2007 Australian Intermediate Mathematics Olympiad

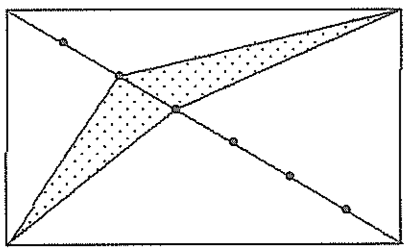
Time allowed: 4 hours
NO calculators are to be used

Questions 1 to 8 only require answers (non-negative integers less than 1000).
Questions 9 and 10 require written solutions and may require arguments or proofs.
The Investigation in Question 10 may be used to determine prize winners.

1. Trevor's trailer has two wheels on its axle and carries a spare wheel. The three wheels are changed around from time to time. The three tyres have been worn for 25000 km, 28000 km, and 31000 km respectively. How many thousand kilometres has Trevor's trailer travelled?

   [2 marks]

2. The rectangle shown has sides of length 28 and 15. The diagonal is divided into 7 equal parts. Find the area of the shaded quadrilateral.

   [3 marks]

3. When 113744 and 109417 are divided by the 3-digit positive integer N, the remainders are 119 and 292 respectively. Find N.

   [3 marks]

4. ABC is a triangle with $AB = 85$, $BC = 75$, and $CA = 40$. A semicircle is tangent to AB and AC and its diameter lies on BC. Find the radius of the semicircle.

   [3 marks]

5. Find $x+y$ where x and y are non-zero solutions of the system of equations

$$\begin{aligned} y^2x &= 15x^2+17xy+15y^2 \\ x^2y &= 20x^2+3y^2. \end{aligned}$$

[3 marks]

6. When a positive integer N is written in base 4 it has three digits. When $3N$ is written in base 6 it also has three digits and has the same middle digit as N to base 4. Find the decimal sum of all such numbers N.

[4 marks]

7. $x^2-19x+94$ is a perfect square where x is an integer. Find the largest possible value of x.

[4 marks]

8. A point P is marked inside a regular hexagon $ABCDEF$ so that $\angle BAP = \angle DCP = 50°$. Find $\angle ABP$.

[4 marks]

9. Find a prime, p, with the property that for some larger prime number, q, both $2q-p$ and $2q+p$ are prime numbers. Prove that there is only one such prime p.

[4 marks]

10. In a triangle ADC, $DC = 65$ and altitudes DB and CE have lengths 33 and 63 respectively. Prove that the lengths of AB and AE can not both be integers.

[5 marks]

Investigation

Find AB and AE.

In a triangle $A'D'C'$, $D'C' = 65k$ and altitudes $D'B'$ and $C'E'$ have lengths $33k$ and $63k$ respectively. Is there a value for k so that both $A'B'$ and $A'E'$ are integers? If not, explain why. If so, find all such values of k.

2007 Australian Intermediate Mathematics Olympiad Solutions

1. *Method 1*

   Each kilometre worn by a tyre is the same kilometre worn by one other tyre. So the total worn distance for the three tyres is exactly twice the distance the trailer travelled.

   Hence the total distance travelled by Trevor's trailer $= (25 + 28 + 31) \div 2 = 42$ thousand kilometres.

   Method 2

   Number the tyres 1, 2, 3.

   Let $x =$ the distance Trevor's trailer travels on tyres 1 and 2.
   Let $y =$ the distance Trevor's trailer travels on tyres 1 and 3.
   Let $z =$ the distance Trevor's trailer travels on tyres 2 and 3.

   Then
   $$\begin{aligned} x + y &= 25000 \\ x + z &= 28000 \\ y + z &= 31000. \end{aligned}$$

   Summing these equations gives $2x + 2y + 2z = 84000$.

   The total distance travelled by Trevor's trailer $= x + y + z = 42$ thousand kilometres.

   Method 3

   Number the tyres 1, 2, 3.

   Let $T =$ total distance travelled by Trevor's trailer.

   Tyre 1 rested for $T - 25000$ km.
   Tyre 2 rested for $T - 28000$ km.
   Tyre 3 rested for $T - 31000$ km.

   Total distance travelled = total resting distance.

   Therefore $T = (T-25000)+(T-28000)+(T-31000) = 3T-84000$.

   So $2T = 84000$ km and $T = 42$ thousand kilometres.

2. *Method 1*

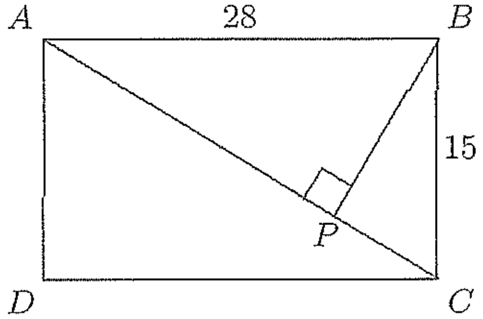
Join all the marks on the diagonal to the same two corners of the rectangle.

The seven triangles on one side of the diagonal have equal bases and the same height, so have the same area.

Hence the rectangle is divided into 7 quadrilaterals of equal area. The area of the shaded region is then $\frac{28\times 15}{7} = 60$.

Method 2

Let P be a point on the diagonal AC so that BP is perpendicular to AC.



From similar triangles APB and ABC we get the following equations.

$$\begin{aligned} \frac{PB}{BC} &= \frac{AB}{AC} \\ \frac{PB}{15} &= \frac{28}{AC} \\ PB &= \frac{15\times 28}{AC}. \end{aligned}$$

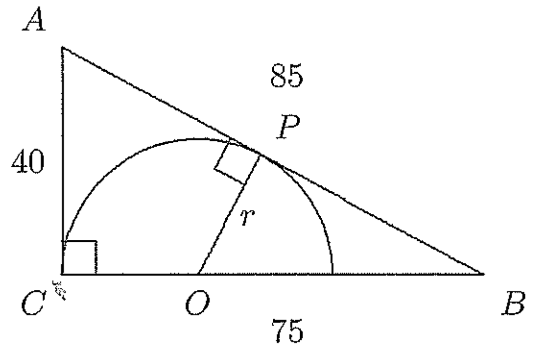
The shaded area $= 2 \times \frac{1}{2} \times \frac{AC}{7} \times PB = 15 \times 4 = 60$.

3. N must be a common divisor of $113744 - 119 = 113625$ and $109417 - 292 = 109125$.

By using Euclid's algorithm or extracting prime factors, we find that the greatest common divisor of these numbers is 1125. So N must be a factor of $1125 = 5^3 \times 3^2$.

The only 3-digit factors of 1125 are $5^3 = 125$, $5^3 \times 3 = 375$, and $5^2 \times 3^2 = 225$. N must be greater than the larger remainder, 292. So $N = 375$.

4. Since $75^2 + 40^2 = 5625 + 1600 = 7225 = 85^2$, $\angle ACB = 90°$ by the converse of Pythagoras' Theorem. Let r be the radius of the semicircle. Draw the radius OP perpendicular to AB as shown.



Method 1

Right-angled triangles OPB and ACB have a common angle at B, hence they are similar. Therefore

$$\begin{aligned}\frac{OP}{OB} &= \frac{AC}{AB}\\ \frac{r}{75-r} &= \frac{40}{85}\\ 85r &= 40(75-r).\end{aligned}$$

So $125r = 3000$ and $r = 24$.

Method 2

AC and AP are tangents to the semicircle. Hence $AP = AC = 40$ and $PB = 45$. Applying Pythagoras' theorem to $\triangle OPB$ gives the following equations.

$$\begin{aligned}OB^2 &= OP^2 + PB^2\\ (75-r)^2 &= r^2 + 45^2\\ 75^2 - 150r &= 45^2.\end{aligned}$$

So $6r = 225 - 81 = 144$ and $r = 24$.

5. *Method 1*

Multiplying the first equation by x and the second equation by y gives

$$\begin{aligned}y^2x^2 &= 15x^3 + 17x^2y + 15xy^2\\ x^2y^2 &= 20x^2y + 3y^3.\end{aligned}$$

Equating these gives

$$\begin{aligned} 15x^3 + 15xy^2 &= 3x^2y + 3y^3 \\ 5x(x^2 + y^2) &= y(x^2 + y^2) \\ 5x &= y. \end{aligned}$$

The second given equation gives $5x^3 = 20x^2 + 75x^2 = 95x^2$. Hence $x = 19$, $y = 95$, and $x + y = 114$.

Method 2

Let x and y be non-zero solutions of the system and put $z = \frac{y}{x}$. Then the system becomes

$$\begin{aligned} x^3z^2 &= 15x^2 + 17x^2z + 15x^2z^2 \\ x^3z &= 20x^2 + 3x^2z^2. \end{aligned}$$

After cancelling x^2 and multiplying the second equation by z, the system becomes

$$\begin{aligned} xz^2 &= 15 + 17z + 15z^2 \\ xz^2 &= 20z + 3z^3. \qquad (*) \end{aligned}$$

Equating these gives

$$\begin{aligned} 15 + 17z + 15z^2 &= 20z + 3z^3 \\ 15 + 15z^2 &= 3z + 3z^3 \\ 5(1 + z^2) &= z(1 + z^2) \\ z &= 5. \end{aligned}$$

From equation $(*)$, $25x = 100 + 375 = 475$.
Hence $x = 19$, $y = xz = 19 \times 5 = 95$, and $x + y = 114$.

6. Suppose N to base 4 is the 3-digit number abc and $3N$ to base 6 is the 3-digit number dbe. Then, in base 10, we have $N = 16a + 4b + c$ and $48a + 12b + 3c = 3N = 36d + 6b + e$.

   Hence 3 divides e, so it must be 0 or 3. Simplifying the equation gives $16a + 2b + c = 12d + f$, where f is 0 or 1. Also $1 \le a \le 3$, $0 \le b \le 3$, $0 \le c \le 3$, and $1 \le d \le 5$.

We tabulate the possible values for the variables, specifying d and f first and then determining all feasible values of a, b, c if any.

d	f	$12d+f$	a	b	c	N
1	0	12	-	-	-	-
1	1	13	-	-	-	-
2	0	24	1	3	2	30
2	1	25	1	3	3	31
3	0	36	2	1	2	38
			2	2	0	40
3	1	37	2	1	3	39
			2	2	1	41
4	0	48	3	0	0	48
4	1	49	3	0	1	49
5	0	60	-	-	-	-
5	1	61	-	-	-	-

Hence the sum of all integers N with the required property is $30+31+38+40+39+41+48+49=316$.

7. *Method 1*

Suppose $x^2-19x+94=m^2$, where m is a non-negative integer. Then $x^2-19x+94-m^2=0$ and $x=(19\pm\sqrt{19^2-4(94-m^2)})/2=(19\pm\sqrt{4m^2-15})/2$. Hence, $4m^2-15=r^2$ for some non-negative integer r.

So $(2m-r)(2m+r)=15$. Thus $2m-r=1$ and $2m+r=15$, or $2m-r=3$ and $2m+r=5$, giving $m=4$ or 2 respectively.

If $m=4$, then $x=(19\pm\sqrt{49})/2=13$ or 6. If $m=2$, then $x=(19\pm\sqrt{1})/2=10$ or 9. Hence the largest value for x is 13.

Method 2

Put $x=y+10$. Then y is an integer and $x^2-19x+94=y^2+y+4=z^2$ where z is a non-negative integer.

Suppose first that y is positive. Then $z>y$ and thus $z=y+a$ for some positive integer a. Hence $y^2+y+4=(y+a)^2=y^2+2ay+a^2$, which simplifies to $(2a-1)y=4-a^2$. Since the left-hand side of this equation is positive, the right-hand side must be too. Hence $a=1$ and $y=3$. It follows that $x=13$ is the only solution in this case.

If y is not positive, then $x\leq 10$.

Therefore the largest value for x is 13.

Method 3

Suppose $x^2 - 19x + 94 = b^2$, where b is an integer. Completing the square on x gives

$$\begin{aligned}
(x - \frac{19}{2})^2 + 94 - (\frac{19}{2})^2 &= b^2 \\
(2x - 19)^2 + 376 - 361 &= 4b^2 \\
4b^2 - (2x - 19)^2 &= 15 \\
(2b + 2x - 19)(2b - 2x + 19) &= 15.
\end{aligned}$$

Hence $\quad 2b+2x-19 \;=\; 1,\; 3,\; 5,\; 15,\; -1,\; -3,\; -5,\; -15$

and $\quad 2b-2x+19 \;=\; 15,\; 5,\; 3,\; 1,\; -15,\; -5,\; -3,\; -1$

respectively.

Subtracting these equations gives

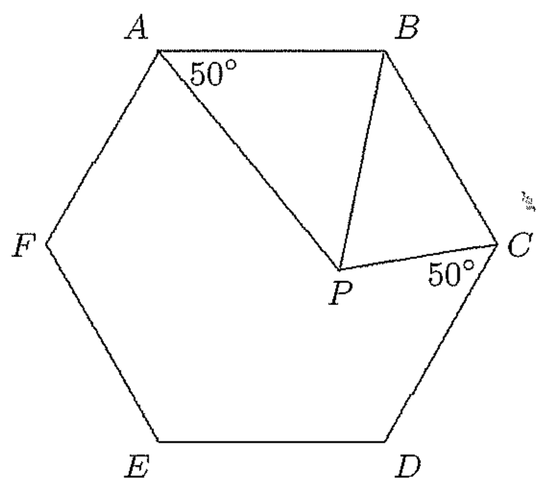
$$4x-38 = -14,\; -2,\; 2,\; 14,\; 14,\; 2,\; -2,\; -14$$

respectively. Hence $4x = 24, 36, 40$, or 52 and $x = 6, 9, 10$, or 13. Therefore the largest value for x is 13.

8. The internal angles of a regular hexagon are $120°$. Therefore $\angle ABC = 120°$ and $\angle BCP = 120° - 50° = 70°$. Since the sum of the angles in a quadrilateral is $360°$, $\angle APC = 120°$. Also the external angle $ABC = 240°$.

   Since $BA = BC$ and the external angle ABC is twice $\angle APC$, P lies on the circle through A and C with centre B. So $BA = BP$ and $\triangle ABP$ is isosceles.

   Therefore $\angle APB = 50°$, and $\angle ABP = 180° - 50° - 50° = 80°$.



9. If $p = 3$ and $q = 5$, then $2q - p = 7$ and $2q + p = 13$ which are both prime. So 3 has the required property. (Other solutions for $p = 3$ include $q = 7, 13, 17$.)

We now show that no other prime number has this property.

If $p = 2$, then $2q + p = 2(q + 1)$ which is never prime for $q > p = 2$.

Suppose $3 < p < q$, where p and q are prime numbers. Each of p and q must have a remainder of 1 or 2 when divided by 3. This gives the following four possibilities.

Case 1. $p = 3j + 1$ and $q = 3k + 1$ for some positive integers j and k. Then $2q + p = 3(2k + j + 1)$.

Case 2. $p = 3j + 1$ and $q = 3k + 2$ for some positive integers j and k. Then $2q - p = 3(2k - j + 1)$.

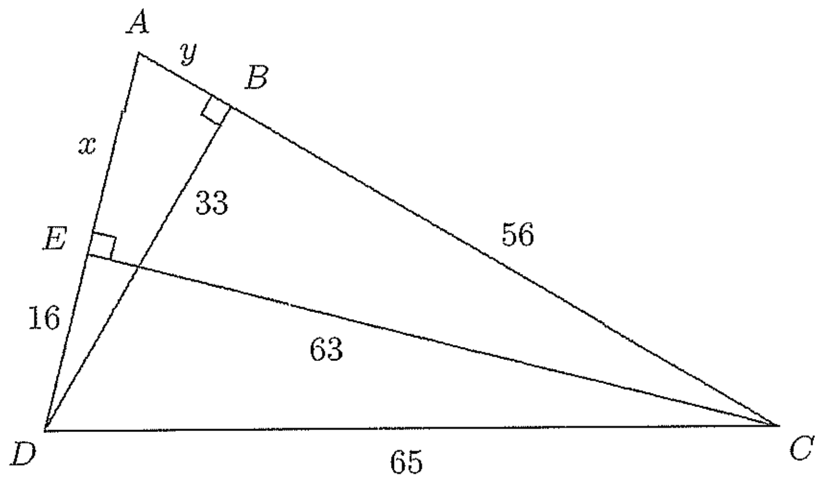
Case 3. $p = 3j + 2$ and $q = 3k + 1$ for some positive integers j and k. Then $2q - p = 3(2k - j)$.

Case 4. $p = 3j + 2$ and $q = 3k + 2$ for some positive integers j and k. Then $2q + p = 3(2k + j + 2)$.

Thus in each case either $2q - p$ or $2q + p$ is a composite number.

Therefore 3 is the only prime number p with the required property.

10. From Pythagoras' theorem we have $DE^2 = DC^2 - CE^2 = 4225 - 3969 = 256$ and $CB^2 = CD^2 - DB^2 = 4225 - 1089 = 3136$. Hence $DE = 16$ and $CB = 56$. Let $x = AE$ and $y = AB$.



Also from Pythagoras' theorem we have $AB^2 + BD^2 = AD^2$ and $AE^2 + EC^2 = AC^2$.

Hence $\quad y^2 + 1089 = (x + 16)^2 = x^2 + 32x + 256$

and $\quad x^2 + 3969 = (y + 56)^2 = y^2 + 112y + 3136$.

Adding these two equations gives

$$\begin{aligned}
5058 &= 32x + 112y + 3392\\
1666 &= 32x + 112y\\
833 &= 16x + 56y. \qquad (1)
\end{aligned}$$

If x and y were integers, then the right hand side would be an even integer and therefore could not equal 833. So x and y can not both be integers.

Investigation

Triangles ABD and AEC are similar because they are right-angled and have a common angle at A. Hence

$$\begin{aligned}
\frac{x+16}{33} &= \frac{y+56}{63}\\
21x + 336 &= 11y + 616\\
21x - 11y &= 280. \qquad (2)
\end{aligned}$$

We solve equations (1) and (2) simultaneously. Multiplying equation (1) by 21 and equation (2) by 16 gives

$$\begin{aligned}
336x + 1176y &= 17493\\
336x - 176y &= 4480.
\end{aligned}$$

Subtracting these equations gives $1352y = 13013$, $104y = 1001$, $8y = 77$, $y = \frac{77}{8}$.

Hence, from equation (1) we get $833 = 16x + 539$, $294 = 16x$, $x = \frac{147}{8}$.

As above, Pythagoras' Theorem gives $D'E' = 16k$ and $C'B' = 56k$. Therefore $\triangle D'B'C'$ is similar to $\triangle DBC$ and $\triangle D'E'C'$ is similar to $\triangle DEC$. Hence $\angle B'C'D' = \angle BCD$ and $\angle E'D'C' = \angle EDC$. So $\angle C'A'D' = \angle CAD$. Therefore $\triangle A'B'D'$ is similar to $\triangle ABD$ and $\triangle A'E'C'$ is similar to $\triangle AEC$. Hence $A'B' = \frac{77}{8}k$ and $A'E' = \frac{147}{8}k$.

Thus $A'B'$ and $A'E'$ are integers if and only k is a positive multiple of 8.