YOU MUST USE THE CONSTRUCTIONS GIVEN IN CLASS

1. Construct a regular expression over {a,b,c} for the language accepted by this nfa: ✓

	a	b	c	
→ A	/	B	/	0
B	B	/	A,C	1
C	/	B,C	/	1

2. Prove that the language L(G) is not regular where G is the following cfg: ✓

$G = (\{S,A,B,C\}, \{a,b,c\}, \{S \to Aa|B|C, A \to aS, B \to a, C \to b\}, S)$.

Note: You <u>must</u> first determine L(G).

3. Construct a reduced dfa for the following extended regular expression over {0,1,2}: ✓

$$[(100^*)^* \cap \overline{1^*}\ ]$$

Note: You <u>must</u> first determine nfas for $(100^*)^*$ and 1^*, then do the intersection. The answer must then be reduced.

4. Construct a Chomsky normal form grammar for L(G) for the following cfg G:

$G = (\{S,B\}, \{a,b,c,d\}, \{S \to SaSbc|Ba, B \to cBd|S|\varepsilon\}, S)$.

Note: You <u>must</u> first remove all ε- and all unit productions.

5. Construct a Greibach normal form grammar for L(G) for the following CNF G:

$G = (\{S,A\}, \{a,d\}, \{S \to AS|A|d, A \to SS|a\}, S)$.

Note: You <u>must</u> first remove all unit productions. You <u>must</u> derive all the productions for S and A; indicate how the result looks for S' and A' as applicable.

6. Prove that the following language L is not contextfree: $L = \{\ 0^{n+1} 1^{n-1} 0^{n} \mid n>0\ \}$. $\{0^{n+2}1^{n}0^{n+1} \mid n \geq 0\}$

7. Consider the class $\mathcal{L}_A$ of all contextfree languages over the fixed alphabet A.

(a) Is $\mathcal{L}_A$ countable? Yes

(b) Is the class $\mathcal{M}_A$ countable where $\mathcal{M}_A$ consists of all languages over A that are not in $\mathcal{L}_A$? No.

(c) Is the class $\mathcal{L}_A \cap \mathcal{M}_A$ countable? Yes.

For each question, you must give a precise argument substantiating your answer.

8. Construct a Turing machine for the language in Question 6, $L = \{\ 0^{n+1} 1^{n-1} 0^{n} \mid n>0\ \}$.

Note: Describe first the process in English; then translate this into moves of the Turing machine.

9. Let L_1 and L_2 be arbitrary languages, subject to the specification in either (i) or (ii). Consider the following four general questions:

$\alpha - \beta = \alpha \cap \overline{\beta}$

(Q1) Does $\overline{L_1} - L_2$ contain a given fixed word w? rec /not re (Q2) Is $\overline{L_1} - L_2$ non-empty? r.e. but not rec/not r

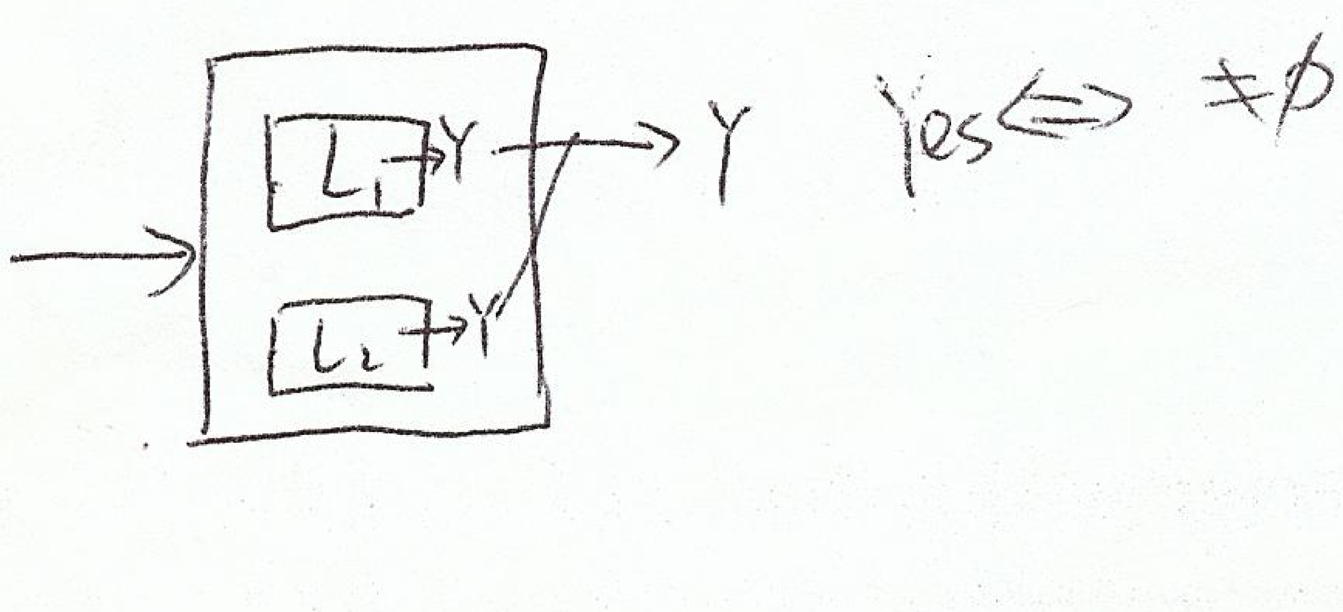
(Q3) Does $L_1 \cap L_2$ contain a given fixed word w? rec (Q4) Is $L_1 \cap L_2$ non-empty? r.e. but not rec. /re but not rec.

For each of these four questions **<u>explain with reasons</u>** whether the problem is <u>recursive</u>, <u>not recursive but r. e.</u>, or <u>non-r. e.</u>, provided /r.e. but not rec.

(i) Both L_1 and L_2 are <u>recursive</u>. (ii) Both L_1 and L_2 are <u>r. e., but not recursive</u>.

Points: 1: 6 2: 8 3: 14 4: 12 5: 12 6: 12 7: 13 8: 8 9:15

L_1 rec ⇒ $\overline{L_1}$ rec



The Power Set of a Countably Infinite Set is Uncountable

Theorem 1
If S is a countably infinite set, 2^S (the power set) is uncountably infinite.
Proof: We show 2^S is uncountably infinite by showing that $2^{\mathbb{N}}$ is uncountably infinite. (Given the natural bijection that exists between $2^{\mathbb{N}}$ and 2^S –because of the bijection that exists from $\mathbb{N}$ to S– it is sufficient to show that $2^{\mathbb{N}}$ is uncountably infinite.)

Assume that the set $2^{\mathbb{N}}$ is countably infinite.

The subsets of $\mathbb{N}$ can be listed $A_0, A_1, A_2, \ldots$ so that every subset is A_i for some i.

We define another set $A = \{i | i \geq 0 \text{ and } i \notin A_i\}$ which contains those integers i which are not members of their namesake set A_i.

But A is a subset of $\mathbb{N}$, and so $A = A_j$ for some j. But this means

1. If $j \in A$, then $j \notin A$.
2. If $j \notin A$, then $j \in A$.

We have a contradiction, since j must either be in the set A or not in the set. Therefore $2^{\mathbb{N}}$ is not countably infinite. $\diamond$

The Theorem that the power set of a countably infinite set is an uncountable set indicates that the set of all languages over any alphabet Σ, $|\Sigma| \neq 0$ is uncountable. (2^{Σ^*} is an uncountable set.)

1.

$$\begin{array}{c|cccc} & a & b & c & \\ \hline A & / & B & / & 0 \\ B & B & / & A,C & 1 \\ C & / & B,C & / & 1 \end{array} \Rightarrow \begin{cases} L_A = bL_B \\ L_B = aL_B \cup cL_A \cup cL_c \cup \varepsilon \\ L_c = bL_B \cup bL_c \cup \varepsilon \end{cases}$$

$L_c = bL_c \cup (bL_B \cup \varepsilon) \Rightarrow L_c = b^*(bL_B \cup \varepsilon) = b^*bL_B \cup b^*$

$L_B = aL_B \cup (cL_A \cup cL_c \cup \varepsilon) \Rightarrow L_B = a^*(cL_A \cup cL_c \cup \varepsilon)$

$= a^*cL_A \cup a^*cL_c \cup a^*$

$= a^*cbL_B \cup a^*c(b^*bL_B \cup b^*) \cup a^*$

$= (a^*cb \cup a^*cb^*b)L_B \cup a^*cb^* \cup a^*$

$\Rightarrow L_B = (a^*cb \cup a^*cb^*b)^*(a^*cb^* \cup a^*)$

$\Rightarrow L_A = bL_B = b(a^*cb \cup a^*cb^*b)^*(a^*cb^* \cup a^*)$

$\Rightarrow L_c = b^*b(a^*cb \cup a^*cb^*b)^*(a^*cb^* \cup a^*) \cup b^*$

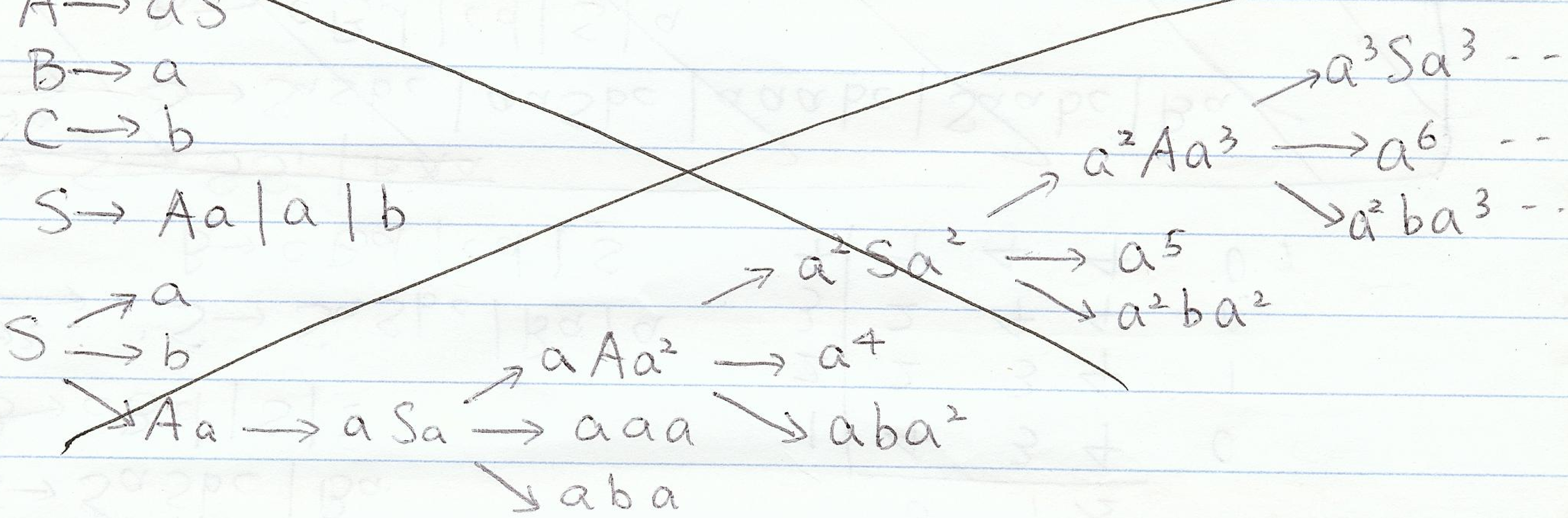
2. $S \to Aa \mid B \mid C$ $\quad 2n+1$

$A \to aS$

$B \to a$

$C \to b$

$\Rightarrow S \to Aa \mid a \mid b$



$L_1 = \{a^n \mid n \geq 0\}$ $\quad L_2 = \{a^{n-1}ba^n \mid n \geq 0\}$ $\quad L_3 = \{a^nba^n \mid n \geq 0\}$

$L = L_1 \cup L_2 \cup L_3$ Proving any of the L_1, L_2, L_3 is not regular will prove L is not regular. e.g. L_2 ...

3. $(100^*)^* \cap \overline{1^*} = \overline{\overline{(100^*)^*} \cup 1^*}$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & \\ \hline 0 & / & 1 & / & 0 \\ 1 & 2 & / & / & 0 \\ 2 & / & / & / & 1 \end{array} \quad \begin{array}{c|cccc} & 0 & 1 & 2 & \\ \hline 0 & 3 & / & / & 0 \\ 3 & / & / & / & 1 \end{array} \quad \begin{array}{c|cccc} & 0 & 1 & 2 & \\ \hline 0 & / & 5 & / & 0 \\ 5 & / & / & / & 1 \end{array} \xrightarrow{*} \begin{array}{c|cccc} & 0 & 1 & 2 & \\ \hline 0 & / & 5 & / & 1 \\ 5 & / & 5 & / & 1 \end{array}$$

$\downarrow *$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & \\ \hline 0 & 3 & / & / & 1 \\ 3 & 3 & / & / & 1 \end{array}$$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & \\ \hline 0 & / & 1 & / & 0 \\ 1 & 2 & / & / & 0 \\ 2 & 3 & / & / & 1 \\ 3 & 3 & / & / & 1 \end{array} \quad \begin{array}{c|cccc} & 0 & 1 & 2 & \\ \hline 0 & 4 & 5 & 4 & 1 \\ 1 & 2 & 4 & 4 & 1 \\ 2 & \cancel{3} & \cancel{1} & 4 & \cancel{1}\,0 \\ 3 & 3 & 1 & 4 & 0 \\ 4 & 4 & 4 & 4 & 1 \\ 5 & / & 5 & / & 1 \end{array}$$

$\cup$

$\downarrow *$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & \\ \hline 0 & / & 1 & / & 1 \\ 1 & 2 & / & / & 0 \\ 2 & 3 & 1 & / & 1 \\ 3 & 3 & 1 & / & 1 \end{array} \xrightarrow{DFA} \begin{array}{c|cccc} & 0 & 1 & 2 & \\ \hline 0 & / & 1 & / & 1 \\ \cancel{1} & / & / & / & 0 \\ 1 & 2 & / & / & 0 \\ 2 & 3 & 1 & / & 1 \\ 3 & 3 & 1 & / & 1 \end{array} \xrightarrow{\text{complement}} \begin{array}{c|cccc} & 0 & 1 & 2 & \\ \hline 0 & 4 & 1 & 4 & 0 \\ 4 & 4 & 4 & 4 & 1 \\ 1 & 2 & 4 & 4 & 1 \\ 2 & 3 & 1 & 4 & 0 \\ 3 & 2 & 1 & 4 & 0 \end{array}$$

	0	1	2	
0	4	15	4	1
1	2	4	4	1
2	3	1	4	0
3	3	1	4	0
4	4	4	4	1
5	/	5	/	1

$\xrightarrow{\text{dfa}}$

	0	1	2	
0	4	15	4	1
4	4	4	4	1
15	2	45	4	1
2	3	1	4	0
45	4	45	4	1
3	3	1	4	0
1	2	4	4	1

$\xrightarrow{\text{complement}}$

		0	1	2	
A	0	4	15	4	0
B	4	4	4	4	0
C	15	2	45	4	0
D	2	3	1	4	1
E	45	4	45	4	0
F	3	3	1	4	1
G	1	2	4	4	0

	0	1	2	
⇒A	B	C	B	0
B	B	B	B	0
C	D	E	B	0
D	F	G	B	1
E	B	E	B	0
F	F	G	B	1
G	D	B	B	0

⇒

F

DF | ABCEG F

DF | CG|AB~~C~~E F

DF | A|CG|BE F

① ② ③ ④ F

2 1 3 4 ①

~~ABCDEG~~

~~D| ABCEG~~

~~CG|D|ABE~~

~~B|CG|D|AE~~

~~|A|B|C|D|E|F|G~~

~~② ③ ④ ⑤ ⑥ ⑦~~

~~7 states.~~

4. $S \to SaSbc \mid Ba$

$B \to cBd \mid S \mid \varepsilon$.

	0	1	2	
1	4	3	4	0
2	2	3	4	1
3	2	4	4	0
4	4	4	4	0

$\xrightarrow{B\to\varepsilon}$ $S \to SaSbc \mid Ba \mid a$

$B \to cBd \mid cd \mid S$

~~⇒ S → SS₁ | BXa~~

$\xrightarrow{S\to a}$ $S \to SaSbc \mid aaSbc \mid aaabc \mid Saabc \mid Ba$

$B \to cBd \mid cd \mid S \mid a$

$\xrightarrow{B\to a}$ $S \to SaSbc \mid aaSbc \mid aaabc \mid Saabc \mid Ba \mid aa$

$B \to cBd \mid cad \mid cd \mid S$

$\xrightarrow{B\to S}$ $S \to SaSbc \mid Ba \mid Sa \mid a$

$B \to cBd \mid cSd \mid cd$

$\Rightarrow S \to SS_1 \mid BX_a \mid SX_a \mid a$

$S_1 \to X_a S_2$

$S_2 \to S S_3$

$S_3 \to X_b X_c$

$B \to CB_1 \mid CB_2 \mid X_c X_d$

$B_1 \to B X_d$

$B_2 \to S X_d$

$X_a \to a$

$X_b \to b$

$X_c \to c$

$X_d \to d$

5. $\underline{S} \to AS \mid \underline{A} \mid d$

$A \to SS \mid a$

$\underrightarrow{S \to A}$ A_1 $S \to AS \mid AA \mid d$

A_2 $A \to SS \mid SA \mid AS \mid AA \mid a$.

renaming. $A_1 \to A_2A_1 \mid A_2A_2 \mid d$

$A_2 \to \underline{A_1}A_1 \mid \underline{A_1}A_2 \mid A_2A_1 \mid A_2A_2 \mid a$

$i=1$, we are done.

$i=2$, $A_2 \to \underline{A_2}\overbrace{A_1A_1}^{\alpha_1} \mid \underline{A_2}\overbrace{A_2A_1}^{\alpha_2} \mid \overbrace{dA_1}^{\beta_1} \mid \underline{A_2}\overbrace{A_1A_2}^{\alpha_3} \mid \underline{A_2}\overbrace{A_2A_2}^{\alpha_4} \mid \overbrace{dA_2}^{\beta_2}$

$\mid \underline{A_2}\underbrace{A_1}_{\alpha_5} \mid \underline{A_2}\underbrace{A_2}_{\alpha_6} \mid \underbrace{a}_{\beta_3}$

$\Rightarrow A_2 \to dA_1 \mid dA_2 \mid a \mid dA_1A_2' \mid dA_2A_2' \mid aA_2'$

$A_2' \to A_1A_1 \mid A_2A_1 \mid A_1A_2 \mid A_2A_2 \mid A_1 \mid A_2 \mid A_1A_1A_2' \mid A_2A_1A_2'$

$\mid A_1A_2A_2' \mid A_2A_2A_2' \mid A_1A_2' \mid A_2A_2'$

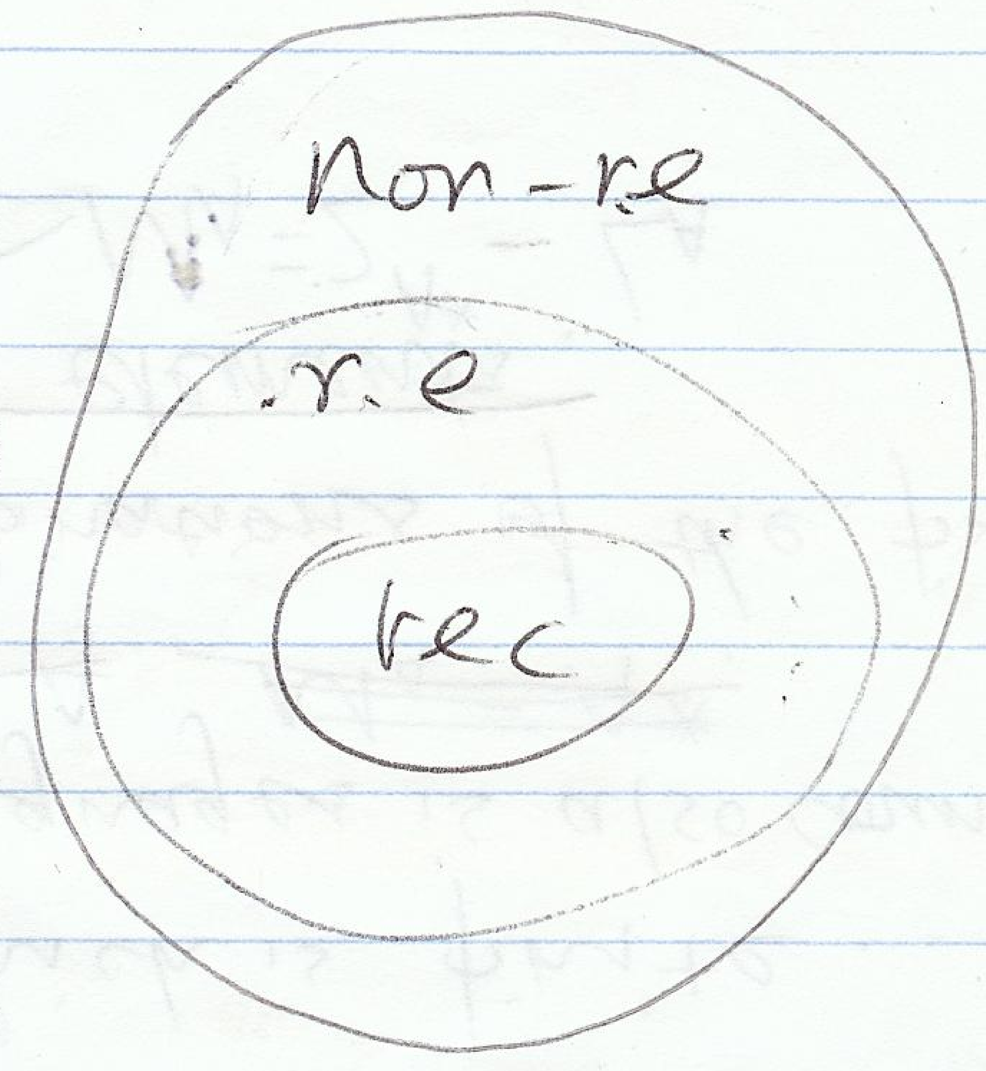
$\Rightarrow A_1 \to dA_1A_1 \mid dA_2A_1 \mid aA_1 \mid dA_1A_2'A_1 \mid dA_2A_2'A_1 \mid aA_2'A_1$

$\mid dA_1A_2 \mid dA_2A_2 \mid aA_2 \mid dA_1A_2'A_2 \mid dA_2A_2'A_2 \mid aA_2'A_2 \mid$

$\underrightarrow{\text{change back to}}$ S A.

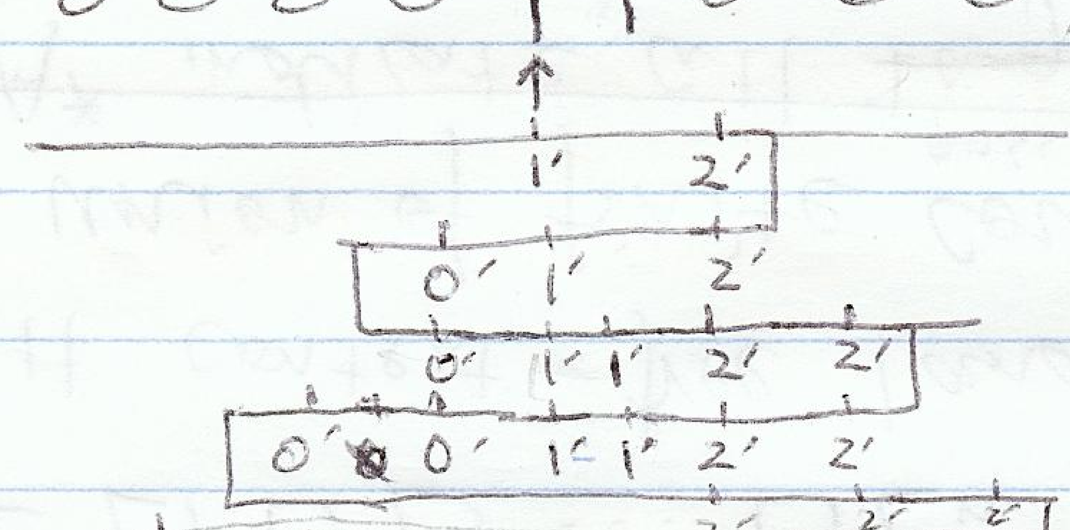
6. $L = \{0^{n+1}1^{n-1}0^n \mid n>0\}$

$\Rightarrow L = \{0^{n+2}1^n0^{n+1} \mid n \geq 0\}$ .....



8. Turing machine $L = \{0^{n+2}1^n0^{n+1} \mid n \geq 0\}$

$\not b$ 0 0 0 0 1 1 0 0 0 $\not b$



	0	1	0'	1'	2'	$\not b$
q_0	$(q_0, 0, R)$	$(q_1, 1', R)$				$(q_{10}, \not b, L)$
q_1	$(q_2, 2', L)$	$(q_1, 1, R)$			$(q_2, 2', R)$	
q_2	$(q_3, 0', R)$	$(q_2, 1, L)$	$(q_2, 0', L)$	$(q_2, 1', L)$	$(q_2, 2', L)$	
q_3		~~(q, , R)~~	$(q_3, 0', R)$	$(q_4, 1', R)$		
q_4		$(q_1, 1', R)$			$(q_5, 2', R)$	
q_5	$(q_6, 2', L)$				$(q_5, 2', R)$	
q_6	$(q_7, 0', R)$		$(q_6, 0', L)$	$(q_6, 1', L)$	$(q_6, 2', L)$	
q_7	$(q_6, 2', L)$		$(q_7, 0', R)$	$(q_7, 1', R)$	$(q_5, 2', R)$	$(q_8, \not b, L)$
q_8	$(q_9, 0', L)$		$(q_8, 0', L)$	$(q_8, 1', L)$	$(q_8, 2', L)$	
q_9	$(q_9, 0', L)$					$(q_f, \not b, L)$
q_{10}	$(q_{11}, 0', L)$					
q_{11}	$(q_{12}, 0', L)$					
q_{12}	$(q_{13}, 0', L)$					
q_{13}						$(q_f, \not b$

need one more state if no $\not b$ on left.

2. $S \to Aa \mid B \mid C$ $\qquad$ $S \to Aa \mid a \mid b$

$A \to aS$ $\qquad$ $\Rightarrow S \to aSa \mid a \mid b$

$B \to a$

$C \to b$

$\Rightarrow S \to aSa,\ a,\ b$

$aSa \to a^2Sa^2,\ a^3,\ aba$

$a^2Sa^2 \to a^3Sa^3,\ a^5,\ a^2ba^2$

$a^3Sa^3 \to a^4Sa^4 \ldots,\ a^7,\ a^3ba^3$

$\Rightarrow L_1 = \{a^n b a^n \mid n \geq 0\}$ $\quad L_2 = \{a^{2n+1} \mid n \geq 0\}$

and $L = L_1 \cup L_2$

7.(1) L_A is a union of context free language if it is accepted by a PDA, and PDA is finite automata, thus countable.

$\Rightarrow L_A = L_1 \cup L_2 \cdots \cup L_n$, where n is the ~~set~~ number of all context-free lanaguages, which is finite.

$\Rightarrow$ the union of finite countable lanaguages is also countab[le]

(2) ~~Let A^* indicate all possible languages over A.~~ ~~$M_A = A^*$~~

$\Rightarrow$ Let A^* indicate all possible Combinations of the fixed alphabets. Then ~~M_A includes 2^{A^*} elements.~~ $M_A = 2^{A^*} - L_A$

$\because A^*$ is countably infinite.

$\therefore 2^{A^*}$ is uncountable

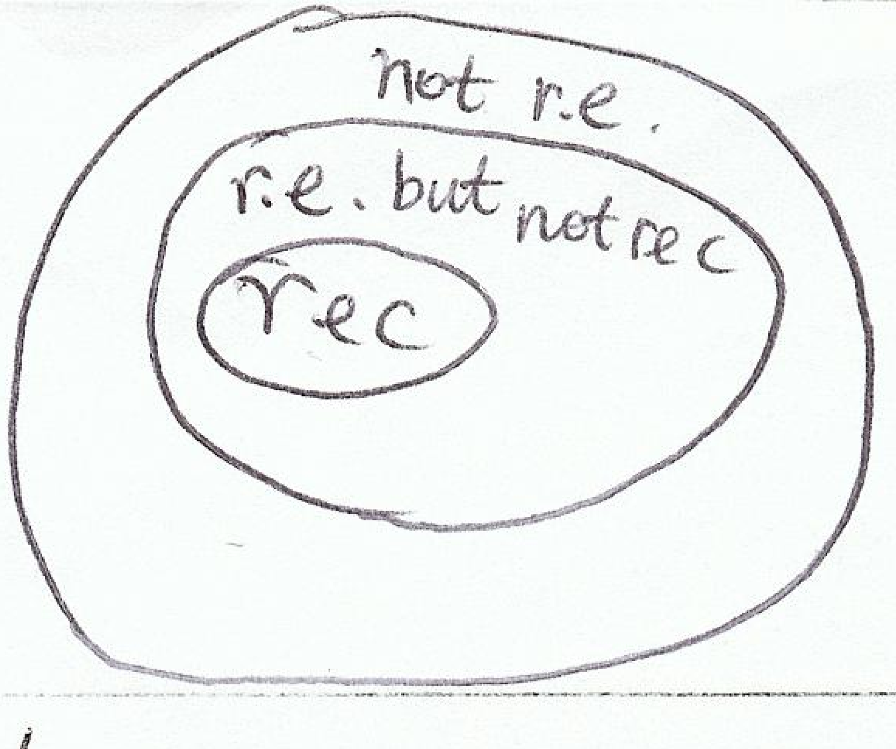
$\therefore 2^{A^*} - L_A$ is uncountable.

(3) $L_A \cap M_A = \{\phi\}$ so it's countable.

example: Is $\overline{L_1} \cdot \overline{L_2} \neq \phi$?

① rec. if L_1, L_2 are rec, then we can go through all the word[s] until we find a word that in the language. if $\overline{L_1} \cdot \overline{L_2} = \phi$, then TM won't halt, so it's r.e, not rec.

② r.e. not rec: It's not r.e.



$\overline{L_1} - L_2 = \overline{L_1} \cap \overline{L_2}$

9. (i) Both L_1 and L_2 are recursive.

(Q1) L_1 is rec.

$\Rightarrow \overline{L_1}$ is also rec.

Since $\overline{L_1}$ and L_2 are recursive

$\therefore \overline{L_1} - L_2$ is recursive, which means always halt for a given fixed word w belongs to $\overline{L_1}$ but not L_2

(Q2) $\overline{L_1} - L_2 \neq \emptyset$?

$\overline{L_1}$ is recursive $\quad$ $\overline{L_1}$ is recursive, $\overline{L_2}$ is recursive

L_2 is recursive

For decide $\overline{L_1} - L_2 = \emptyset$, the TM for $\overline{L_1} - L_2$ has to run forever to enumerate all the possibilities with no guarantee to stop. But for $\overline{L_1} - L_2 \neq \emptyset$, once we find a word in $\overline{L_1} - L_2$ and not empty, the TM will halt on the input and say yes.

$\Rightarrow \overline{L_1} - L_2 \neq \emptyset$? is r.e. but not recursive.

(Q3) $L_1 \cap L_2$ Contain a given fixed word w?

Since L_1 and L_2 are recursive

TM for $L_1 \cap L_2$ can always answer yes for word w in $L_1 \cap L_2$ and say no when $w \notin (L_1 \cap L_2)$

$\therefore L_1 \cap L_2$ is recursive.

(Q4) $L_1 \cap L_2 \neq \emptyset$?

TM for $L_1 \cap L_2$, simulate input word w for $L_1 \cap L_2$, when $w \in L_1 \cap L_2$ and $w \notin \emptyset$, the TM halts and answer yes. but for $L_1 \cap L_2 = \emptyset$, the TM has to go through all the cases for $L_1 \cap L_2$ and runs forever with no guarantee to stop and say no.

$\therefore L_1 \cap L_2 \neq \emptyset$? is r.e but not recursive.

(ii) Both L_1 and L_2 are r.e. but not recursive.

(Q1) $\overline{L_1} - L_2$

$\because L_1$ is r.e but not recursive

∴ there is no way for $\overline{L_1}$ halts on input w and no such TM for $\overline{L_1} - L_2$

∴ $\overline{L_1} - L_2$ is not r.e.

(Q2) $\overline{L_1} - L_2 \neq \emptyset$?

∵ L_1 is r.e. but not recursive

∴ no TM for $\overline{L_1}$

∴ $\overline{L_1} - L_2$ is non r.e.

(Q3) $L_1 \cap L_2$ contain a given fixed word w.

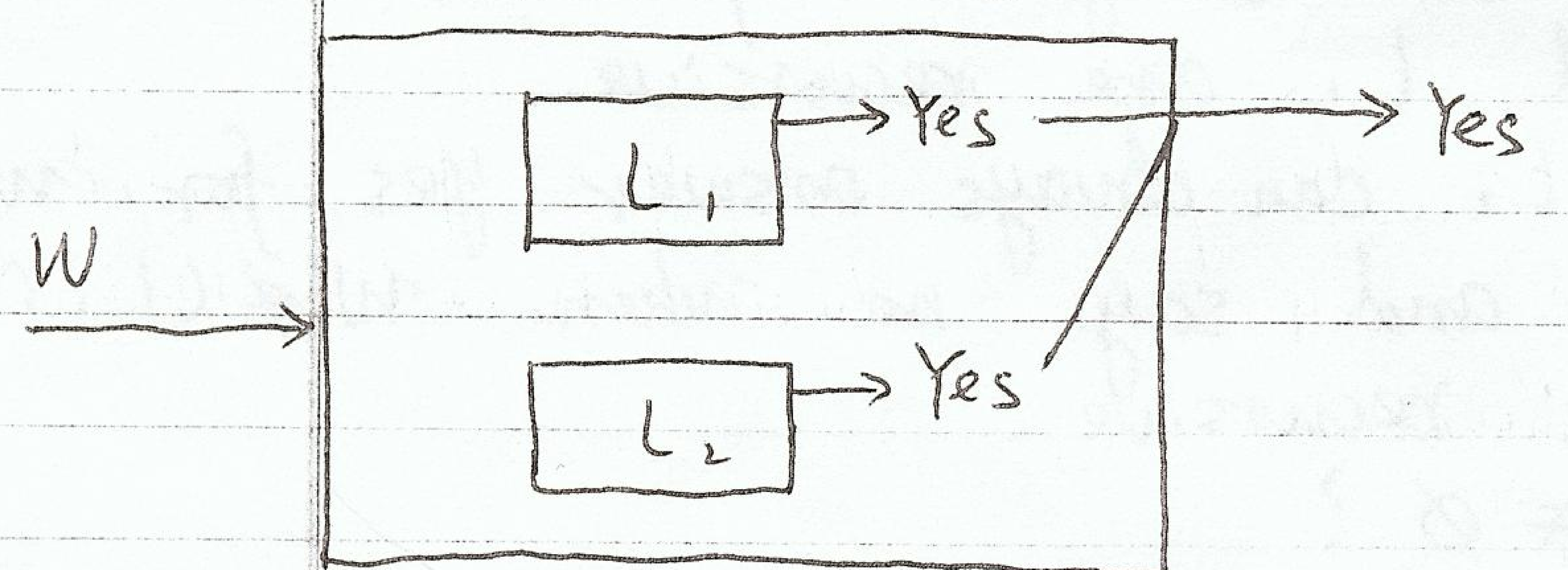
∵ L_1 and L_2 is r.e. but not recursive

∴ when $w \in L_1$ and $w \in L_2$, the TM for $L_1 \cap L_2$ always answers yes on input w.

∴ $L_1 \cap L_2$ is r.e. but not recursive.

(Q4) $L_1 \cap L_2 \neq \emptyset$?

L_1 and L_2 are r.e. but not recursive



∴ $L_1 \cap L_2 \neq \emptyset$ is r.e but not recursive.