Australian Intermediate Mathematics Olympiad
2008

Time allowed: 4 hours
NO calculators are to be used

Questions 1 to 8 only require answers (non-negative integers less than 1000).
Questions 9 and 10 require written solutions and may require arguments or proofs.
The Investigation in Question 10 may be used to determine prize winners.

1. Consider a circular sector of radius 360 which is one-sixth of a circle. A circle is drawn inside this sector so that it is tangent to the two radii and to the circular arc. Calculate the radius of this smaller circle. **[2 marks]**

2. Find the 3-digit number at the right-hand end of the number $1! + 2! + 3! + \cdots + 2007! + 2008!$, where $n!$ stands for the number $1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.
For example, $7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$. **[2 marks]**

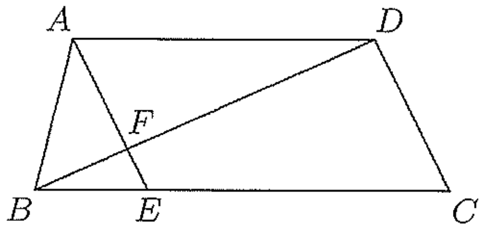
3. If $\frac{2}{35} = \frac{1}{x} + \frac{1}{y}$ and x and y are different positive integers, find the minimum value of $x + y$. **[3 marks]**

4. Find the largest prime factor of $7^{14} - 56 + 7^{13}$. **[3 marks]**

5. Each interior angle in a 16-sided convex polygon is an integer number of degrees. When arranged in ascending order of magnitude, these angles form an arithmetic progression. How many degrees are there in the largest interior angle in the polygon? **[3 marks]**

6. Impatient Imran always walks down the moving escalator outside his office. It moves at a constant but annoyingly slow speed. Once he got from top to bottom in 16 seconds taking 28 steps. Another time he got from top to bottom in 24 seconds with 21 steps. How many steps high is the escalator? **[3 marks]**

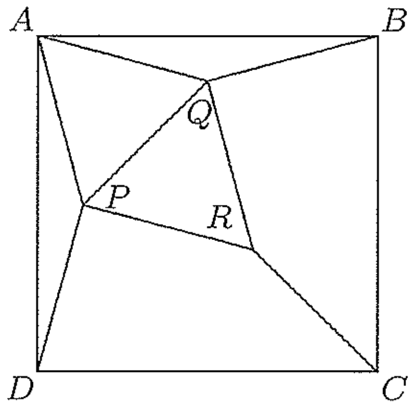
7. In the trapezium $ABCD$, AD is parallel to BC. Also BD is perpendicular to DC. The point F is chosen on the line BD so that AF is perpendicular to BD. AF is extended to meet BC at the point E. If $AB = 41$, $BF = 9$ and the area of quadrilateral $FECD$ is 960, what is the length of AD?



[4 marks]

8. Curious Kate calculated the sum of all positive integers each of which equals 101 times the sum of its digits. Find the remainder when her sum is divided by 1000. **[5 marks]**

9. $ABCD$ is a square. P, Q, R are points such that triangles APQ and PQR are equilateral and $AQ = QB$ and $AP = PD$. Prove that $RC = PD$.



[5 marks]

10. Real numbers a, b, c, d, e are linked by the two equations:

$$\begin{aligned} e &= 40 - a - b - c - d \\ e^2 &= 400 - a^2 - b^2 - c^2 - d^2 \end{aligned}$$

Determine the largest value for e. **[5 marks]**

Investigation

Find all integer solutions, if any, of the following pair of equations.

$$\begin{aligned} e &= 30 - a - b - c - d \\ e^2 &= 200 - a^2 - b^2 - c^2 - d^2 \end{aligned}$$

Australian Intermediate Mathematics Olympiad Solutions 2008

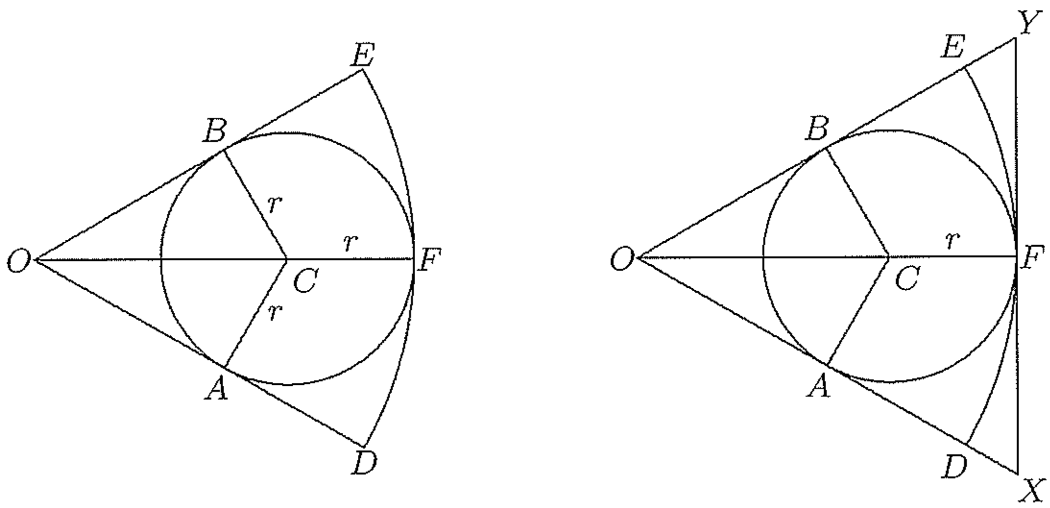
1. Let ODE be the sector with radius 360 and centre O. Thus $\angle EOD = 60°$. Let C be the centre of the smaller circle and r its radius. The smaller circle touches OD at A, OE at B, and arc DE at F. Now O, C, F are collinear because both circles have a common tangent at F and that tangent is perpendicular to OF and CF.

   Angles OAC and OBC are 90° and $CA = CB = r$, hence triangles OCA and OCB are congruent. Therefore OC bisects $\angle AOB$.

   Method 1

   $\triangle AOC$ has internal angles 30°, 60°, 90°. Hence its hypotenuse $OC = 2r$.

   Therefore $OF = 3r = 360$ and $r = \mathbf{120}$.



   Method 2

   Let the common tangent at F intersect OD and OE extended at X and Y respectively. $\triangle XOY$ is equilateral so its angle bisectors and medians are the same. The small circle is inscribe in $\triangle XOY$ so C is the intersection of its medians. Therefore $r = CF = \frac{1}{3}OF = \mathbf{120}$.

2. The product $1 \times 2 \times 3 \times \cdots \times 14 \times 15$ has factors 4, 5, 10 and 15 and therefore is divisible by 1000. Hence the last three digits of $n!$ are zeros for $n \geq 15$. So the 3-digit number at the right-hand end of $1!+2!+3!+\cdots+2007!+2008!$ is the 3-digit number at the right-hand end of $1!+2!+3!+\cdots+14!$. The 3-digit numbers at the right-hand end of $1!$, $2!$, $3!$, …, $14!$ are easily calculated by starting with 1 and multiplying consecutively by the next highest integer. They are respectively 001, 002, 006, 024, 120, 720, 040, 320, 880, 800, 800, 600, 800, 200. Therefore the 3-digit number at the right-hand end of the sum of these numbers is **313**.

3. *Method 1*

We have $\frac{2}{35} = \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}$.
Therefore $x + y = 2k$ and $xy = 35k$ for some positive integer k.
So $x(2k - x) = 35k$ and $x^2 - 2kx + 35k = 0$.
Hence $x = (2k \pm \sqrt{4k^2 - 140k})/2 = k \pm \sqrt{k^2 - 35k}$ and $y = k \mp \sqrt{k^2 - 35k}$. Since x and y are integers and not equal, $k^2 - 35k$ must be a positive square.
Hence the least value of k is 36 and the minimum value of $x + y$ is **72**.

Method 2

Suppose $x < y$.
If $x > 34$ then $\frac{1}{x} + \frac{1}{y} < \frac{1}{35} + \frac{1}{35} = \frac{2}{35}$. So $x \leq 34$.
If $x < 18$ then $\frac{1}{x} + \frac{1}{y} \geq \frac{1}{17} > \frac{2}{35}$. So $x \geq 18$.

We seek poitive integer solutions for $35x + 35y = 2xy$ with $18 \leq x \leq 34$.

x	$35x = (2x-35)y$	y	$x+y$
18	$35(18) = 1y$	630	648
19	$35(19) = 3y$	—	
20	$35(20) = 5y$	140	160
21	$35(21) = 7y$	105	126
22	$35(22) = 9y$	—	
23	$35(23) = 11y$	—	
24	$35(24) = 13y$	—	
25	$35(25) = 15y$	—	
26	$35(26) = 17y$	—	
27	$35(27) = 19y$	—	
28	$35(28) = 21y$	—	
29	$35(29) = 23y$	—	
30	$35(30) = 25y$	42	72
31	$35(31) = 27y$	—	
32	$35(32) = 29y$	—	
33	$35(33) = 31y$	—	
34	$35(34) = 33y$	—	

Thus the minimum value of $x + y$ is **72**.

4. We use some standard algebraic factorisations.

$$
\begin{aligned}
&7^{14} - 56 + 7^{13} \\
&= 7^{13}(7+1) - 56 \\
&= (56)(7^{12} - 1) \\
&= (56)(7^6 - 1)(7^6 + 1) \\
&= (56)(7^3 - 1)(7^3 + 1)((7^2)^3 + 1) \\
&= (56)(7-1)(7^2+7+1)(7+1)(7^2-7+1)(7^2+1)((7^2)^2 - 7^2 + 1) \\
&= (56)(6)(57)(8)(43)(50)(2353) \\
&= (56)(6)(57)(8)(43)(50)(13)(181).
\end{aligned}
$$

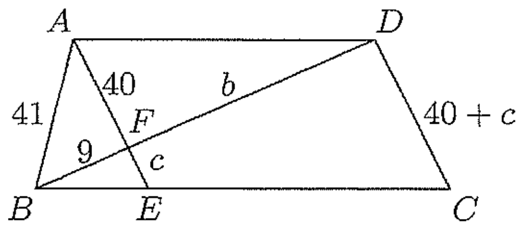
We see that 181 is prime by testing for prime factors up to its square root, that is, up to 13. Hence the largest prime factor of $7^{14} - 56 + 7^{13}$ is **181**.

5. Let S be the sum of the interior angles in the polygon in degrees. So $S = 16 \times 180 - 360 = 2520$. Let a be the smallest and p the largest interior angle of the polygon in degrees. From the sum of the terms in an arithmetic progression we get $S = \frac{16}{2}(a+p) = 8(a+p)$. Hence $a+p = 315$. As the polygon is convex, $p < 180$ and so $a > 135$. If d is the common difference of the arithmetic progression, then $p = a+15d$ so the equation becomes $2a + 15d = 315$. Since the left side must be odd, d must be odd. If d is 3 or more, then $2a \leq 270$, a contradiction. So $d = 1$, $a = 150$, and $p = 150 + 15 = \mathbf{165}$.

6. Let s be the height of the escalator in steps and e be the speed of the escalator in steps per second. In 16 seconds the escalator moved $16e$ steps and Imran took 28 steps. Since Imran moved from the top of the escalator to the bottom in this time, we have $16e + 28 = s$. Similarly, $24e + 21 = s$. Hence $16e + 28 = 24e + 21$, $8e = 7$, and $s = (\frac{7}{8})16 + 28 = \mathbf{42}$.

7. From Pythagoras theorem $AF^2 = 41^2 - 9^2 = 1681 - 81 = 1600$, hence $AF = 40$. Let $FD = b$ and $FE = c$. DC and AE are both perpendicular to FD, so $ADCE$ is a parallelogram. Therefore $DC = AE = 40 + c$.

From similar triangles BFE and BDC, $\frac{9+b}{9} = \frac{40+c}{c}$. So $\frac{b}{9} = \frac{40}{c}$ or $bc = 360$.



Method 1

The area of trapezium $FDCE$ is $960 = FD(DC + FE)/2$.
So $1920 = b(40 + 2c) = 40b + 2bc = 40b + 720$ and $b = 30$. From Pythagoras theorem $AD = \mathbf{50}$.

Method 2

Let the area of $\triangle BFE = \alpha$. Then $\frac{\alpha}{180} = \frac{c}{40}$ and $\frac{\alpha+960}{\alpha} = \frac{(40+c)^2}{c^2}$.
Therefore $960c = \frac{\alpha}{c}(80c + 1600) = \frac{180}{40}(1600 + 80c) = 9(800 + 40c)$.
Hence $600c = 7200$ and $c = 12$.
Since $bc = 360$, we have $b = 30$. From Pythagoras theorem $AD = \mathbf{50}$.

8. Suppose $x_0+10x_1+100x_2+\cdots+10^n x_n = 101(x_0+x_1+x_2+\cdots+x_n)$.
Then $0 = -100x_0 - 91x_1 - x_2 + 899x_3 + 9899x_4 + \cdots$.
If $n \le 2$, then the right side of this equation is negative.
If $n \ge 4$, then the right side is positive since
$100x_0 + 91x_1 + x_2 \le (100 + 91 + 1)9 = 1728$.
Therefore $n = 3$ and we have $899x_3 = 100x_0 + 91x_1 + x_2$.
Now $100x_0 + 91x_1 + x_2 \le 1728$ so $x_3 = 1$.
We tabulate the possibilities for x_0, x_1, x_2.

x_0	$91x_1 + x_2$	x_1	x_2
8	99	1	8
7	199	≤ 2	—
6	299	≤ 3	—
5	399	≤ 4	—
4	499	≤ 5	—
3	599	≤ 6	—
2	699	≤ 7	—
1	799	≤ 8	—
0	899	≤ 9	—

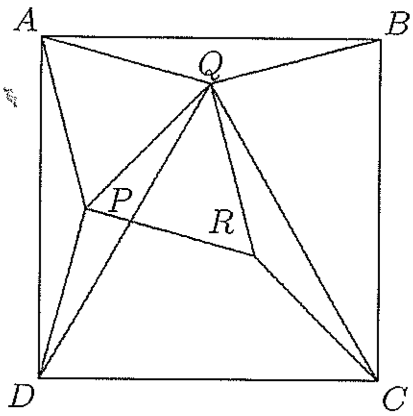
Thus 1818 is the only number with the given property.

So the sum of all positive integers each of which equals 101 times the sum of its digits is 1818 and when this is divided by 1000 the remainder is **818**.

9. Triangles APD and AQB are congruent because $DP = PA = AQ = QB$ and $AD = AB$.
Since $\angle PAD = \angle QAB$ and $\angle PAQ = 60°$, $\angle PAD = \angle QAB = \frac{1}{2}(90 - 60) = 15°$.
Because $\triangle APD$ is isosceles, $\angle PDA = 15°$ and $\angle APD = 150°$.

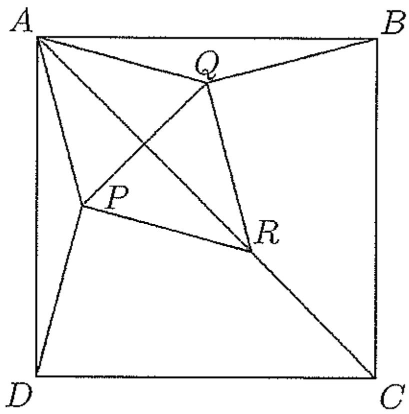
Method 1

$\angle QPD = 360 - 150 - 60 = 150° = \angle APD$ and $PA = PQ = PD$, so triangles DPA and DPQ are congruent. Hence $DQ = DA = DC$.
Also $\angle ADQ = 30°$, so $\angle QDC = 60°$ and $\triangle QDC$ is equilateral.
Now triangles DPQ and CRQ are congruent since $QP = QR$, $QD = QC$ and $\angle PQD = 60° - \angle DQR = \angle RQC$.
Hence $RC = PD$ as required.



Method 2

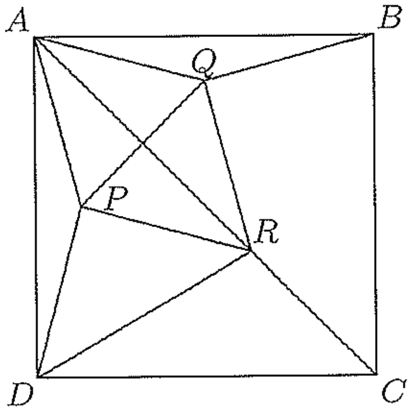
From $\triangle APD$ we have $AD = 2AP\cos 15° = 2AP\cos(45-30)°$
$= 2AP(\cos 45° \sin 30° + \sin 45° \sin 30°) = 2AP(\frac{1}{\sqrt{2}}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{1}{2})$
$= AP(1+\sqrt{3})/\sqrt{2}$.
$APRQ$ is a rhombus so AR bisects $\angle PAQ$ and hence $\angle DAB$.
Therefore R lies on the diagonal AC of $ABCD$.
Hence $AR + RC = AC = AD\sqrt{2} = AP(1+\sqrt{3})$.
Also AR is the perpendicular bisector of PQ.
Therefore $AR = AP\sqrt{3}$.
Hence $RC = AP = PD$.



Method 3

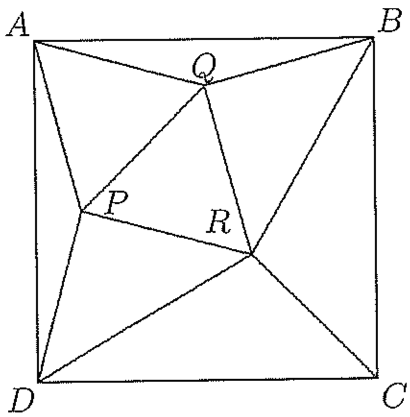
$\angle DPR = 360 - 150 - 60 - 60 = 90°$. Hence $DR = PD\sqrt{2}$. Also $DP = PA = PQ = PR$. Therefore $\triangle DPR$ is isosceles, $\angle PDR = 45°$ and $\angle RDC = 30°$.

$APRQ$ is a rhombus so AR bisects $\angle PAQ$ and hence $\angle DAB$.
Therefore R lies on the diagonal AC of $ABCD$. Hence $\angle RCD = 45°$.
From the sine rule, $RC/\sin 30° = DR/\sin 45° = 2PD$.
Therefore $RC = PD$.



Method 4

$\angle DPR = 360 - 150 - 60 - 60 = 90°$ and $DP = PA = PQ = PR$.
Hence $DR = PD\sqrt{2}$ and $\angle PDR = \angle PRD = 45°$.
Similarly, $BR = QB\sqrt{2} = PD\sqrt{2}$. Therefore triangles DRC and BRC are congruent.
Hence $\angle RCD = 45°$.
Since $\angle RDC = 30°$, the sine rule gives $RC/\sin 30° = DR/\sin 45° = 2PD$.
Therefore $RC = PD$.



10. From the first equation we get

$(40-e)^2 = (a+b+c+d)^2 = a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)$.

Now, for any numbers x and y, $0 \le (x-y)^2 = x^2-2xy+y^2$. Therefore $2xy \le x^2+y^2$.

Hence $1600-80e+e^2 = (40-e)^2 \le 4(a^2+b^2+c^2+d^2) = 4(400-e^2) = 1600-4e^2$.

It follows that $e(80-5e) \ge 0$, implying that $0 \le e \le 16$.

The value $e = 16$ can be attained by taking $a = b = c = d = 6$.

Thus the largest value for e that satisfies the given equations is **16**.

Investigation

As above we get $900-60e+e^2 = (30-e)^2 \le 4(a^2+b^2+c^2+d^2) = 4(200-e^2) = 800-4e^2$.

So $5e^2-60e+100 \le 0$, $e^2-12e+20 \le 0$, $(e-2)(e-10) \le 0$, or $2 \le e \le 10$.

The equations can be written as

$$\begin{aligned} 30 &= a+b+c+d+e \\ 200 &= a^2+b^2+c^2+d^2+e^2 \end{aligned}$$

Hence they are symmetric in a, b, c, d, e. So all variables lie between 2 and 10 and we may assume that $10 \ge a \ge b \ge c \ge d \ge e \ge 2$. We tabulate all possible integer solutions for the second equation and check if they satisfy the first equation.

$200 = a^2+b^2+c^2+d^2+e^2$	$a+b+c+d+e = 30?$
$200 = 100+64+16+16+4$	$10+8+4+4+2 < 30$
$200 = 100+25+25+25+25$	$10+5+5+5+5 = 30$
$200 = 81+81+25+9+4$	$9+9+5+3+2 < 30$
$200 = 81+49+36+25+9$	$9+7+6+5+3 = 30$
$200 = 64+64+64+4+4$	$8+8+8+2+2 < 30$
$200 = 49+49+49+49+4$	$7+7+7+7+2 = 30$

So the only integer solutions of the given equations are permutations of 10, 5, 5, 5, 5 and 9, 7, 6, 5, 3 and 7, 7, 7, 7, 2.