Australian Mathematical Olympiad Committee

2020 School of Excellence

Welcome to the 2020 Australian Mathematical Olympiad Committee (AMOC) School of Excellence! To get the most out of the school, please go through the following information carefully.

Information about the School

This school will be held online using the Zoom video conferencing platform. To participate you will need a reliable internet connection and an electronic device such as a tablet or computer capable of running Zoom. We will provide you with the necessary Zoom links to access the School closer to the start date.

A detailed timetable will be provided closer to the start of the School. All times will be listed in Australian Eastern Daylight Time (AEDT). Please take care to account for time zone differences when reading this timetable. There will be an introductory session to welcome everyone and to test everyone's Zoom access from **3pm on Friday 4 December**. The main part of the School will run from **Monday 7 December – Saturday 12 December** where the program will commence at 10:45am (AEDT) and conclude at approximately 8pm (AEDT) each day. There will be intermittent breaks each day so that the total daily contact time will be about 6 hours overall.

There will be four groups of students at the school. You are invited as part of the following group.

☑ Junior A ☐ Junior B ☐ Intermediate ☐ Senior

The main purposes of this school are to select the reserve team member for 2021 European Girls Mathematical Olympiad (EGMO) from among the girls in the Intermediate group (the 2021 EGMO team has already been finalised), and more generally, train students in all groups for possible future selection in an EGMO or International Mathematical Olympiad (IMO) team.

In addition to preparing students for selection in an Australian Olympiad team, the program will help you to develop your mathematical abilities. It is also a great opportunity to make new friends and enjoy the company of other like-minded people.

Please come properly equipped to your online classes. Here is a checklist of essential items.

- ☐ Pens, including a few different colours
- ☐ Ruler
- ☐ Compass (circle drawing instrument)
- ☐ 128 page exercise book (for use in lectures and evening problem sessions)
- ☐ Your solutions to the preparation problems
- ☐ Ream of blank paper for use in exams (Seniors and Intermediates only)

We highly recommend that all students obtain the book *Problem Solving Tactics* available from the Australian Maths Trust (AMT) at shop.amt.edu.au. It will be a handy reference to have. You can get a head start on the school by starting to familiarise yourself with the material in this book. To Seniors in particular, revise chapter 5 entitled *Important configurations in geometry*.

Preparation Problems

Enclosed are a set of preparation problems for the school. Try to solve as many of these challenging problems as you can in the weeks leading up to this year's school.

It is important that you **write up your attempts**, and have them handy for the evening problems sessions where we will discuss solutions. To those who have not attended an AMOC school before, do what you can; you will learn many techniques for problem solving at the school anyway.

Note that some students may find the material very challenging. This is quite normal and we encourage you to adopt a positive attitude and work diligently and give it your best. Some material may need to be revisited several times before progress is made.

The preparation problems are assigned as follows.

Junior Problems				**Intermediate Problems**				**Senior Problems**			
N1	N2	N3	N4	N3	N4	N5	N6	N5	N6	N7	N8
C1	C2	C3	C4	C3	C4	C5	C6	C5	C6	C7	C8
G1	G2	G3	G4	G3	G4	G5	G6	G5	G6	G7	G8
A1	A2	A3	A4	A3	A4	A5	A6	A5	A6	A7	A8

All students, please scan or, even better, typeset (LaTeX is great!) your work and email it in a single file to amoc@amt.edu.au by no earlier than **Monday 23 November** and no later than **Sunday 29 November**. Please use the following naming convention for your file: If you are in the junior group and your name is John Smith, then name your file "JUNIOR John Smith". Intermediates and Seniors are similarly named in the obvious way.

Intermediates and Seniors will have their work marked and be given feedback. Juniors will not have their work marked, however it will be useful for us to see some of your work prior to the evening problem sessions, and we may incorporate some of your solutions in the evening presentations.

We look forward to seeing you all onscreen at the 2020 AMOC School of Excellence!

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N1 Let a, b, c, and d be four distinct integers.

Prove that $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ is divisible by 12.

N2 An example of *Clayton's cancelling* is $\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$. That is, the correct result is obtained using the incorrect method of "cancelling" the 6s.

Find all instances of Clayton's cancelling which simultaneously satisfy the following three criteria.

(i) Both numerator and denominator are strictly two-digit numbers with the numerator smaller than the denominator.

(ii) The units digit of the numerator is equal to the tens digit of the denominator.

(iii) Crossing out the units digit of the numerator and the tens digit of the denominator yields the correct lowest terms simplification of the original fraction.

N3 Prime numbers p, q and r satisfy the following two conditions.

$$p + q < 111 \quad \text{and} \quad \frac{p+q}{r} = p - q + r.$$

Find the largest possible value of pqr.

N4 Which positive integers can be written in the form

$$a^2 + b^2 - c^2$$

for positive integers $1 \le a < b < c$?

N5 Find all solutions in positive integers to the equation

$$2^m + m = n!.$$

N6 Let k and m be given positive integers with m odd.

Show that there exists a positive integer n such that

$$m^n + n^m$$

has at least k distinct prime factors.

N7 For any positive integer $n > 1$ a permutation $(a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$ is called *square* if for each positive integer i with $1 \le i \le n-1$ the number $a_i a_{i+1} + 1$ is a perfect square. A permutation $(a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$ is called *cubic* if for each positive integer i with $1 \le i \le n-1$ the number $a_i a_{i+1} + 1$ is a perfect cube.

(a) Prove that there are infinitely many positive integers n for which there exists a square permutation of $(1, 2, \ldots, n)$.

(b) Is there a positive integer n for which there exists a cubic permutation of $(1, 2, \ldots, n)$?

N8 We say a number n is *special* if it can be written in the form $n = a^b + b$ for integers $a, b > 1$.

(a) Are there 2015 consecutive positive integers, exactly 2000 of which are special?

(b) Are there 2015 consecutive positive integers, exactly 1 of which is special?

G1 Let $ABCD$ be a parallelogram such that $AB = BD = CD$. Point K ($K \neq A$) on side AB satisfies $DA = DK$. Let M and N be points in the plane with the property that K and A are the midpoints of segments CM and BN, respectively.

Prove that $DN = DM$.

G2 Let X, Y and Z be points on the sides AD, AB and BC of rectangle $ABCD$.

Given that $AX = CZ$, prove that $XY + YZ \geq AC$.

G3 Let M be a point on side AB of equilateral triangle ABC. The point N is such that triangle AMN is also equilateral but N does not lie on AC. Let D be the intersection of lines AC and BN. Let K be the intersection of lines CM and AN.

Prove that $KA = KD$.

G4 Let Γ be the circumcircle of acute triangle ABC. Let ω be a circle passing through A and tangent to BC at X. Suppose that ω intersects Γ for a second time at Y where Y lies on the minor arc AC of Γ. The line AX intersects Γ for a second time at W. The line XY intersects Γ for a second time at Z.

Prove that the minor arcs CW and ZB of Γ are equal in length.

G5 Let P be a point lying outside a circle Γ. Points A and B lie on Γ and are such that PA and PB are tangent to Γ. Point C lies on segment AB. Circle PBC intersects circle Γ for a second time at point D. Point Q is located in such a way that A is the midpoint of PQ.

Prove that AD is parallel to QC.

G6 Circle $ABCDEF$ is externally tangent to circle FGH at point F. Furthermore, line AB is tangent to circle FGH at point H and line CE is tangent to circle FGH at point G. Also D lies on line FH. The bisector of $\angle EAH$ intersects segment GH at point P.

Prove that D is the circumcentre of triangle ABP.

G7 Let triangle ABC be an acute, scalene triangle. Let A_1 and B_1 be the feet of the altitudes from A and B to BC and CA, respectively. Let Z be the intersection of the tangents at A and B to the circumcircle of triangle ABC. Let line A_1B_1 intersect line AZ at point X and line BZ at point Y.

Prove that the circumcircles of triangles ABC and XYZ are tangent.

G8 The incircle ω of triangle ABC touches the sides AB and AC of the triangle at P and Q, respectively. Let J be the excentre opposite A. Let T be the second point of intersection of circles JBP and JCQ.

Prove that circle BCT is tangent to ω.

A1 Determine the total number of pairs of positive integers (p, q) such that the roots of the equation

$$x^2 - px - q = 0$$

are not more than 10.

A2 Do there exist three different quadratic polynomials $f(x)$, $g(x)$, $h(x)$ such that the roots of the equation $f(x) = g(x)$ are 1 and 4, the roots of the equation $g(x) = h(x)$ are 2 and 5, and the roots of the equation $h(x) = f(x)$ are 3 and 6?

A3 Suppose that x, y and z are real numbers satisfying the following three equations.

$$\begin{aligned} x + y + z &= 2 \\ x^2 + y^2 + z^2 &= 6 \\ x^3 + y^3 + z^3 &= 8 \end{aligned}$$

Find all possible values of $x^{2020} + y^{2020} + z^{2020}$.

A4 Find all functions defined for real numbers and taking real numbers as values such that for all real numbers x and y

$$f(xf(y) + x) = xy + f(x).$$

A5 Let x, y, z be positive real numbers satisfying $xy + yz + zx = 3xyz$.

Prove that

$$x^2y + y^2z + z^2x \geq 2(x + y + z) - 3$$

and determine when equality holds.

A6 Suppose that a function $f\colon \mathbb{R} \to \mathbb{R}$ has the property that

$$f(x)^2 \leq f(y)$$

for every $x > y$.

Prove that $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$.

A7 For each integer n where $|n| \geq 2$, let $d(n)$ denote the smallest prime divisor of n.

Find all nonconstant polynomials $F(x)$ with integer coefficients such that

$$F(n + d(n)) = n + d(F(n))$$

whenever n is an integer satisfying $|n| \geq 2014$ and $|F(n)| \geq 2$.

A8 Let $f\colon \mathbb{N}_0 \to \mathbb{N}^+$ be a given strictly increasing function, and let $x_0 > x_1 > x_2 > \cdots$ be a given sequence of positive real numbers converging to zero.

Prove that there exists a sequence $y_0 > y_1 > y_2 > \cdots$ of positive real numbers converging to zero such that

$$x_n \leq y_n \leq 2y_{f(n)}$$

for each $n \in \mathbb{N}_0$.

($\mathbb{N}_0$ denotes the set of non-negative integers, and $\mathbb{N}^+$ denotes the set of positive integers.)

C1 For each integer $n \geq 3$, we define an n-ring to be a circular arrangement of n (not necessarily different) positive integers such that the product of every three neighbouring integers is n.

Find the number of integers n in the range $3 \leq n \leq 2020$ for which it is possible to form an n-ring.

C2 Michelle is an expert in tiling square bathroom floors of side length n, where n is a positive integer. The tiles she uses are in the shape of a 4×1 rectangle. The only jobs she is willing to take are those in which she does not have to cut any tiles.

For which values of n is Michelle willing to take the job?

C3 Sampson has a collection of squares whose sides are parallel to the coordinate axes and whose vertices have integer coordinates. Moreover, each such square has side length 2020, and each pair of squares have exactly two points in common.

What is the maximal number of squares in Sampson's collection?

C4 Johnny owns 49 coins, one each of value $\$1, \$2, \$3, \ldots, \$49$. He wishes to put all these coins in his money box, one at a time. However, Johnny suffers from triskaphobia (fear of the number three), so he never wants the amount of money in his money box to be a multiple of three.

In how many different ways can Johnny put the coins in his money box?

C5 Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line, all of which contain a counter, A wins.

Find the minimum value of k for which A cannot guarantee being able to win in a finite number of moves, or prove that no such minimum value exists.

C6 Let S be the set of 3-digit positive integers, that is, from 100 up to 999. For any two members x and y of S we write $x \to y$ if x can be changed into y by choosing a digit d of x where $d \leq 8$, and replacing it with $d+1$. A *quartet* is a list $r \to s \to t \to u$ of four elements of S. For example, $517 \to 527 \to 537 \to 637$ is quartet.

(a) Is it possible to partition the elements of S into quartets?

(b) For each positive integer n, solve the same problem of partitioning S into quartets, where S is the set of n-digit positive integers.

C7 Consider 148 points around a circle. Some pairs of these points are connected by line segments. It is observed that no two line segments intersect strictly inside the circle.

Prove that one can find 50 of the 148 points such that no two of them are connected.

C8 Amy chooses an odd number $k = 2l + 1$, places k plates around a circle and distributes 2013 beads on these plates as she pleases. After that Bob marks one of the plates. After this Amy chooses l unmarked plates, no two of which are neighbours, and collects the beads from these plates.

Given that Amy is free to choose the value of l, what is the maximal number of beads that Amy is guaranteed to be able to collect at the end?