AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

# Australian Intermediate Mathematics Olympiad 2012

## Questions

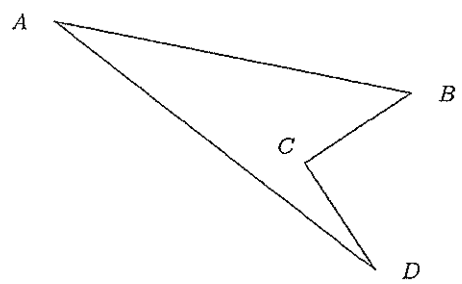
1. Each letter in the grid represents a positive integer.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 31 | $B$ | $C$ | $D$ | $E$ | 7 | $G$ | $H$ | $I$ |

The sum of any three consecutive integers in the grid is 164. Find the value of $H$.

[2 marks]

2. A yacht race takes place on the course depicted in the diagram below. The starting and finishing point is $A$, with marker buoys at $B$, $C$ and $D$.




The distance $AB$ is 6 km and the distance $DA$ is 6.5 km. Buoys $B$ and $C$ are 2 km apart and buoy $C$ is exactly the halfway mark for the race.

For a yacht moving directly from buoy $B$ to buoy $C$, its heading would be southwest. For a yacht moving directly from buoy $C$ to buoy $D$, its heading would be southeast. If the area of water bounded by the course is $R\,\text{km}^2$, find the value of $R$.

[2 marks]

3. Two identical bottles are filled with weak alcohol solutions. In one bottle, the ratio of the volume of alcohol to the volume of water is 1:25. In the other bottle, the ratio of the volume of alcohol to the volume of water is 1:77. If the entire contents of the two bottles are mixed together, the ratio of the volume of alcohol to the volume of water in the mixture is 1:$N$. Find the value of $N$.

[3 marks]

4. What is the maximum number of terms in a series of consecutive even positive integers whose sum is 1974?

[3 marks]

PLEASE TURN OVER THE PAGE FOR QUESTIONS 5, 6, 7, 8, 9 AND 10

5. Let $x$ and $y$ be positive integers satisfying $x^{5x} = y^y$. What is the largest value for $x$?

[4 marks]

6. The lengths of the sides of triangle $T$ are 190, 323 and 399. What is the length of the shortest altitude of $T$?

[4 marks]

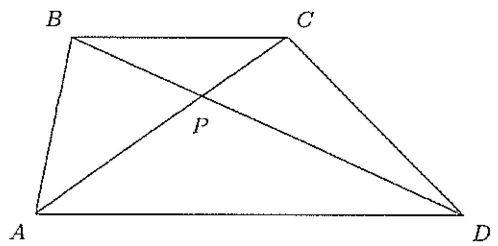
7. The non-zero real numbers $x, y, z$ satisfy the system of equations:

$$\begin{aligned} xy &= 2(x+y) \\ yz &= 3(y+z) \\ zx &= 4(z+x) \end{aligned}$$

Determine $5x + 7y + 9z$.

[4 marks]

8. $ABCD$ is a trapezium with $AD$ parallel to $BC$. The area of $ABCD$ is 225. The area of $\triangle BPC$ is 49. What is the area of $\triangle APD$?




[4 marks]

9. The $n$th triangular number is the sum of the first $n$ positive integers. Let $T_n$ denote the sum of the first $n$ triangular numbers. Derive a formula for $T_n$ and, hence or otherwise, prove that $T_n + 4T_{n-1} + T_{n-2} = n^3$.

[5 marks]

10. A bag contains a certain number of 5-cent, 10-cent, 20-cent, 50-cent and one-dollar coins of more than one denomination. For example, if the bag contains nine 5-cent coins and one 20-cent coin, then it contains exactly ten coins and two denominations. Notice in this example that, if any single coin is removed from the bag, then the remaining coins may be divided into three heaps of equal value. Determine all possible combinations of distinct *denominations* the bag may contain so that after removing any coin from the bag, its contents can be divided into three heaps of equal value.

[4 marks]

*Investigation*

Suppose the bag contains a certain number of 5-cent, 10-cent, 20-cent, 50-cent, one-dollar, and two-dollar coins of more than one denomination. Determine all possible combinations of distinct denominations the bag may contain so that after removing any coin from the bag, its contents can be divided into two heaps of equal value.

[4 bonus marks]


Australian Government

The Australian Mathematics and Science Olympiads are supported by the Australian Government
Department of Innovation, Industry, Science and Research
Department of Education, Employment and Workplace Relations