AIMO SAMPLE PAPER

Directions to Students

- You have 4 hours and may answer as many questions as you can. This paper contains 10 questions worth a total of 35 marks as indicated.

- **Questions 1 to 8 have answers which are non-negative integers less than 1000.** Only the answers are required. Write them in the spaces provided on this coversheet. **If an answer is incorrect but correct comprehensible working is provided, then some marks may still be awarded.** Questions 9 and 10 require written answers with proofs.

- Please be sure to write your name, year level, and school name on each page of submitted written answers and attach these to this coversheet.

- Question 10 has an investigation associated with it which may be used to determine prize winners. Marks for the investigation will only be awarded if there are two or more students with the same highest score. You are strongly advised to answer all the other questions before spending time on the investigation.

- You must work on your own. You may use a ruler, compass, rough paper, graph paper and eraser but **calculators, books and other aids are not allowed.**

1. A point P lies inside a square $ABCD$ of side 120. Let F be a point on CD so that PF is perpendicular to CD. P is equidistant from A, B, F. Find PA. [**2 marks**]

2. If $935712 \times N$ is a perfect cube for some positive integer N, find the minimum value of N. [**2 marks**]

3. A, B, C are digits. The 3-digit number ACB is divisible by 3, BAC is divisible by 4, BCA is divisible by 5, and CBA has an odd number of factors. Find ABC. [**3 marks**]

4. While waiting for the bishop to arrive at St Stephen's Anglican Church to lead a service, every person present greeted every other person with a hand shake. Arriving late, the bishop shook hands with only some of the people as he made his way in. Altogether, 1933 handshakes had taken place. How many people shook hands with the bishop? [**3 marks**]

5. In the grid shown, a number is to be placed in each small square so that the product of all three numbers in any row, column, or diagonal is the same positive number. Find the sum of x and y.

x	32	y
16	8	

[**3 marks**]

6. An integer a has just two digits. When the digits are reversed the resulting number b is $p\%$ larger than a. Given that p is an odd integer find the largest value of p. [**4 marks**]

7. Let a, b, c, d, e be a five-term geometric sequence where a, b, c, d, e are integers and $0 < a < b < c < d < e < 100$. What is the sum of all possible values for c? [**4 marks**]

8. In triangle ABC, $\angle ABC = 138°$ and $\angle ACB = 24°$. Point D is on AC so that $\angle BDC = 60°$ and point E is on AB so that $\angle ADE = 60°$. If $\angle DEC = x°$, find x. [**4 marks**]

9. Let ABC be an equilateral triangle with $AB = x$. On the extension of side BC, we define points A' (on the same side as B) and A'' (on the same side as C) such that $A'B = CA'' = y$. Similarly, on the extension of side CA, we define B' (on the same side as C) and B'' (on the same side as A) such that $B'C = AB'' = y$, while on the extension of side AB, we define C' (on the same side as A) and C'' (on the same side as B) such that $C'A = BC'' = y$.

(a) Prove that the points A', B'', C', A'', B', C'' lie on a circle.

(b) If x and y are positive integers, determine the smallest integer value for R^2 where R is the radius of that circle. [**5 marks**]

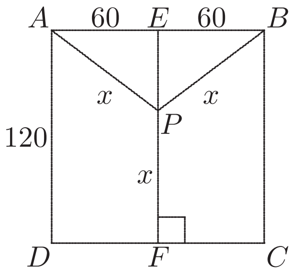
10. What is the maximum number of terms in an arithmetic sequence of primes with common difference 6? [**5 marks**]

Investigation

Find the minimum common difference for an increasing arithmetic sequence of 6 primes.

AIMO SAMPLE PAPER: SOLUTIONS

1. Let E be the midpoint of AB and let $PA = x$.



Triangles AEP and BEP are congruent (SSS). Hence $\angle AEP = \angle BEP = 90°$.
So the line through P parallel to AB and CD must be perpendicular to PE and PF.
Therefore EPF is a straight line of length 120 and $EP = 120 - x$.
From Pythagoras Theorem, $(120 - x)^2 + 60^2 = x^2$. $\boxed{1}$
Hence $14400 - 240x + x^2 + 3600 = x^2$, $18000 = 240x$, and $x =$ **75**. $\boxed{1}$

2. Since $935712 = 2^5 \times 3^4 \times 19^2$ and $935712 \times N$ is a perfect cube, $\boxed{1}$
at least one prime factor of N must be 2, at least two prime factors must be 3, and at least one prime factor must be 19.
Hence the minimum value of N is $2 \times 3^2 \times 19 =$ **342**. $\boxed{1}$

3. Since BCA is divisible by 5, $A = 0$ or 5. But ACB is a 3-digit number, so $A = 5$. $\boxed{1}$
Since 4 divides BAC, $10A + C = 50 + C$ is a multiple of 4, so $C = 2$ or 6.
Since 3 divides ACB, 3 also divides $A + B + C = 5 + B + C$.
If $C = 2$, then $B = 2$ or 5 or 8.
If $C = 6$, then $B = 1$ or 4 or 7. $\boxed{1}$
Thus there are 6 possibilities for CBA: 225, 255, 285, 615, 645, 675.
Of these only 225, a perfect square, has an odd number of factors.
Hence ABC is **522**. $\boxed{1}$

4. If there were n people at the church waiting for the bishop and every person shook hands with every other person, then there were $n(n-1)/2$ handshakes before he arrived. $\boxed{1}$
Now $n(n-1)/2 \approx 1933$. Hence $n^2 \approx 3866$. So $n \approx 62$.
If $n \leq 61$, then the bishop shook hands with at least $1933 - 61 \times 60/2 = 103$ people, which is more than 61 and therefore impossible. $\boxed{1}$
If $n \geq 63$, then the number of handshakes before the bishop arrived was at least $63 \times 62/2 = 1953$, which is more than 1933 and therefore impossible.
So the number of people the bishop shook hands with was $1933 - 62 \times 61/2 = 1933 - 1891 =$ **42**. $\boxed{1}$

5. *Method 1*

Let A, B, C, D be the missing numbers as indicated.

A	B	C
x	32	y
16	8	D

Then

$$\begin{array}{ll} ABC = 32AD & (1) \\ ABC = 32 \times 16 \times C = 512C & (2) \\ ABC = 32xy & (3) \\ ABC = 16 \times 8 \times D = 128D & (4) \\ ABC = 16Ax & (5) \\ ABC = 32 \times 8 \times B = 256B & (6) \end{array}$$

From (4) and (1), $ABC = 128D = 128 \times BC/32 = 4BC$. So $A = 4$. $\boxed{1}$

From (6), $AC = 256$. So $C = 64$.

From (2), $AB = 512$. So $B = 128$. $\boxed{1}$

From (1), $D = BC/32 = 256$.

From (5), $x = BC/16 = 512$.

From (3), $y = ABC/32x = 2$.

Thus $x + y = \mathbf{514}$. $\boxed{1}$

Method 2

Let P be the common product and A, B, C, D the missing numbers as indicated above.

Then $B = \frac{P}{8\times 32}$, $C = \frac{P}{16\times 32}$, $D = \frac{P}{8\times 16}$, and $A = \frac{P}{32\times D} = \frac{8\times 16}{32} = 4$. $\boxed{1}$

Also $A = \frac{P}{B\times C} = \frac{8\times 32\times 16\times 32}{P}$.

So $P = \frac{8\times 32\times 16\times 32}{4} = 32 \times 32 \times 32$, $\boxed{1}$

$x = \frac{P}{16\times A} = \frac{32\times 32\times 32}{16\times 4} = 16 \times 32 = 512$, and

$y = \frac{P}{32\times x} = \frac{32\times 32\times 32}{32\times 512} = 2$.

Hence $x + y = \mathbf{514}$. $\boxed{1}$

Method 3

If we express each number in the given grid as a power of 2, then solving the original grid with constant products is equivalent to solving the corresponding grid of exponents with constant row, column, and diagonal sums.

Let S be the common sum and let a, b, c, d, X, Y be the exponents base 2 of the original corresponding missing numbers as indicated.

a	b	c
X	5	Y
4	3	d

Then $b = S - 8$, $c = S - 9$, $d = S - 7$, $a = S - 5 - d = S - 5 - S + 7 = 2$. $\boxed{1}$

Also $a = S - b - c = S - S + 8 - S + 9 = 17 - S$.
Therefore $S = 15$, $X = S - a - 4 = 9$, $Y = S - X - 5 = 1$. $\boxed{1}$
So $x = 2^9 = 512$ and $y = 2^1 = 2$.
Hence $x + y = \mathbf{514}$. $\boxed{1}$

6. Let $a = 10x + y$ and $b = 10y + x$ where x and y are non-zero digits.

Then $y > x > 0$ and $b = (1 + \frac{p}{100})a$.

So $100b = (100 + p)a$, $1000y + 100x = 1000x + 100y + p(10x + y)$, $900(y - x) = p(10x + y)$. $\boxed{1}$

Now $y - x = 1, 2, 3, 4, 5, 6, 7$, or 8. Also $10x + y$ must divide $900(y - x)$. $\boxed{1}$
If $y - x = 1$, then $10x + y$ can only be 12 or 45.
If $y - x = 2$, then $10x + y$ can only be 24.
If $y - x = 3$, then $10x + y$ can only be 25 or 36.
If $y - x = 4$, then $10x + y$ can only be 15 or 48.
If $y - x = 7$, then $10x + y$ can only be 18.
If $y - x = 5, 6$, or 8, then $10x + y$ has no possible value. $\boxed{1}$

Of these possible values for $10x + y$, only 12, 24, 36, and 48 give an odd value of p and in each case that is 75. Hence the largest value of p is **75**. $\boxed{1}$

7. The common ratio of the sequence is the fraction $\frac{b}{a}$, which we write as $\frac{n}{m}$ in lowest terms with $n > m$.

Then $e = a \times \frac{n^4}{m^4}$. So m^4 divides a. Hence $a = km^4$, where k is a positive integer, and the sequence becomes km^4, km^3n, km^2n^2, kmn^3, kn^4 with $kn^4 < 100$. $\boxed{1}$

If $n > 4$, then $kn^4 \geq n^4 > 256 > 100$. So $n \leq 3$.

If $n = 3$, then $81k < 100$, so $k = 1$. Also $m = 1$ or 2.

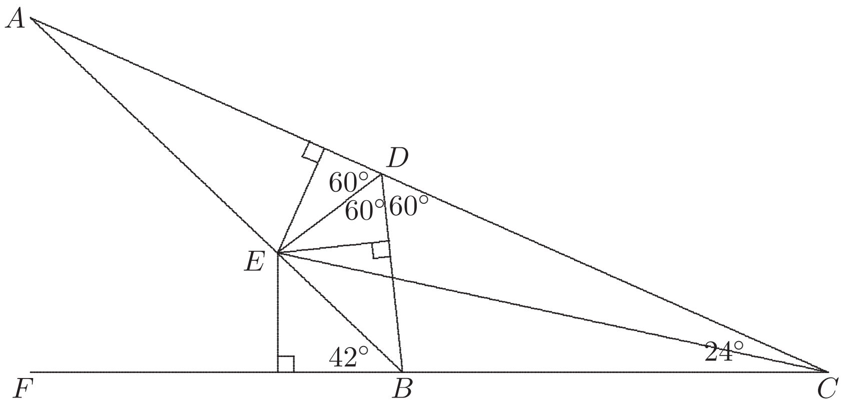
If $m = 1$, then the sequence is (1, 3, 9, 27, 81).

If $m = 2$, then the sequence is (16, 24, 36, 54, 81). $\boxed{1}$

If $n = 2$, then $m = 1$ and $16k < 100$. So $k = 1, 2, 3, 4, 5$ or 6. Thus there are six sequences: (1, 2, 4, 8, 16), (2, 4, 8, 16, 32), (3, 6, 12, 24, 48), (4, 8, 16, 32, 64), (5, 10, 20, 40, 80) and (6, 12, 24, 48, 96). $\boxed{1}$

The sum of all values of c is $9 + 36 + 4 + 8 + 12 + 16 + 20 + 24 = \mathbf{129}$. $\boxed{1}$

8. *Method 1*



The diagram shows the given information with CB extended to F and $\angle ABF = 180° - \angle ABC = 42°$.

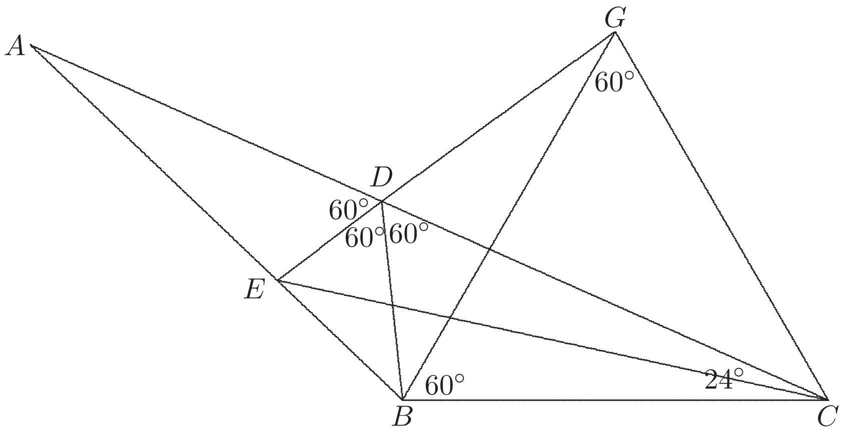
From the construction of points D and E, $\angle BDE = 60°$. $\boxed{1}$

Thus DE is the bisector of $\angle BDA$. Therefore E is equidistant from DA and DB. $\boxed{1}$

Now $\angle DBF = \angle BDC + \angle BCD = 60° + 24° = 84°$. So $\angle DBA = 42°$ and AB is the bisector of $\angle DBF$. Therefore E is equidistant from BF and BD. $\boxed{1}$

Thus E is equidistant from CA and CF. Hence CE is the bisector of $\angle ACF$, so $\angle DCE = 12°$. Therefore, $\angle DEC = 180° - 120° - 12° = 48°$. So $x =$ **48**. $\boxed{1}$

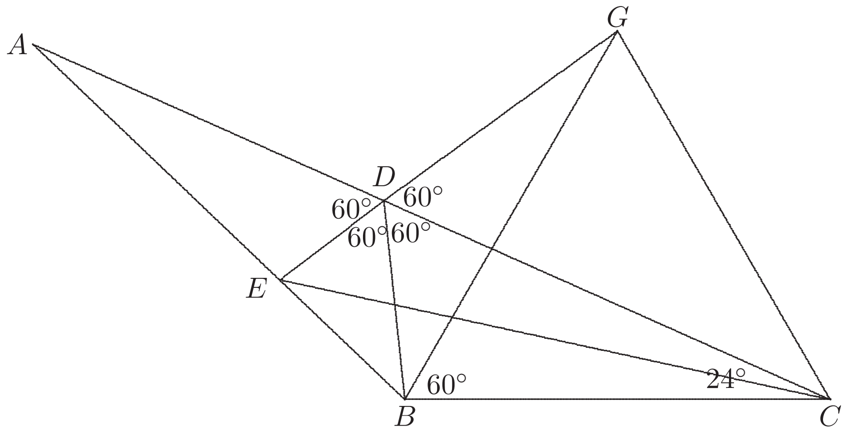
Method 2



From the construction of points D and E, $\angle BDE = 60°$. $\boxed{1}$
Construct an equilateral triangle BCG with G and D on the same side of BC.
Since $\angle BDC = 60° = \angle BGC$, $BDGC$ is a cyclic quadrilateral. $\boxed{1}$
Then $\angle GDC = \angle GBC = 60°$. Hence GDE is a straight line.
From the sum of the angles in quadrilateral $DEBC$, $\angle DEB = 360° - 138° - 120° - 24° = 78°$.
Also $\angle GBE = \angle ABC - \angle GBC = 138° - 60° = 78°$.
Therefore triangle BGE is isosceles and so $GE = GB = GC$. $\boxed{1}$
Thus E, B, C lie on a circle with centre G. So $\angle BEC = \frac{1}{2}\angle BGC = 30°$.
Finally, $\angle DEC = \angle DEB - \angle CEB = 78° - 30° = 48°$. So $x =$ **48**.

$\boxed{1}$

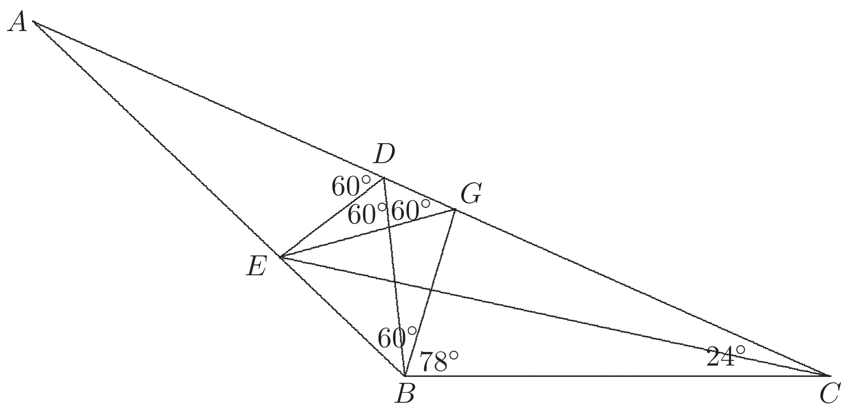
Method 3



From the construction of points D and E, $\angle BDE = 60°$. $\boxed{1}$
Let G be the intersection of the extension of line ED and the line through B at $60°$ to BC. Since $\angle GDC = 60° = \angle GBC$, $BDGC$ is a cyclic quadrilateral. $\boxed{1}$
Then $\angle BGC = \angle BDC = 60°$. Hence $\angle GCB = 180° - 60° - 60° = 60°$ and $GB = GC$.
From the sum of the angles in quadrilateral $DEBC$, $\angle DEB = 360° - 138° - 120° - 24° = 78°$.
Also $\angle GBE = \angle ABC - \angle GBC = 138° - 60° = 78°$.
Therefore triangle BGE is isosceles and so $GE = GB$. $\boxed{1}$
Thus E, B, C lie on a circle with centre G. So $\angle BEC = \frac{1}{2}\angle BGC = 30°$.
Finally, $\angle DEC = \angle DEB - \angle CEB = 78° - 30° = 48°$. So $x = \mathbf{48}$.

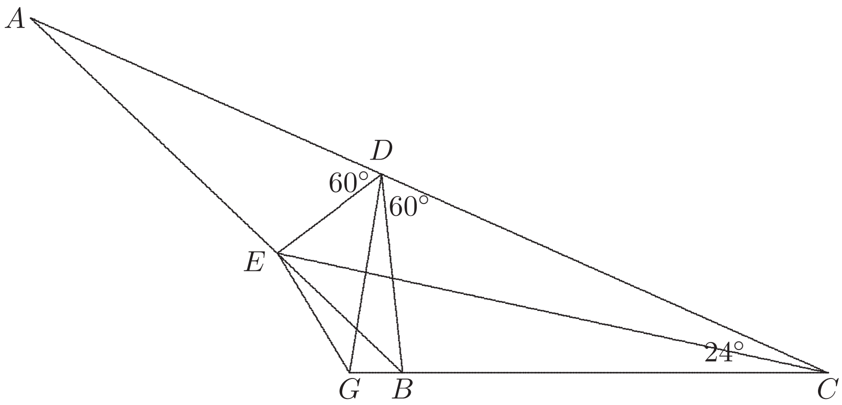
$\boxed{1}$

Method 4



From the construction of points D and E, $\angle BDE = 60°$. $\boxed{1}$
Let G be the point on AC so that $\angle EBG = 60°$.
Then $\angle GBC = 138° - 60° = 78°$ and $\angle BGC = 180° - 78° - 24° = 78°$.
So $\triangle BCG$ is isosceles. $\boxed{1}$
Since $\angle EBG + \angle EDG = 180°$, quadrilateral $EBGD$ is cyclic.
Hence $\angle BEG = \angle BDG = 60°$ and $\angle BDE = \angle BGE = 60°$.
So $\triangle BEG$ is equilateral. $\boxed{1}$
Therefore triangles ECB and ECG are congruent. Hence EC bisects $\angle BCG$.
From $\triangle DEC$, $\angle DEC = 180° - 120° - 12° = 48°$. So $x = \mathbf{48}$. $\boxed{1}$

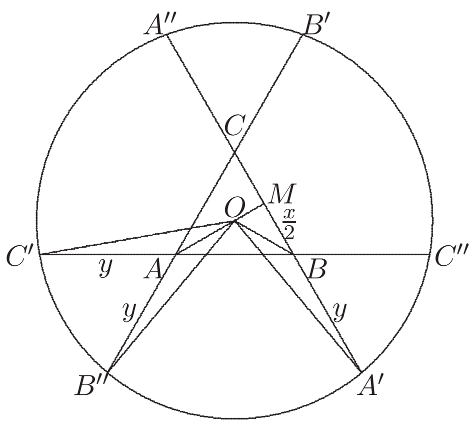
Method 5



From the construction of points D and E, $\angle BDE = 60°$. $\boxed{1}$
Let G be the point on the extension of CB so that $\angle EGB = 120°$.
Then $\angle GEB = 138° - 120° = 18°$.
Since $\angle EDB + \angle EGB = 180°$, quadrilateral $EDBG$ is cyclic. $\boxed{1}$
Hence $\angle GDB = \angle GEB = 18°$, $\angle GDC = 60° + 18° = 78°$, $\angle DGC = 180° - 78° - 24° = 78°$.
So $\triangle CDG$ is isosceles.
Also $\angle EDG = 60° - 18° = 42°$ and $\angle EGD = 120° - 78° = 42°$.
So $\triangle DEG$ is isosceles. $\boxed{1}$
Therefore triangles ECD and ECG are congruent. Hence EC bisects $\angle DCG$.
From $\triangle DEC$, $\angle DEC = 180° - 120° - 12° = 48°$. So $x = \mathbf{48}$. $\boxed{1}$

9. *Method 1*

(a) In the equilateral triangle ABC, the three medians are also the three angle bisectors and they meet at a point O. $\boxed{1}$
Hence triangles OAB'' and OAC' are congruent. Therefore $OB'' = OC'$.
Similarly $OA' = OC''$ and $OA'' = OB'$.
Since $OA = OB$ and $\angle OAC' = \angle OBA'$, triangles OAC' and OBA' are congruent.
Therefore $OC' = OA'$. Similarly $OA' = OB'$.
Thus $OA' = OB' = OC' = OA'' = OB'' = OC''$ and so the points A', B'', C', A'', B', C'' lie on a circle with centre O. $\boxed{1}$



(b) Let M be the midpoint of BC. Then AM is perpendicular to BC.
From Pythagoras Theorem, $AM^2 = x^2 - (\frac{x}{2})^2 = \frac{3x^2}{4}$.
As O is also the centroid of $\triangle ABC$, we have $OM = \frac{1}{3}AM = \frac{1}{3} \times \frac{\sqrt{3}x}{2}$. $\boxed{1}$
Hence

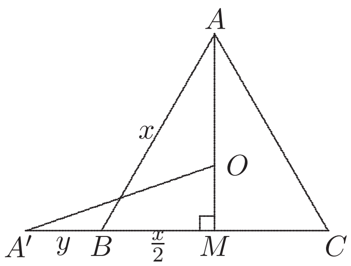
$$R^2 = OA'^2 = OM^2 + MA'^2 = \frac{x^2}{12} + (\frac{x}{2} + y)^2 = \frac{x^2}{12} + \frac{x^2}{4} + xy + y^2 = \frac{x^2}{3} + xy + y^2.$$

$\boxed{1}$

For this to be an integer, x must be a multiple of 3. The smallest integer value for R^2 is obtained when $x = 3$ and $y = 1$ and we get $R^2 = 3 + 3 + 1 = 7$. $\boxed{1}$

Method 2

(a) In the equilateral triangle ABC, the three medians are also the altitudes and they meet at a point O. $\boxed{1}$
Let M be the midpoint of BC.



From Pythagoras Theorem, $AM^2 = x^2 - (\frac{x}{2})^2 = \frac{3x^2}{4}$.
As O is the centroid of $\triangle ABC$, we have $OM = \frac{1}{3}AM = \frac{1}{3} \times \frac{\sqrt{3}x}{2}$. $\boxed{1}$
Hence

$$OA'^2 = OM^2 + MA'^2 = \frac{x^2}{12} + (\frac{x}{2} + y)^2 = \frac{x^2}{12} + \frac{x^2}{4} + xy + y^2 = \frac{x^2}{3} + xy + y^2.$$

$\boxed{1}$

Thus $OA' = \sqrt{\frac{x^2}{3} + xy + y^2}$.

Similary all of $OB', OC', OA'', OB'', OC''$ equal $\sqrt{\frac{x^2}{3} + xy + y^2}$.

So the points A', B'', C', A'', B', C'' lie on a circle with centre O. $\boxed{1}$

(b) For $R^2 = \frac{x^2}{3} + xy + y^2$ to be an integer, x must be a multiple of 3. The smallest integer value for R^2 is obtained when $x = 3$ and $y = 1$ and we get $R^2 = 3+3+1 = 7$.
$\boxed{1}$

10. *Method 1*

The primes 5, 11, 17, 23, 29 form an arithmetic sequence with common difference 6. $\boxed{1}$
So 5 terms are possible.
All terms in an arithmetic sequence of more than one prime with common difference 6 must be odd. So the last digit of each term is 1, 3, 5, 7 or 9. $\boxed{1}$

The following table shows the last digit of all the terms in a 6-term sequence.

First term	Second	Third	Fourth	Fifth	Sixth
1	7	3	9	5	1
3	9	5	1	7	3
5	1	7	3	9	5
7	3	9	5	1	7
9	5	1	7	3	9

$\boxed{1}$

In each case there is a term which is greater than 5 and ends in 5 and therefore is not a prime. $\boxed{1}$
So no 6-term sequence of primes is possible and the maximum number of terms is 5. $\boxed{1}$

Method 2

Suppose p, $p+6$, $p+12$, ... , is an arithmetic sequence of primes with common difference 6. $\boxed{1}$
We write the remainders when the terms are divided by 5. $\boxed{1}$
There are 5 possible sequences of remainders: 0, 1, 2, 3, 4, 0, ... ; 1, 2, 3, 4, 0, 1, ... ;
2, 3, 4, 0, 1, 2, ... ; 3, 4, 0, 1, 2, 3, ... ; 4, 0, 1, 2, 3, 4, ... $\boxed{1}$
The term corresponding to remainder 0 will not be prime unless it is 5. So the only way to get an arithmetic sequence of primes with common difference 6 is to start with 5.
Thus the sequence is 5, 11, 17, 23, 29, 35, ... $\boxed{1}$
Only the first 5 terms are primes. Hence 5 is the maximum number of terms in an arithmetic sequence of primes with common difference 6. $\boxed{1}$

Investigation

An arithmetic sequence of 6 primes has at least 3 consecutive terms that are odd, so the common difference is even.
If the common difference is not a multiple of 3, then the corresponding sequence of remainders from division by 3 is all of 0, 1, 2 repeated in some order. So the sequence of primes would have at most 3 terms.
If the common difference is not a multiple of 5, then the corresponding sequence of remainders from division by 5 is all of 0, 1, 2, 3, 4 repeated in some order. So the sequence of primes would have at most 5 terms.
The least even number that is a multiple of 3 and 5 is 30. Therefore the common difference is at least 30.

The sequence 7, 37, 67, 97, 127, 157 is an arithmetic sequence of primes with common difference 30.
Hence an arithmetic sequence of 6 primes has a minimum common difference of 30.