Australian Intermediate Mathematics Olympiad 1999

1. How many digits are there in the smallest number which is composed entirely of fives (e.g. 55555) and which is divisible by 99?

[2 marks, 493/834]

2. In the addition sum $TAP + BAT + MAN$, each letter represents a different digit and no first digit is zero. What is the smallest sum that can be obtained?

[2 marks, 603/834]

3. Chord AB subtends an angle of $153°$ at the centre O of a circle. Point P is the point of trisection of the major arc AB closer to B. If Q is a point on the minor arc AB, find in degrees $\angle AQP$.

[3 marks, 231/834]

4. How many different integers satisfy the equation

$$(x^2 - 5x + 5)^{(x^2 - 11x + 30)} = 1?$$

[3 marks, 172/834]

5. Which three-digit number has the greatest number of different factors?

[4 marks, 186/834]

6. My father had to wash some nappies in a strong bleach solution and wanted to rinse them so that they contained as small a concentration of bleach as possible. He can wring them out so there is just half a litre of liquid left. He wrings them out, adds 12 litres of water, mixes thoroughly, wrings them out again, adds a further 8 litres of water, and mixes thoroughly, reducing the concentration of bleach to $\frac{1}{k}$ of its original. Find k.

[4 marks, 185/834]

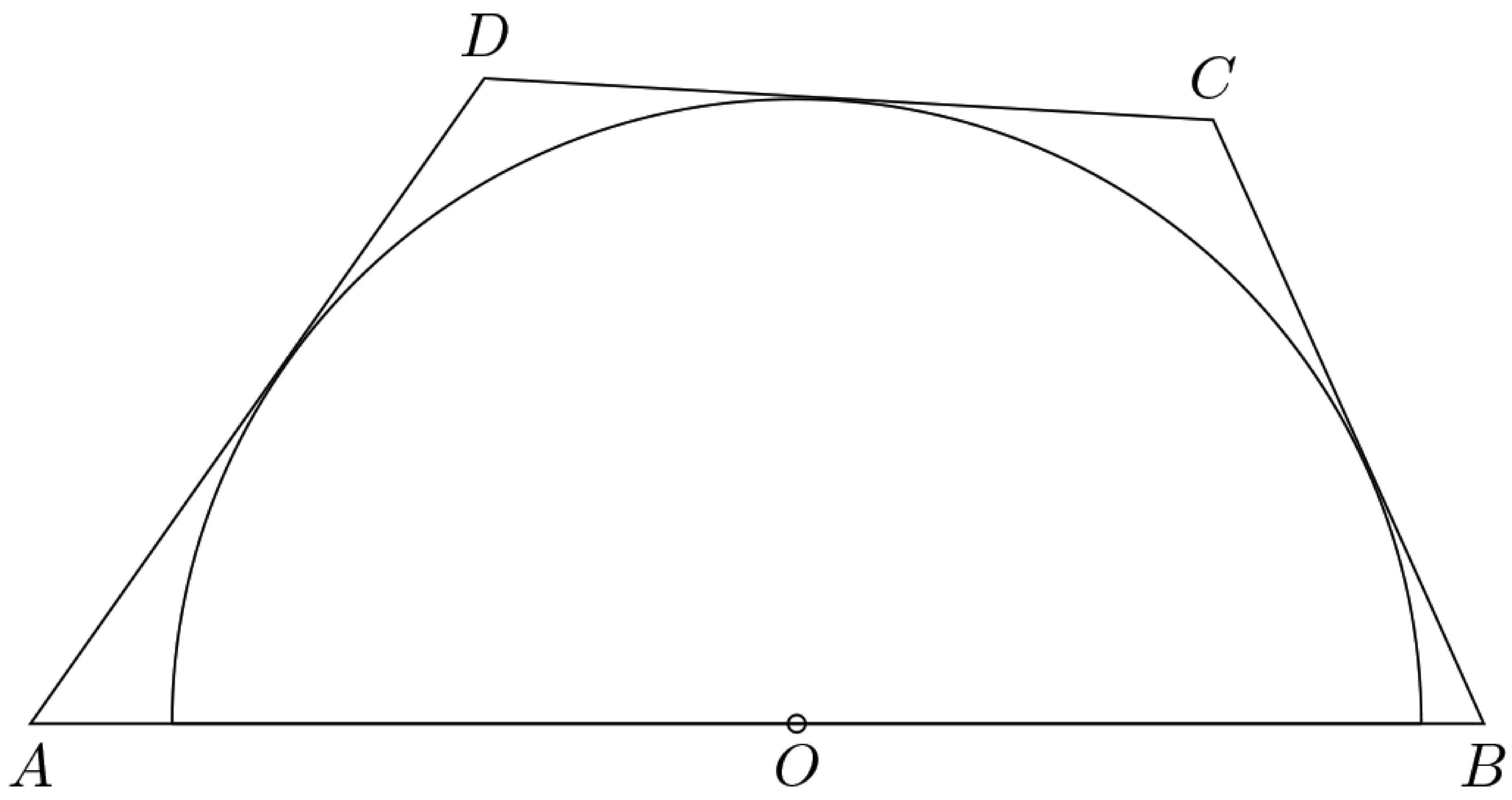
7. A three-digit number N has first digit a $(\neq 0)$, second digit b and third digit c. $N = b(10c + b)$ where b and $(10c + b)$ are primes. Find N.

[5 marks, 625/834]

8. 121 is a *palindrome*. It is an integer which is the same number when it is reversed. Find an integer n such that n^2 is a palindrome with 6 digits.

[5 marks, 180/834]

9. A semicircle, centre O, is inscribed in quadrilateral $ABCD$ as shown and $AO = OB$. Prove that $AO^2 = AD \cdot BC$.



[*5 marks, mean 0.3*]

10. N is the smallest positive integer such that the sum of the digits of N is 18 and the sum of the digits of $2N$ is 27. Find N.

[*5 marks, mean 0.9*]

Investigation

a M is the largest positive integer containing no zeros such that the sum of the digits of M is 18 and that of $2M$ is 27. Find M.

b Find other numbers N with the property that the sum of the digits of N is 18 and the sum of the digits of $2N$ is 27.

c Let $S(n)$ = sum of the digits of n, where n is a positive integer. If $S(N) = a$ and $S(2N) = b$, find rules relating a and b.

If N is a 4-digit number, what is the largest possible value of $a - b$ and for what values of N does this occur ?

Australian Intermediate Mathematics Olympiad 2000

1. The integer n is the smallest positive multiple of 15 such that every digit of n is either 0 or 2. Compute $n/15$.

[*2 marks, 583/716*]

2. Evaluate the product

$$(\sqrt{2}+\sqrt{11}+\sqrt{13})(\sqrt{2}+\sqrt{11}-\sqrt{13})(\sqrt{2}-\sqrt{11}+\sqrt{13})(-\sqrt{2}+\sqrt{11}+\sqrt{13}).$$

[*2 marks, 326/716*]

3. Each of the interior angles of a heptagon (a seven-sided polygon) is obtuse and the number of degrees in each angle is a multiple of 9. No two angles are equal. Find in degrees the sum of the largest two angles in the heptagon.

[*3 marks, 465/716*]

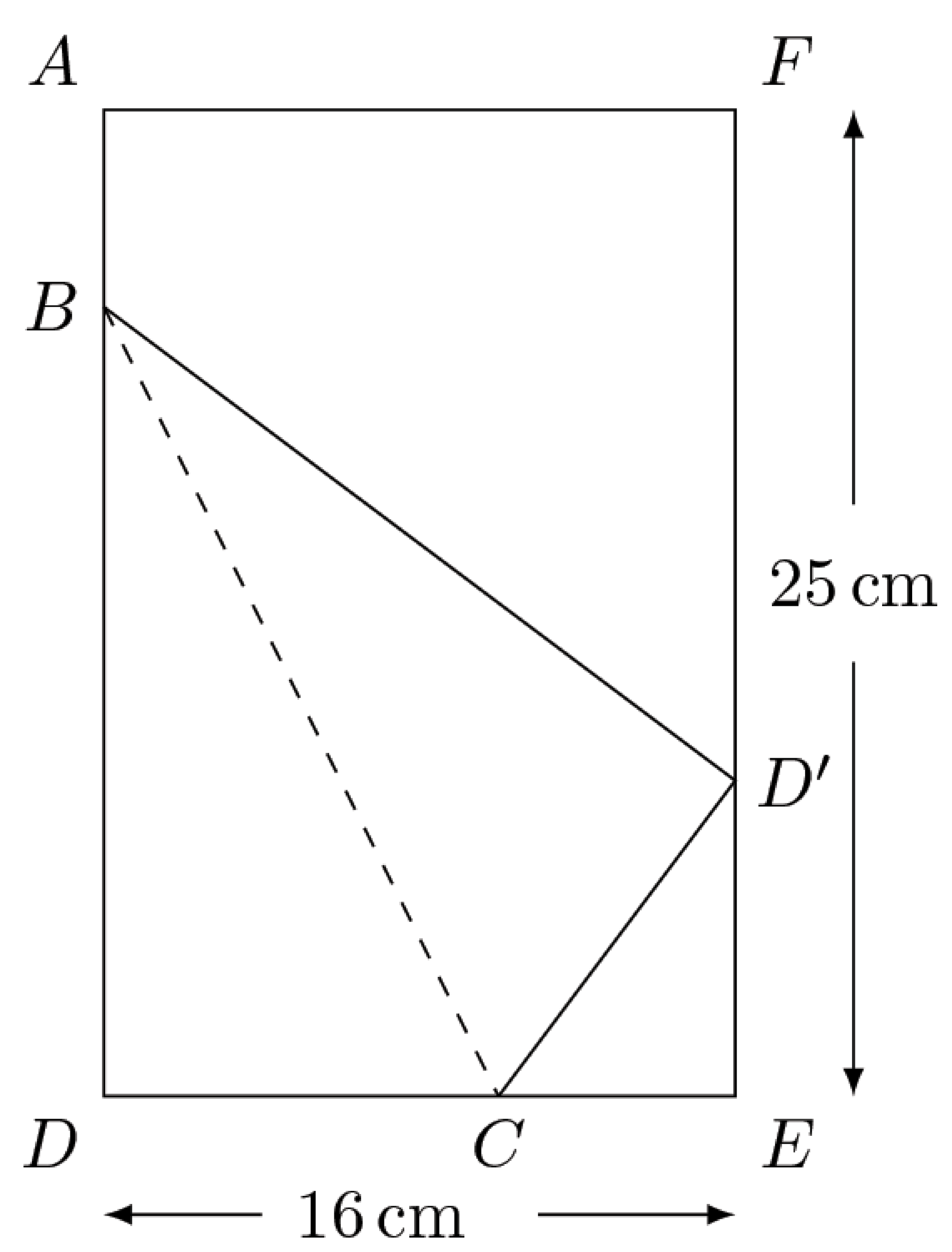
4. Briony takes a standard pack of 52 cards and throws some cards away. However she makes sure she keeps all four aces among the remaining cards. She then selects four cards at random from these remaining cards. If the probability of her selecting the four aces is $\frac{1}{1001}$, how many cards did she throw away?

[*3 marks, 314/716*]

5. Let $abcd$ be a four-digit number with the following properties: $a+b+c+d = ab = c \times d$, where ab denotes the two-digit number $10a+b$. What is the remainder when the largest number with these properties is divided by 1000?

[*3 marks, 330/716*]

6. A 16 cm$\times$25 cm rectangular piece of paper is folded so that a corner touches the opposite side as shown. The distance $AB = 5$ cm. Find BC^2.



[*3 marks, 349/716*]

7. Postumo is a stamp collector who specialises in stamps from Aruba, Mayotte and Svalbard. From time to time he rings his friends in these countries to talk about the latest stamp issues. The international phone codes for these countries are: Aruba 297, Mayotte 269, and Svalbard 47. So far Postumo has collected a total of 30 stamps from these countries. He has more stamps from Aruba than from Mayotte but fewer stamps from Aruba than from Svalbard. The number of his Svalbard stamps is more than four times but less than five times the number of his Aruban stamps. While this information is insufficient to determine the exact number of stamps from two countries in Postumo's collection, it is possible to determine the number of stamps he has from the remaining country. What is the result when you add this number to the country's international phone code?

[*4 marks, 498/716*]

8. A function f is defined for all real numbers and, for all real x, satisfies the equations

$$f(2+x) = f(2-x) \quad \text{and} \quad f(7+x) = f(7-x).$$

If $x = 0$ is a solution of $f(x) = 0$, what is the smallest number of solutions that $f(x) = 0$ could have in the interval $-2000 \le x \le 2000$?

[*4 marks, 15/716*]

9. A trapezium is divided into four triangles by its two diagonals. If X and Y are the areas of the triangles adjacent to the parallel sides, find in terms of X and Y the area of the trapezium.

[*4 marks, mean 0.4*]

10. For any positive integer n, let $g(n)$ denote the number of ordered pairs (x, y) of positive integers such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}.$$

a Find $g(10)$.

b Find $g(2000)$.

[*5 marks, mean 0.3*]

Investigation

a For any positive integer n, let $h(n)$ denote the number of ordered pairs (x, y) of positive integers such that

$$\frac{2}{x} + \frac{1}{y} = \frac{1}{n}.$$

Find $h(2000)$.

b Find $g(p)$ if p is prime.

Australian Intermediate Mathematics Olympiad 2001

1. When a three-digit number is multiplied by 3 and 1 added, the result is the reverse of the original number. What is the original number?

[2 marks, 659/751]

2. Let a and b be real numbers, with $a > 1$ and $b \neq 0$. If $ab = a^b$ and $\frac{a}{b} = a^{3b}$, find b^{-a}.

[3 marks, 177/751]

3. The telephone numbers in a large office all have three digits, running from 000 to 999, but not all are in use. If any two digits of a functioning number are interchanged, the resulting number either remains the same or becomes one that is not in use. What is the maximum number of functioning numbers possible?

[3 marks, 32/751]

4. Triangle ABC is an acute-angled triangle with $AB = 15$ and $AC = 13$. Let D be the foot of the perpendicular from A to BC and area $\triangle ADC = 30$. If the area of $\triangle ABC$ is an integer, what is it?

[3 marks, 330/751]

5. Find the number of ordered pairs of integers (x, y) which satisfy

$$x^2 + 2001 = y^2.$$

[3 marks, 26/751]

6. Find the smallest positive integer n such that there exists an integer m with

$$0.33 < \frac{m}{n} < \frac{1}{3}.$$

[3 marks, 57/751]

7. In triangle ABC, $\angle BAC = 22°$. A circle, centre O, has AB produced, AC produced and BC all as tangents. Find the number of degrees in $\angle BOC$.

[3 marks, 77/751]

8. Mr. Bean outsourced one of his soy plots, sized 10800 square metres, for processing to three companies specialised in insect treatment: Mount Rid, Legionnaire Futures and Ross River Limited. Mount Rid was able to offer a process that would take three hours to complete, while Legionnaire Futures quoted four hours and Ross River had an offer of six hours. The three companies agreed that they all should work together, each at their processing capabilities, to complete the treatment. Mr. Bean agreed with this plan and work commenced. When half the soy plot had been treated, Legionnaire Futures had to withdraw. The other two companies, sticking to the agreement, together completed the work. Find the total number of minutes taken to complete the treatment.

[4 marks, 448/751]

9. A circle, centre O, has AB as a diameter. Let C be a point on the circle different from A and B, D be the point on AB such that $\angle CDB = 90°$ and M be the point on BC such that $\angle BMO = 90°$. If DB is $3 \times OM$, calculate $\angle ABC$.

[5 marks, mean 0.5]

10. The nth triangular number is given by $t_n = 1+2+3+\cdots+n$. So the first 12 triangular numbers are: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78.
Notice that the integer 16 can be written as the sum of two triangular numbers in two different ways: $16 = 1 + 15$ (i.e. $t_1 + t_5$) and $16 = 6 + 10$ (i.e. $t_3 + t_4$). The integer 16 is the smallest integer with this property.

a i. Find the next smallest integer which can be written as the sum of two triangular numbers in two different ways.

ii. Given that $t_{k(n)} = t_n + t_{2n+3} - t_{n+2}$, find a formula for $k(n)$. Hence show that there are infinitely many integers each of which is the sum of two triangular numbers in at least two ways.

b Notice that $46 = 1 + 45$ (i.e. $t_1 + t_9$) and $46 = 10 + 36$ (i.e. $t_4 + t_8$). Find a general formula for which this is the first case and use this to find another infinite set of integers, each of which is the sum of two triangular numbers in at least two different ways.

[6 marks, mean 0.9]

Investigation

Are there any integers which are the product of two triangular numbers in two different ways? Are there infinitely many?

Australian Intermediate Mathematics Olympiad 2002

1. The number 888888 is written as the product of two 3-digit numbers. Find the larger.

[*2 marks, 520/791*]

2. $ABCD$ is a trapezium in which AB is parallel to DC, $AB = 84$ and $DC = 25$. A circle can be drawn inside the trapezium so that it just touches all four sides. Find the perimeter of the trapezium.

[*2 marks, 187/791*]

3. Start with three consecutive positive integers. Leave the first unchanged, add 10 to the second and add a prime number to the third. The three numbers are now in geometric progression. (That is, they are of the form a, ar, ar^2.) What was the prime number added to the third of the consecutive integers?

[*3 marks, 388/791*]

4. In 3-D space, points P and Q, 136 metres apart, are on the same side of a wall and are the same horizontal distance from it. At P a sound signal is generated. When travelling directly, it arrives at Q one-tenth of a second earlier than after being reflected on the wall. Assuming that sound travels at 340 metres per second and that the angles made by the sound beam and its reflection with the wall are equal, determine the distance, in metres, of the point P from the wall.

[*3 marks, 283/791*]

5. The currency of a small island republic has three different kinds of coins each worth a different integral amount of dollars. Heather received four coins of the republic worth \$28 while David collected five with a total value of \$21. Each had at least one coin of each kind. Find, in dollars, the total worth of the three coins.

[*3 marks, 602/791*]

6. ABC is a triangle with $AB = 360$, $BC = 240$ and $AC = 180$. The internal and external bisectors of $\angle CAB$ meet BC and BC produced at P and Q respectively. Find the radius of the circle which passes through A, P and Q.

[*3 marks, 28/791*]

7. A mathematics student read a telephone number $abc - defg$ and thought that it was a normal base 10 subtraction problem. The answer came out to be -95. Given that the digits of the telephone number were distinct, determine the smallest possible 7-digit telephone number with this property. What then is abc?

[*4 marks, 124/791*]

8. Sarah constructs a large cube from a pile of unit cubes. She then paints some (at least one) of the faces of her large cube blue, then some (at least one) of the remaining faces yellow, and then all of the remaining faces red. Later on, Brett comes along and pulls the large cube apart and discards the unit cubes with no paint on them. Brett takes the remaining unit cubes (those with some paint on them) and separates them into three piles as follows: he removes all the unit cubes that have some blue on them to form pile 1; from the unit cubes that remain, he removes all those which have some yellow on them to form pile 2; the remaining unit cubes form pile 3. Brett is surprised to find that the number of unit cubes in piles 1 and 3 is the same. Brett puts the three piles together again and forms three new piles as follows: he removes all the unit cubes which have some red on them to form pile 1; from the unit cubes that remain, he removes all those which have some yellow on them to form pile 2; the remaining unit cubes form pile 3. There are now more unit cubes in pile 1 than pile 3, but pile 2 contains the most. Brett puts the three piles together again and forms three new piles as follows: he removes all the unit cubes which have some blue on them to form pile 1; from the unit cubes that remain, he removes all the unit cubes which have some red on them to form pile 2; the remaining unit cubes form pile 3. There are now more unit cubes in pile 2 than pile 1, but pile 3 contains the most. How many unit cubes have some yellow on them?

[*4 marks, 56/791*]

9. Note that

(1) $(3n-1)^2 + (3n)^2 + (3n+1)^2 = (5n)^2 + (n-1)^2 + (n+1)^2$;

(2) $(3n)^2 + (3n+1)^2 + (3n+2)^2 = (5n+2)^2 + (n-1)^2 + n^2$.

a Find three positive integers which are not consecutive, the sum of whose squares is equal to:

i. $12^2 + 13^2 + 14^2$

ii. $11^2 + 12^2 + 13^2$

iii. $10^2 + 11^2 + 12^2$.

($\{3,4,5\}$ are consecutive numbers, $\{3,5,8\}$ and $\{3,4,6\}$ are non-consecutive numbers.)

b Prove that the sum of the squares of three consecutive positive integers, each greater than 3, can be written as the sum of squares of three non-consecutive positive integers.

c Find all the ways to write $7^2+8^2+9^2$ as the sum of three squares of positive integers.

[*5 marks, mean 0.9*]

10. Let $ABCD$ be a parallelogram and let P be a point inside it such that $\angle APB + \angle CPD = 180°$. Prove that $\angle PBC = \angle CDP$.

[*5 marks, mean 0.1*]

Investigation

If $ABCD$ is a rhombus, what is the locus of P?

Australian Intermediate Mathematics Olympiad 2003

1. Whee Bin's Chinese restaurant serves dishes in three sizes: small dishes cost \$$x$, medium dishes cost \$$y$ and large dishes cost \$$z$, where $x < y < z$ and x, y and z are positive integers.
Last Friday, Giovanna, Peter and John ordered a total of 9 small dishes, 6 medium dishes and 10 large dishes.
John remarked: 'This bill is exactly twice as much as when I was here 2 nights ago'.
Peter remarked: 'This bill is exactly three times as large as when I was here last night'.
Giovanna said, 'It is still a good price; the total is less than \$100'.
The prices have not been changed this week. How many dollars was the bill?

[*2 marks, 597/756*]

2. Annie, Bruce and Ken play a game of cards, each beginning and ending the game with a whole number of dollars. Annie starts with \$3 for every \$5 Bruce has, and finishes with twice as much money as Bruce. Ken starts with the same amount of money as Annie and ends with the same amount of money as Bruce. What is the least number of dollars that Bruce could have lost?

[*2 marks, 498/756*]

3. For how many integers between 1 and 2003 is the improper fraction $\dfrac{n^2+4}{n+5}$ not in simplest form?

[*3 marks, 18/756*]

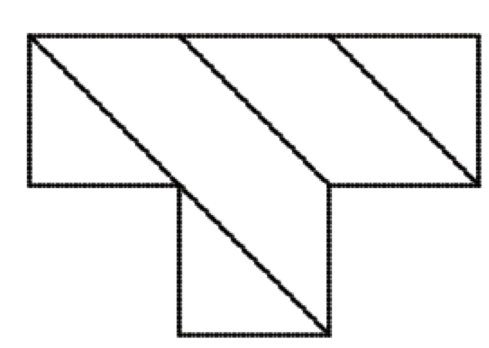
4. ABC is a triangle. AB is produced to P so that $PB = AB$. BC is produced to Q so that $QC = CB$. CA is produced to R so that $RA = AC$. The area of ABC is 51. What is the area of PQR?

[*3 marks, 283/756*]

5. Rina sets off on her bike to Con's place. At exactly the same moment, Con sets off to Rina's place along the same straight road in his car. A while later, they pass each other (neither spotting the other) and shortly after, Con arrives at Rina's place to find that she is not there. Con waits 22 minutes and then heads back along the same road, arriving at his place at exactly the same time as Rina. Rina travelled at the same speed the whole time whereas Con travelled 4 times as fast as Rina on the way to her place and 5 times as fast on the way back. How many minutes did it take Rina to reach Con's place?

[*3 marks, 495/756*]

6. Faith designs the following logo for a telephone company. The logo consists of a T divided into 5 regions. Faith has to colour each region with one of three colours: red, green or gold. No two regions with a common edge can have the same colour; and each colour must be used at least once. How many possible colour schemes are available to Faith for her logo?



[*3 marks, 190/756*]