemma B.11
Problem (B.10) is equivalent to Problem (6.10) in the following sense: let $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ be feasible for Problem (B.10). Then for every $p \in \Omega_p = [p_l, p_u]$, the tuple $\left(T, u(\cdot), p, \mathbf{x}(\cdot;p)\right)$ is feasible for Problem (6.10) and the values of the objective functions coincide. Vice versa, let $\left(T, u(\cdot), p, \mathbf{x}(\cdot;p)\right)$ be feasible for Problem (6.10). Then $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ is feasible for Problem (B.10) and the values of the objective functions coincide. In particular, $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ is a local minimum of Problem (B.10) if and only if $\left(T, u(\cdot), p, \mathbf{x}(\cdot;p)\right)$ is a local minimum of Problem (6.10) for each $p \in \Omega_p$. The same holds for global minima.

Proof The feasibility assertions follow from the Equivalence (B.9) and the fact that every pair $\left(p, \mathbf{x}(\cdot;p)\right)$ is a (global) maximizer of the lower level problem of Problem (6.10) for given T and $u(\cdot)$. The accordance of the respective values of the objective functions is clear.

Let $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ be a local minimum of Problem (B.10), and $p \in \Omega_p$. If the tuple $\left(T, u(\cdot), p, \mathbf{x}(\cdot;p)\right)$ would not be a local minimum of Problem (6.10), then for every $\varepsilon > 0$ we find a feasible $\left(T'_\varepsilon, u'_\varepsilon(\cdot), p'_\varepsilon, \mathbf{x}'_\varepsilon(\cdot;p)\right)$ with $|T - T'_\varepsilon| < \varepsilon$, $\|u(\cdot) - u'_\varepsilon(\cdot)\|_\infty < \varepsilon$ and $T'_\varepsilon < T$. Let $\mathbf{x}'_\varepsilon\left(\cdot;\Omega_p\right)$ denote the solution of IVP (B.10b-B.10c) which is determined by T'_ε, $u'_\varepsilon(\cdot)$, and Ω_p. By the already proven part of the lemma, $\left(T'_\varepsilon, u'_\varepsilon(\cdot), \mathbf{x}'_\varepsilon\left(\cdot;\Omega_p\right)\right)$ is feasible for Problem (B.10). By Lemma B.10, $\left(T'_\varepsilon, u'_\varepsilon(\cdot), \mathbf{x}'_\varepsilon\left(\cdot;\Omega_p\right)\right)$ lies in an arbitrary small neighborhood of $\left(T, u(\cdot), \mathbf{x}\left(\cdot,\Omega_p\right)\right)$ if ε is small enough. Thus, $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ is no local minimum of Problem (B.10) which is a contradiction.

The transfer of local minima from Problem (6.10) to Problem (B.10) can be shown similarly, and the transfer of global minima from Problem (6.10) to Problem (B.10) and vice versa follows from the first part of the lemma. □

Justified by the previous Lemma, we focus on Problem (B.10) in the following.

Non-Emptiness of Feasible Set and Global Optimum of Problem (B.10)

First, we investigate the feasible set of Problem (B.10). We start with

Lemma B.12
Let $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ be feasible for Problem (B.10). Then $T > \frac{4}{10+p_l} + \frac{4}{10-p_l}$.

Proof We proceed similar as in the proof of Lemma B.2 for $p = p_l$ and make use of $\mathbf{x}_3\left(t;\Omega_p\right) \le \mathbf{x}_1\left(t;\Omega_p\right)$. □

Let $T > \frac{4}{10+p_l} + \frac{4}{10-p_l}$. We set

$$t_1(T,\Omega_p) = \frac{4}{T(10-p_l)} \quad \text{and} \quad t_2(T,\Omega_p) = 1 - \frac{4}{T(10+p_l)}\,.$$

Then $0 < t_1(T,\Omega_p) < t_2(T,\Omega_p) < 1$. We define

$$u_T\left(t;\Omega_p\right) = \begin{cases} 10 & \text{for } 0 \le t < t_1(T,\Omega_p)\,, \\ p_l & \text{for } t_1(T,\Omega_p) \le t < t_2(T,\Omega_p)\,, \\ -10 & \text{for } t_2(T,\Omega_p) \le t \le 1\,. \end{cases}$$

Let $\mathbf{x}_T\left(\cdot;\Omega_p\right) \in \mathbb{R}^4$ denote the differential states which are determined by Ω_p, T, and $u_T\left(\cdot;\Omega_p\right)$. Then we have

$$\mathbf{x}_{T,2}\left(t;\Omega_p\right) = \begin{cases} T(10-p_l)t & \text{for } 0 \le t < t_1(T,\Omega_p), \\ 4 & \text{for } t_1(T,\Omega_p) \le t < t_2(T,\Omega_p), \\ 4 - T(10+p_l)\left(t - t_2(T,\Omega_p)\right) & \text{for } t_2(T,\Omega_p) \le t \le 1. \end{cases}$$

In particular, $\mathbf{x}_{T,2}\left(t;\Omega_p\right) \le 4$ for all $t \in [0,1]$ and $\mathbf{x}_{T,2}\left(1;\Omega_p\right) = 0$. Similar as in Appendix B.2.4, Equation (B.6), we get

$$\mathbf{x}_{T,1}\left(1;\Omega_p\right) = 4T - \frac{8}{10-p_l} - \frac{8}{10+p_l}.$$

Furthermore, $\mathbf{x}_{T,4}\left(t;\Omega_p\right) = \mathbf{x}_{T,2}(t) - (p_u - p_l)Tt$ and consequently

$$\begin{aligned} \mathbf{x}_{T,3}\left(1;\Omega_p\right) &= \mathbf{x}_{T,1}\left(1;\Omega_p\right) - \frac{1}{2}(p_u - p_l)T^2 \\ &= -\frac{1}{2}(p_u - p_l)T^2 + 4T - \frac{8}{10-p_l} - \frac{8}{10+p_l}. \end{aligned}$$

Thus, the tuple $\left(T, u_T\left(\cdot;\Omega_p\right), \mathbf{x}_T\left(\cdot;\Omega_p\right)\right)$ is feasible if and only if $\mathbf{x}_{T,3}\left(1;\Omega_p\right) \ge 10$. Similar to Appendix B.2.4, Proposition B.3, we have

Proposition B.13
Let $\left(T, u(\cdot), \mathbf{x}(\cdot;\Omega_p)\right)$ be feasible for Problem (B.10) with $u(\cdot) \neq u_T\left(\cdot;\Omega_p\right)$ (as elements of $L^\infty([0,1],\mathbb{R})$). Then, $\left(T, u_T\left(\cdot;\Omega_p\right), \mathbf{x}_T\left(\cdot;\Omega_p\right)\right)$ is feasible and we have

$$10 \le \mathbf{x}_3\left(1;\Omega_p\right) < \mathbf{x}_{T,3}\left(1;\Omega_p\right).$$

Proof We have $\mathbf{x}_2\left(t;\Omega_p\right) = \mathbf{x}_4\left(t;\Omega_p\right) + (p_u - p_l)Tt$, and equally for $\mathbf{x}_T\left(\cdot;\Omega_p\right)$. Hence $\mathbf{x}_2(t;\Omega_p) \le \mathbf{x}_{T,2}(t;\Omega_p)$ implies $\mathbf{x}_4(t;\Omega_p) \le \mathbf{x}_{T,4}(t;\Omega_p)$ and the same holds for strict inequality. We use this and proceed similarly as in the proof of Proposition B.3. □

We define

$$g: \mathbb{R} \to \mathbb{R}, \ T \longmapsto g\left(T;\Omega_p\right) = -\frac{1}{2}(p_u - p_l)T^2 + 4T - \frac{8}{10-p_l} - \frac{8}{10+p_l} - 10.$$

Let $\underline{T} = \frac{4}{10-p_l} + \frac{4}{10+p_l}$. Then $\underline{T} = \frac{80}{100-p_l^2} \le \frac{80}{19} < 5$ and, as $p_u > p_l$,

$$g\left(\underline{T};\Omega_p\right) = -\frac{1}{2}(p_u - p_l)\underline{T}^2 + 2\underline{T} - 10 \le 2\underline{T} - 10 < 0. \tag{B.11}$$

For $T > \underline{T}$ we have

$$g\left(T;\Omega_p\right) = \mathbf{x}_{T,3}\left(1;\Omega_p\right) - 10\,.$$

Since $p_u > p_l$, the map $g\left(\cdot\,;\Omega_p\right)$ is concave quadratic function with argument of the maximum $T_{\max} = \frac{4}{p_u - p_l}$ and corresponding value

$$g\left(T_{\max};\Omega_p\right) = \frac{8}{p_u - p_l} - \frac{8}{10 - p_l} - \frac{8}{10 + p_l} - 10\,. \tag{B.12}$$

A straightforward calculation shows

$$g\left(T_{\max};\Omega_p\right) \geq 0 \iff p_u \leq p_l + \frac{8}{10 + \frac{8}{10 - p_l} + \frac{8}{10 + p_l}}\,. \tag{B.13}$$

We can now characterize the non-emptiness of the feasible set of Problem (B.10).

Proposition B.14
The feasible set of Problem (B.10) is non-empty if and only if

$$p_u \leq p_l + \frac{8}{10 + \frac{8}{10 - p_l} + \frac{8}{10 + p_l}}\,. \tag{B.14}$$

The same holds for the feasible set of Problem (6.10). If the feasible sets are non-empty, we have

$$T_{\max} = \frac{4}{p_u - p_l} \geq 5 + \frac{4}{10 - p_l} + \frac{4}{10 + p_l}\,. \tag{B.15}$$

Proof By Lemma B.11, the feasible set of Problem (B.10) is non-empty if and only if the feasible set of Problem (6.10) is non-empty. If the feasible set of Problem (B.10) is non-empty, there is a feasible tuple $\left(T, u(\cdot), \mathbf{x}\left(\cdot\,;\Omega_p\right)\right)$. By Proposition B.13, the tuple $\left(T, u_T\left(\cdot\,;\Omega_p\right), \mathbf{x}_T\left(\cdot\,;\Omega_p\right)\right)$ is feasible. In particular, we have

$$0 \leq \mathbf{x}_{T,3}\left(1;\Omega_p\right) - 10 = g\left(T;\Omega_p\right) \leq g\left(T_{\max};\Omega_p\right) = \frac{8}{p_u - p_l} - \frac{8}{10 - p_l} - \frac{8}{10 + p_l} - 10\,,$$

and (B.14) follows from the Equivalence (B.13). Conversely, we assume that (B.14) holds. From the Equivalence (B.13) and Equation (B.12) we get

$$g\left(T_{\max};\Omega_p\right) = 2\left(\frac{4}{p_u - p_l} - \frac{4}{10 - p_l} - \frac{4}{10 + p_l} - 5\right) \geq 0\,,$$

and in particular

$$T = T_{\max} = \frac{4}{p_u - p_l} \geq 5 + \frac{4}{10 - p_l} + \frac{4}{10 + p_l} > \frac{4}{10 - p_l} + \frac{4}{10 + p_l}.$$

Thus, $u_T\left(\cdot;\Omega_p\right)$ is well-defined and $\mathbf{x}_{T,3}\left(1;\Omega_p\right) = g\left(T_{\max};\Omega_p\right) + 10 \geq 10$. In particular, $\left(T, u_T\left(\cdot;\Omega_p\right), \mathbf{x}_T\left(\cdot;\Omega_p\right)\right)$ is feasible for Problem (B.10) and the feasible set is non-empty. $\square$

Next, we compute the unique global optimum of Problem (B.10). Let the feasible set of Problem (B.10) be non-empty. Then by the previous Proposition and Equation (B.13), we have $g\left(\frac{4}{p_u-p_l};\Omega_p\right) \geq 0$ and $\frac{4}{p_u-p_l} > \frac{4}{10-p_l} + \frac{4}{10+p_l}$. On the other hand, $g\left(\frac{4}{10-p_l} + \frac{4}{10+p_l};\Omega_p\right) < 0$, see (B.11). Let $\mathcal{Z} = \left\{T \in \mathbb{R} \,|\, g\left(T;\Omega_p\right) = 0\right\}$. As $g\left(\cdot;\Omega_p\right)$ is a (non-constant) concave quadratic function, for the cardinality of $\mathcal{Z}$ we have $|\mathcal{Z}| \in \{1,2\}$. We set

$$z\left(\Omega_p\right) \stackrel{\text{def}}{=} \min_{z \in \mathcal{Z}} z > \frac{4}{10 - p_l} + \frac{4}{10 + p_l}. \tag{B.16}$$

By definition of $z\left(\Omega_p\right)$ and the properties of $g\left(\cdot;\Omega_p\right)$, we get the implication

$$g\left(T;\Omega_p\right) \geq 0 \implies z\left(\Omega_p\right) \leq T. \tag{B.17}$$

We define

$$T^* = T^*\left(\Omega_p\right) \stackrel{\text{def}}{=} z\left(\Omega_p\right), \quad u^*(\cdot) = u^*\left(\cdot;\Omega_p\right) \stackrel{\text{def}}{=} u_{T^*}\left(\cdot;\Omega_p\right), \quad \text{and} \quad \mathbf{x}^*\left(\cdot;\Omega_p\right) \stackrel{\text{def}}{=} \mathbf{x}_{T^*}\left(\cdot;\Omega_p\right).$$

By construction, $\left(T^*, u^*(\cdot), \mathbf{x}^*\left(\cdot;\Omega_p\right)\right)$ is feasible for Problem (B.10) and we get

Corollary B.15
Let the feasible set of Problem (B.10) be non-empty. Then $\left(T^*, u^*(\cdot), \mathbf{x}^*\left(\cdot;\Omega_p\right)\right)$ is the unique global optimum of Problem (B.10).

Proof As the feasible set of Problem (B.10) is non-empty, $\left(T^*, u^*(\cdot), \mathbf{x}^*\left(\cdot;\Omega_p\right)\right)$ is well-defined, see the preceding considerations. Let $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ be any other feasible point. We note that $u(\cdot) = u_T(\cdot;\Omega_p)$ is not excluded. Thus, by Proposition B.13 we have

$$0 \leq \mathbf{x}_3\left(1;\Omega_p\right) - 10 \leq \mathbf{x}_{T,3}\left(1;\Omega_p\right) - 10 = g\left(T;\Omega_p\right)$$

and consequently $T^* = z\left(\Omega_p\right) \leq T$, see (B.17). Hence, $\left(T^*, u^*(\cdot), \mathbf{x}^*\left(\cdot;\Omega_p\right)\right)$ is a global optimum. To show the uniqueness, we assume that there is another feasible point

$\left(T^*, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ with $u(\cdot) \neq u^*(\cdot) = u_{T^*}\left(\cdot;\Omega_p\right)$. We apply Proposition B.13 again and see

$$0 \leq \mathbf{x}_3\left(1;\Omega_p\right) - 10 < \mathbf{x}_{T^*,3}\left(1;\Omega_p\right) - 10 = g\left(T^*;\Omega_p\right) = 0$$

which is a contradiction. Thus, such a point does not exist and the global minimum is unique. □

The Unique Solvability of Problem (B.10)

In the following, we show that the global optimum of Problem (B.10) is the only local optimum in the considered normed space. We proceed similar as in Appendix B.2.4 and adapt the proofs where needed.

Lemma B.16
Let $\delta_u > 0$ and $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ be feasible for Problem (B.10) with

$$\mathbf{x}_2\left(1;\Omega_p\right) < 0\,.$$

Then there is a feasible tuple $\left(T', u'(\cdot), \mathbf{x}'\left(\cdot;\Omega_p\right)\right)$ with $T' = T$, $\|u'(\cdot) - u(\cdot)\|_\infty < \delta_u$, and $\mathbf{x}'_3\left(1;\Omega_p\right) > 10$.

Proof We have $\mathbf{x}_2\left(t;\Omega_p\right) = \mathbf{x}_4\left(t;\Omega_p\right) + (p_u - p_l)T\,t$, and equally for $\mathbf{x}'\left(\cdot;\Omega_p\right)$. Hence, $\mathbf{x}_2(t;\Omega_p) \leq \mathbf{x}'_2(t;\Omega_p)$ implies $\mathbf{x}_4(t;\Omega_p) \leq \mathbf{x}'_4(t;\Omega_p)$ and the same holds for strict inequality. We exploit this fact and proceed similarly as in the proof of Lemma B.6. □

Lemma B.17
Let $\delta_T > 0$ and $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ be feasible for Problem (B.10) with

$$\mathbf{x}_3\left(1;\Omega_p\right) > 10\,.$$

Then there is a feasible $\left(T', u'(\cdot), \mathbf{x}'(\cdot)\right)$ with $u'(\cdot) = u(\cdot)$, $|T' - T| < \delta_T$, and $T' < T$.

Proof Similar to proof of Lemma B.7. □

It remains to investigate the case of a feasible, non-optimal tuple $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ for which the terminal constraints are satisfied with equality, see Proposition B.20. Proposition B.20 can be seen as the counterpart of Proposition B.8 in Appendix B.2.4. However in contrast to Appendix B.2.4, $T > T^*$ does not imply $\mathbf{x}_{T,3}\left(1;\Omega_p\right) > 10$. This is because $\mathbf{x}_{T,3}\left(1;\Omega_p\right) - 10 = g\left(T;\Omega_p\right)$ for $T \geq T^*$, $g\left(T^*;\Omega_p\right) = 0$, and $g\left(\cdot;\Omega_p\right)$ is a concave quadratic function. Therefore, the proof of Proposition B.8 cannot be transferred directly to Proposition B.20 and needs to be adapted. For this, we need two auxiliary results:

Lemma B.18
Let $T > T^*$ such that $\left(T, u_T\left(\cdot;\Omega_p\right), \mathbf{x}_T\left(\cdot;\Omega_p\right)\right)$ is feasible for Problem (B.10) and $\mathbf{x}_{T,3}\left(\cdot;\Omega_p\right) = 10$. Then

$$T > \frac{4}{p_u - p_l}\,.$$

Furthermore, for $t_1 = \frac{4}{T(10-p_l)}$ and $t_2 = 1 - \frac{4}{T(10+p_l)}$ we have

$$T\left(t_2 - t_1\right) > 5\,.$$

Proof We have $g\left(T;\Omega_p\right) = \mathbf{x}_{T,3}\left(1;\Omega_p\right) - 10 = 0$, i. e., T is a zero of $g\left(\cdot;\Omega_p\right)$. Since $T > T^* = \min\left\{T \in \mathbb{R} \,\middle|\, g\left(T;\Omega_p\right) = 0\right\}$, the concave quadratic function $g(\cdot)$ has two zeros, namely T and T^*. The vertex of $g\left(\cdot;\Omega_p\right)$ is given by $T_{\max} = \frac{4}{p_u - p_l}$ and we have $T^* < T_{\max} < T$. Furthermore,

$$T(t_2 - t_1) = T - \frac{4}{10 + p_l} - \frac{4}{10 - p_l} > T_{\max} - \frac{4}{10 + p_l} - \frac{4}{10 - p_l} \geq 5\,,$$

where the latter inequality is due to Proposition B.14, Inequality (B.15). □

Proposition B.19
Let $\delta_T, \delta_u > 0$, and $T > T^*$ such that $\left(T, u_T\left(\cdot;\Omega_p\right), \mathbf{x}_T\left(\cdot;\Omega_p\right)\right)$ is feasible for Problem (B.10) with $\mathbf{x}_{T,3}\left(1;\Omega_p\right) = 10$. Then there is a feasible $\left(T', u'(\cdot), \mathbf{x}'\left(\cdot;\Omega_p\right)\right)$ with $\mathbf{x}'_3\left(1;\Omega_p\right) > 10$, $\left\|u'(\cdot) - u_T\left(\cdot;\Omega_p\right)\right\|_\infty < \delta_u$, $|T' - T| < \delta_T$, and $T' < T$.

Proof We start with technical preparations which become relevant later in the proof. Let $t_1 = \frac{4}{T(10-p_l)}$ and $t_2 = 1 - \frac{4}{T(10+p_l)}$. We have $T\left(t_2 - t_1\right) > 5$ according to Lemma B.18. Consequently, we can choose a $n \in \mathbb{N}$, $n > 2$, with

$$T\left(t_2 - t_1\right) > \frac{5n}{n-1}\,.$$

Next, we consider the quadratic function

$$h : \mathbb{R} \to \mathbb{R},\;\; y \mapsto (T - y)^2 \frac{10}{T^2} + (T - y)\, y \frac{4(n-1)}{Tn} \left(t_2 - t_1\right). \tag{B.18}$$

Then $h(0) = 10$ and

$$\frac{\mathrm{d}}{\mathrm{d}y}h(y)\Big|_{y=0} = -\frac{20}{T} + \frac{4(n-1)}{Tn}T\,(t_2 - t_1) > -\frac{20}{T} + \frac{4(n-1)}{Tn}\frac{5n}{n-1} = 0$$

by the choice of n. In particular, there is a $\varepsilon' > 0$ such that $h(y) > 10$ for all $0 < y < \varepsilon'$.

In the following, we construct a feasible $\left(T', u'(\cdot), \mathbf{x}'\left(\cdot;\Omega_p\right)\right)$ with the desired properties. Let

$$t_1' = t_1 + \frac{1}{n}\,(t_2 - t_1) \quad \text{and} \quad t_2' = t_1 + \frac{n-1}{n}\,(t_2 - t_1)\,. \tag{B.19}$$

Then $t_1 < t_1' < t_2' < t_2$ (since $n > 2$). Furthermore, we set

$$\varepsilon = \frac{1}{2}\min\left(\delta_T, \frac{5}{4(n-1)}\delta_u, \frac{5}{4(n-1)}(10 - p_l), \varepsilon', 1\right)$$

and define

$$T' = T - \varepsilon \quad \text{and} \quad \delta = \frac{4n}{T'T\,(t_2 - t_1)}\varepsilon\,.$$

Then $|T' - T| < \delta_T$. From Lemma B.18 and Proposition B.14, Inequality (B.15), we get

$$T' = T - \varepsilon > T - 1 > \frac{4}{p_u - p_l} - 1 > 5 - 1 = 4\,,$$

and therefore

$$0 < \delta < \frac{n\varepsilon}{T\,(t_2 - t_1)} < \frac{4n\varepsilon}{T\,(t_2 - t_1)} < \frac{n-1}{5n}4n\varepsilon = \frac{4(n-1)}{5}\varepsilon\,.$$

In particular, by the definition of ε we get

$$\delta < \delta_u \quad \text{and} \quad \delta < 10 - p_l\,. \tag{B.20}$$

We define

$$u'(t) = u_T\left(t;\Omega_p\right) + \delta\chi_{[t_1,t_1']}(t) - \delta\chi_{[t_2',t_2]}(t)\,.$$

From (B.20) we get $\left\|u'(\cdot) - u_T\left(\cdot;\Omega_p\right)\right\|_\infty < \delta_u$ and, as $u_T\left(t;\Omega_p\right) = p_l$ for $t \in [t_1, t_1'] \cup [t_2', t_2]$, $u'(t) \in [-10, 10]$ for $t \in [0, 1]$.

Let $\mathbf{x}'\left(\cdot;\Omega_p\right)$ denote the differential states which are determined by T', $u'(\cdot)$, and Ω_p. Since $u'(t) = u_T\left(t;\Omega_p\right)$ in $[0, t_1]$, we have

$$\mathbf{x}_2'\left(t;\Omega_p\right) = T'\int_0^t u'(\tau) - p_l \,\mathrm{d}\tau = \frac{T'}{T}T\int_0^t u_T\left(\tau;\Omega_p\right) - p_l \,\mathrm{d}\tau = \frac{T'}{T}\mathbf{x}_{T,2}\left(t;\Omega_p\right)$$

for all $t \in [0, t_1]$.

For $t \in [t_1, t_1']$ we have $\mathbf{x}_{T,2}\left(t;\Omega_p\right) = \mathbf{x}_{T,2}\left(t_1;\Omega_p\right) = 4$ and therefore

$$\begin{aligned}
\mathbf{x}_2'\left(t;\Omega_p\right) &= \mathbf{x}_2'\left(t_1;\Omega_p\right) + T'\delta\left(t - t_1\right) = \frac{T'}{T}\mathbf{x}_{T,2}\left(t_1;\Omega_p\right) + T'\delta\left(t - t_1\right) \\
&= \frac{T'}{T}\mathbf{x}_{T,2}\left(t;\Omega_p\right) + T'\delta\left(t - t_1\right) \le \frac{T'}{T}\mathbf{x}_{T,2}\left(t;\Omega_p\right) + T'\delta(t_1' - t_1) \\
&= 4\frac{T'}{T} + \frac{4n\varepsilon}{T\left(t_2 - t_1\right)}\frac{t_2 - t_1}{n} = 4\frac{T'}{T} + 4\frac{\varepsilon}{T} = 4\,.
\end{aligned}$$

At $t = t_1'$ we have equality, $\mathbf{x}_2'\left(t_1';\Omega_p\right) = 4 = \frac{T'}{T}\mathbf{x}_{T,2}\left(t_1';\Omega_p\right) + 4\frac{\varepsilon}{T}$.

In $[t_1', t_2']$ we have $u'(t) = u_T\left(t;\Omega_p\right) = p_l$ and $\mathbf{x}_{T,2}\left(t;\Omega_p\right) = 4$. Thus we can rewrite $\mathbf{x}_2'\left(\cdot;\Omega_p\right)$ as

$$\mathbf{x}_2'\left(t;\Omega_p\right) = 4 = \frac{T'}{T}\mathbf{x}_{T,2}\left(t;\Omega_p\right) + 4\frac{\varepsilon}{T} \quad \text{for } t \in [t_1', t_2']\,.$$

For $t \in [t_2', t_2]$ we compute

$$\begin{aligned}
\mathbf{x}_2'\left(t;\Omega_p\right) &= \mathbf{x}_2'\left(t_2';\Omega_p\right) - T'\delta(t - t_2') \\
&= \frac{T'}{T}\mathbf{x}_{T,2}\left(t;\Omega_p\right) + 4\frac{\varepsilon}{T} - T'\delta(t - t_2') \\
&\ge \frac{T'}{T}\mathbf{x}_{T,2}\left(t;\Omega_p\right) + 4\frac{\varepsilon}{T} - T'\delta(t_2 - t_2') \\
&= \frac{T'}{T}\mathbf{x}_{T,2}\left(t;\Omega_p\right) + 4\frac{\varepsilon}{T} - \frac{4n\varepsilon}{T\left(t_2 - t_1\right)}\frac{t_2 - t_1}{n} = \frac{T'}{T}\mathbf{x}_{T,2}\left(t;\Omega_p\right)\,,
\end{aligned}$$

as $\mathbf{x}_{T,2}\left(t;\Omega_p\right) = \mathbf{x}_{T,2}\left(t_2';\Omega_p\right)$ in $[t_2', t_2]$. In particular, since $\mathbf{x}_2'\left(\cdot;\Omega_p\right)$ is monotonically decreasing in $[t_2', t_2]$ we get $\mathbf{x}_2'\left(t;\Omega_p\right) \le \mathbf{x}_2'\left(t_2';\Omega_p\right) = 4$ for $t \in [t_2', t_2]$. Furthermore, at $t = t_2'$ we have the equality

$$\mathbf{x}_2'\left(t_2;\Omega_p\right) = \frac{T'}{T}\mathbf{x}_{T,2}\left(t_2;\Omega_p\right)\,.$$

For $t \in [t_2, 1]$ we have $u'(t) = u_T\left(t;\Omega_p\right)$ and therefore

$$\begin{aligned}\mathbf{x}_2'\left(t;\Omega_p\right) &= \mathbf{x}_2'\left(t_2;\Omega_p\right) + T'\int_{t_2}^{t} u'(\tau) - p_l \,\mathrm{d}\tau \\ &= \frac{T'}{T}\mathbf{x}_{T,2}\left(t_2;\Omega_p\right) + \frac{T'}{T}\left[\mathbf{x}_{T,2}\left(t;\Omega_p\right) - \mathbf{x}_{T,2}\left(t_2;\Omega_p\right)\right] = \frac{T'}{T}\mathbf{x}_{T,2}\left(t;\Omega_p\right).\end{aligned}$$

Altogether, we have

$$\mathbf{x}_2'\left(t;\Omega_p\right) = \frac{T'}{T}\mathbf{x}_{T,2}\left(t;\Omega_p\right) + D(t)$$

with the continuous, non-negative function

$$D(t) = \begin{cases} 0 & \text{for } t \in [0, t_1], \\ T'\delta(t - t_1) & \text{for } t \in (t_1, t_1'], \\ 4\frac{\varepsilon}{T} & \text{for } t \in (t_1', t_2'], \\ 4\frac{\varepsilon}{T} - T'\delta(t - t_2') & \text{for } t \in (t_2', t_2], \\ 0 & \text{for } t \in (t_2, 1]. \end{cases}$$

From what we have seen above we conclude

$$\mathbf{x}_2'\left(t;\Omega_p\right) \le 4 \;\text{ for all } t \in [0,1] \quad \text{and} \quad \mathbf{x}_2'\left(1;\Omega_p\right) = \frac{T'}{T}\mathbf{x}_{T,2}\left(1;\Omega_p\right) = 0.$$

In the remainder of the proof, we investigate $\mathbf{x}_3'\left(1;\Omega_p\right)$. By (B.19) we get

$$t_1' - t_1 = t_2 - t_2' = \frac{1}{n}(t_2 - t_1).$$

Hence,

$$T'\int_0^1 D(t)\,\mathrm{d}t = \frac{1}{2}T'^2\delta\left(t_1' - t_1\right)^2 + 4\frac{T'}{T}\varepsilon\left(t_2 - t_1'\right) - \frac{1}{2}T'^2\delta\left(t_2 - t_2'\right)^2 = 4\frac{T'}{T}\varepsilon\frac{n-1}{n}(t_2 - t_1).$$

Since $\mathbf{x}_4'\left(t;\Omega_p\right) = \mathbf{x}_2'\left(t;\Omega_p\right) - T'(p_u - p_l)\,t$, we have

$$\mathbf{x}_3'\left(1;\Omega_p\right) = \mathbf{x}_1'\left(1;\Omega_p\right) - \frac{1}{2}T'^2(p_u - p_l),$$

and similarly $\mathbf{x}_{T,3}\left(1;\Omega_p\right) = \mathbf{x}_{T,1}\left(1;\Omega_p\right) - \frac{1}{2}T^2(p_u - p_l)$. Thus,

$$
\begin{aligned}
\mathbf{x}'_3\left(1;\Omega_p\right) &= \frac{T'^2}{T^2}\mathbf{x}_{T,1}\left(1;\Omega_p\right) + T'\int_0^1 D(t)\,\mathrm{d}t - \frac{1}{2}T'^2(p_u - p_l)\\
&= \frac{T'^2}{T^2}\mathbf{x}_{T,1}\left(1;\Omega_p\right) + 4\frac{T'}{T}\varepsilon\frac{n-1}{n}(t_2 - t_1) - \frac{1}{2}T'^2(p_u - p_l)\\
&= \frac{T'^2}{T^2}\left(\mathbf{x}_{T,1}\left(1;\Omega_p\right) - \frac{1}{2}T^2(p_u - p_l)\right) + 4\frac{T'}{T}\varepsilon\frac{n-1}{n}(t_2 - t_1)\\
&= \frac{T'^2}{T^2}\mathbf{x}_{T,3}\left(1;\Omega_p\right) + 4\frac{T'}{T}\varepsilon\frac{n-1}{n}(t_2 - t_1)\\
&= 10\frac{(T-\varepsilon)^2}{T^2} + 4\frac{T-\varepsilon}{T}\varepsilon\frac{n-1}{n}(t_2 - t_1).\\
&= h(\varepsilon)
\end{aligned}
$$

by definition of $h(\cdot)$, see (B.18). From the properties of $h(\cdot)$, since $0 < \varepsilon < \varepsilon'$ we get

$$\mathbf{x}'_3\left(1;\Omega_p\right) = h(\varepsilon) > 10.$$

To sum up, the tuple $\left(T', u'(\cdot), \mathbf{x}'\left(\cdot;\Omega_p\right)\right)$ is feasible, and we have $\mathbf{x}'_3\left(1;\Omega_p\right) > 10$, $\left\| u'(\cdot) - u_T\left(\cdot;\Omega_p\right)\right\|_\infty < \delta_u$, $|T' - T| < \delta_T$, and $T' < T$, as desired. $\square$

Now, we are able to consider the case of a non-optimal tuple for which the Terminal Constraints (B.10e) and (B.10f) are satisfied with equality:

Proposition B.20
Let $\delta_T, \delta_u > 0$, and $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right) \neq \left(T^*, u^*(\cdot), \mathbf{x}^*\left(\cdot;\Omega_p\right)\right)$ be feasible for Problem (B.10) with

$$\mathbf{x}_3\left(1;\Omega_p\right) = 10 \quad \text{and} \quad \mathbf{x}_2\left(1;\Omega_p\right) = 0.$$

Then there is a feasible $\left(T', u'(\cdot), \mathbf{x}'\left(\cdot;\Omega_p\right)\right)$ with $\| u'(\cdot) - u(\cdot)\|_\infty < \delta_u$, $|T' - T| < \delta_T$, $T' \leq T$, and $\mathbf{x}'_3\left(1;\Omega_p\right) > 10$.

Proof Since the global optimum of Problem (B.10) is unique (see Corollary B.15) we have $T > T^*$. Let $t_1 = \frac{4}{T(10-p_l)}$ and $t_2 = 1 - \frac{4}{T(10+p_l)}$. We make a case distinction as in the proof of Proposition B.8.

Case 1): $u(\cdot) \not\equiv 10$ on $[0, t_1]$ (almost surely). Similar to Case 1) in the proof of Proposition B.8, there is a control function $u'(\cdot)$ with $\| u'(\cdot) - u(\cdot)\|_\infty < \delta_u$ such that $\left(T, u'(\cdot), \mathbf{x}'\left(\cdot;\Omega_p\right)\right)$ is feasible, where $\mathbf{x}'\left(\cdot;\Omega_p\right)$ denotes the differential states which are determined by T, $u'(\cdot)$, and Ω_p. We have $\mathbf{x}_2\left(t;\Omega_p\right) \leq \mathbf{x}'_2\left(t;\Omega_p\right)$ for all $t \in [0,1]$ and $\mathbf{x}_2\left(t';\Omega_p\right) < \mathbf{x}'_2\left(t';\Omega_p\right)$ for some $t' \in [0,1]$. Due to $\mathbf{x}_2\left(t;\Omega_p\right) = \mathbf{x}_4\left(t;\Omega_p\right) + (p_u - p_l)Tt$

(and similarly for $\mathbf{x}'\left(\cdot;\Omega_p\right)$) and $\mathbf{x}_3\left(1;\Omega_p\right) = 10$ we get $\mathbf{x}_3'\left(1;\Omega_p\right) > 10$.

Case 2): $u(\cdot) \equiv 10$ on $[0, t_1]$ (almost surely). If we have $\mathbf{x}_2\left(t;\Omega_p\right) = 4$ for $t \in [t_1, t_2]$, then $u(\cdot) \equiv p_l$ on $[t_1, t_2]$ (almost surely) and furthermore $u(\cdot) \equiv -10$ on $[t_2, 1]$ (almost surely) due to the terminal constraint $\mathbf{x}_2\left(1;\Omega_p\right) \leq 0$. In particular, $u(\cdot) = u_T\left(\cdot;\Omega_p\right)$. We can apply Proposition B.19 to achieve the targeted result. For the remainder, we assume that there is a $t \in [t_1, t_2]$ with $\mathbf{x}_2\left(t;\Omega_p\right) < 4$. We can proceed similar to Case 2) in the proof of Proposition B.8 to show that there is a control function $u'(\cdot)$ with $\|u'(\cdot) - u(\cdot)\|_\infty < \delta_u$ such that $\left(T, u'(\cdot), \mathbf{x}'\left(\cdot;\Omega_p\right)\right)$ is feasible, if $\mathbf{x}'\left(\cdot;\Omega_p\right)$ denotes the differential states which are determined by T, $u'(\cdot)$, and Ω_p. We have $\mathbf{x}_2\left(t;\Omega_p\right) \leq \mathbf{x}_2'\left(t;\Omega_p\right)$ for all $t \in [0,1]$ and $\mathbf{x}_2\left(t';\Omega_p\right) < \mathbf{x}_2'\left(t';\Omega_p\right)$ for some t'. This implies $\mathbf{x}_3'\left(1;\Omega_p\right) > 10$ as in Case 1), see above. □

Next, we show that Problem (B.10) has exactly one local minimum.

Proposition B.21
Let the feasible set of Problem (B.10) be non-empty. The tuple $\left(T^*, u^*(\cdot), \mathbf{x}^*\left(\cdot;\Omega_p\right)\right)$ is the only local optimum of Problem (B.10) in the considered normed space.

Proof The proof works similar to the proof of Proposition B.9. We know that the tuple $\left(T^*, u^*(\cdot), \mathbf{x}^*\left(\cdot;\Omega_p\right)\right)$ is the unique global minimum of Problem (6.9), see Corollary B.15. Let $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right) \neq \left(T^*, u^*(\cdot), \mathbf{x}^*\left(\cdot;\Omega_p\right)\right)$ be feasible for Problem (6.9). Since the global minimum is unique, we have $T > T^*$. Let $\delta_T, \delta_u > 0$. We distinct two cases:

Case 1): $\mathbf{x}_3\left(1;\Omega_p\right) = 10$. Depending on whether Constraint (B.10f) is satisfied with equality or not, we can apply Lemma B.16 or Proposition B.20 to get a feasible tuple $\left(T', u'(\cdot), \mathbf{x}'\left(\cdot;\Omega_p\right)\right)$ with $\|u'(\cdot) - u(\cdot)\|_\infty < \delta_u$, $|T' - T| < \frac{\delta_T}{2}$, $T' \leq T$, and $\mathbf{x}_3'\left(1;\Omega_p\right) > 10$. Subsequently, we apply Lemma B.17 and find a feasible $\left(T'', u''(\cdot), \mathbf{x}''\left(\cdot;\Omega_p\right)\right)$ with $u''(\cdot) = u'(\cdot)$, $|T'' - T| \leq |T'' - T'| + |T' - T| < \delta_T$, and $T'' < T' \leq T$.

Case 2): $\mathbf{x}_3\left(1;\Omega_p\right) > 10$. In this case, we can directly apply Lemma B.17 and find a feasible $\left(T', u'(\cdot), \mathbf{x}'\left(\cdot;\Omega_p\right)\right)$ with $u'(\cdot) = u(\cdot)$, $|T' - T| < \delta_T$, and $T' < T$.

Using Lemma B.10 we conclude that $\left(T, u(\cdot), \mathbf{x}\left(\cdot;\Omega_p\right)\right)$ is not a local minimum. □

It remains to transfer the established results for Problem (B.10) to the original Problem (6.10). We get

Corollary B.22
Let the feasible set of Problem (B.10) be non-empty. Let T^* and $u^*(\cdot)$ be the globally optimal controllable parameter and control function of Problem (B.10). For given T^*, $u^*(\cdot)$, and any $p \in \Omega_p$, we denote the solution of the resulting IVP (6.10b-6.10c) by $\mathbf{x}^*(\cdot\,;p)$. Then the global solutions of Problem (6.10) are given by

$$\left\{\left(T^*, u^*(\cdot), p, \mathbf{x}^*(\cdot\,;p)\right) \,\middle|\, p \in \Omega_p\right\} .$$

Furthermore, if $\left(T, u(\cdot), p, \mathbf{x}(\cdot\,;p)\right)$ is a local solution of Problem (6.10) in the considered normed space, then $T = T^*$ and $u(\cdot) = u^*(\cdot)$. In particular, every local solution is a global solution.

Proof Due to Corollary B.15 and Lemma B.11 the global solutions of Problem (6.10) are given by

$$\left\{\left(T^*, u^*(\cdot), p, \mathbf{x}^*(\cdot\,;p)\right) \,\middle|\, p \in \Omega_p\right\} .$$

If $\left(T, u(\cdot), p, \mathbf{x}(\cdot\,;p)\right)$ is a local solution of Problem (6.10), again by Lemma B.11 we know that $\left(T, u(\cdot), \mathbf{x}\left(\cdot\,;\Omega_p\right)\right)$ is a local solution of Problem (B.10). From Proposition B.21 we get $T = T^*$ and $u(\cdot) = u^*(\cdot)$, as claimed. $\square$