5 Budget equations for domain mean quantities

The evolution of the Paluch diagram can be understood by looking at the budget equation for the domain-mean h and s_{vl}. Horizontal domain averaged is denoted by the overline $\overline{()}$.

5.1 Derivation

Starting from two budget equations in BB2017 (Eq. 3 and 7)

$$\frac{\partial s_{vl}}{\partial t} + \vec{v} \cdot \nabla s_{vl} = -\frac{1}{\rho_o}\frac{\partial}{\partial z}\left[\mu L_v \overline{P} + \overline{F_R}\right] + \left(\frac{\partial \overline{s_{vl}}}{\partial t}\right)_{LS} - w_{LS}\frac{\partial s'_{vl}}{\partial z} + \left(\frac{\partial s_{vl}}{\partial t}\right)_{SGS} \tag{12}$$

$$\frac{\partial q_t}{\partial t} + \vec{v} \cdot \nabla q_t = -\frac{1}{\rho_o}\frac{\partial P}{\partial z} + \left(\frac{\partial \overline{q_t}}{\partial t}\right)_{LS} - w_{LS}\frac{\partial q'_t}{\partial z} + \left(\frac{\partial q_t}{\partial t}\right)_{SGS} \tag{13}$$

Note:

1. $\vec{v} = (u\hat{i}, v\hat{j}, w\hat{k}) = u_i$ with i=1 3.
2. $f = \overline{f} + f'$, where $f' = f_m + f_c$
3. The $\left(\frac{\partial \overline{f}}{\partial t}\right)_{LS}$ term is the horizontally uniform component ($f(z)$) of the large scale advective forcing, which is prescribed in the study.

Perform domain averaging on the horizontal plane (x,y), I obtained the following equations (Eq. (17) to Eq. (20)) The part that confused me a bit is the domain averaging of the advection term in the material derivative.

Using the mass conservation or continuity equation:

$$\begin{aligned}
\vec{v} \cdot \nabla s_{vl} &= \nabla \cdot (\vec{v} s_{vl}) \\
\overline{\vec{v} \cdot \nabla s_{vl}} &= \nabla \cdot \overline{(\vec{v} s_{vl})} \\
&= \frac{\partial}{\partial x_i}\overline{(\overline{s_{vl}} + s')(\overline{u_i} + u')} \\
&= \frac{\partial \overline{s_{vl}}\,\overline{u_i}}{\partial x_i} + \frac{\partial \overline{s'_{vl} u'_i}}{\partial x_i}
\end{aligned} \tag{14}$$

Because we know: $\frac{\partial u_i}{\partial x_i} = \frac{\partial \overline{u}_i}{\partial x_i} + \frac{\partial \overline{u'_i}}{\partial x_i} = 0$, $\frac{\partial \overline{u'_i}}{\partial x_i} = 0$, and so $\frac{\partial \overline{u}_i}{\partial x_i} = 0$.

Also, domain-mean quantities only have structure in the vertical direction, so the continuity equation for the mean velocity means $\frac{\partial \overline{u}(z)}{\partial x} + \frac{\partial \overline{v}(z)}{\partial y} + \frac{\partial \overline{w}(z)}{\partial z} = \frac{\partial \overline{w}(z)}{\partial z} = 0$

Since w is expected to be 0 at the surface, so $d\overline{w}/dz = 0$ means that the domain-mean vertical velocity is zero at all level..

Therefore, the first term in the domain-mean advection term Eq. (14) is reduced to

$$\frac{\partial \overline{s_{vl}}\,\overline{u_i}}{\partial x_i} = \frac{\partial \overline{s_{vl}}\,\overline{w}}{\partial z} \tag{15}$$

$$= \overline{w}\frac{\partial \overline{s_{vl}}}{\partial z} = 0 \tag{16}$$

$$\frac{\partial \overline{s_{vl}}}{\partial t} = \left(\frac{\partial \overline{s_{vl}}}{\partial t}\right)_{LS} - \frac{1}{\rho_o}\frac{\partial}{\partial z}\left[\mu L_v \overline{P} + \overline{F_R}\right] - \frac{\partial \overline{w' S'_{vl}}}{\partial z} + \left(\frac{\partial \overline{s_{vl}}}{\partial t}\right)_{SGS} \tag{17}$$

$$\frac{\partial \overline{q_t}}{\partial t} = \left(\frac{\partial \overline{q_t}}{\partial t}\right)_{LS} + \frac{1}{\rho_o}\frac{\partial \overline{P}}{\partial z} - \frac{\partial \overline{w' q'_t}}{\partial z} + \left(\frac{\partial \overline{q_t}}{\partial t}\right)_{SGS} \tag{18}$$

$$h = s_{vl} + \mu L_v q_t \tag{19}$$

Combining Eq. (17), Eq. (18) using Eq. (19), we obtain,

$$\frac{\partial \overline{h}}{\partial t} = \left(\frac{\partial \overline{h}}{\partial t}\right)_{LS} - \frac{1}{\rho_o}\frac{\partial \overline{F_R}}{\partial z} - \frac{\partial \overline{w' h'}}{\partial z} + \left(\frac{\partial \overline{h}}{\partial t}\right)_{SGS} \tag{20}$$

where $u' = u_m + u_c$, $i = 1, 2, 3$, SGS denotes the sub-grid-scale turbulent effect on the tendency of domain mean quantity, and is likely small compared to the rest of the terms.

5.2 Interpretation and Evaluation

There are three common processes that contribute to the domain-mean tendency of s_{vl} and MSE, if assuming that the SGS turbulent effect is insignificant.

i) the large scale advective forcing (e.g., cold advection)

ii) the domain mean radiative forcing (e.g., radiative cooling)

iii) the "turbulence flux" term (mixing effect), which include the "mesoscale turbulence eddies", the cumulus scale eddies (microsacle), and their cross terms.

For the s_{vl}, the precipitation induced domain-mean latent heating is another process that affect the liquid virtual static energy (surrogate for buoyancy according to BB2017).

5.2.1 How do I compute these source terms from the LES output?

I found the following domain-mean quantities from the "OUT_STAT/*.nc", but would like to confirm with Peter.

terms	variable name	long name (units)
$\overline{P}$	PRECIP	precipitation flux (mm/day)
$\overline{F_R}(z)$	RADQR	Radiative heating rate (K/day)
$-\frac{\partial \overline{w'h'}}{\partial z}$	$Q1C - Q2$	$Q1C = Q1 - QR$ (K/day), apparent heat source as in Yanai Q2: apparent moisture source sink
$-\frac{\partial \overline{w's'_{vl}}}{\partial z}$	$Q1C - \frac{1}{\rho_o}\frac{\partial \overline{P}}{\partial z}$ (?)	according to Yanai's definition. Q1C: apparent heat source
$(\frac{\partial \overline{q_t}}{\partial t})_{LS}$	QVTEND + QHTEND	large-scale vertical, horizontal advection moisture tendency (g/kg/day)
$(\frac{\partial \overline{s_{vl}}}{\partial t})_{LS}$	THTEND + TVTEND	large-scale vertical, horizontal advection temperature tendency (K/day)