Math Logic Assignment #3 💯

— 饶翔云 520030910366

— Wed, Oct 19, 2022 1:51 PM

1.

Solution:

Suppose $\mathcal{B}$ is effectively decidable. Assume there is a set C produced by the following algorithm:

```
for i in N:
    if i not in Ai:
        i in C
```

Obviously C is a effectively decidable subset of $\mathbb{N}$. $C = A_k$ for some k.Then:
(k,k) ∈ B
⇔k in A_k
⇔k not in C
⇔k not in A_k
⇔(k,k) ∉ B
Therefore, B is not effectively decidable for if the input is (k,k) then B can output a valid answer.

2.

Solution:

We will prove it by induction:

Base case:For α, it has 1 sentences symbol and 0 binary connectives. So $s(\alpha) = 1 = c(\alpha) + 1$ is true in this case.

Inductive case:Suppose for two complex propositions α and β the conclusion is true. That is, $s(\alpha) = c(\alpha) + 1$ and $s(\beta) = c(\beta) + 1$. Then:

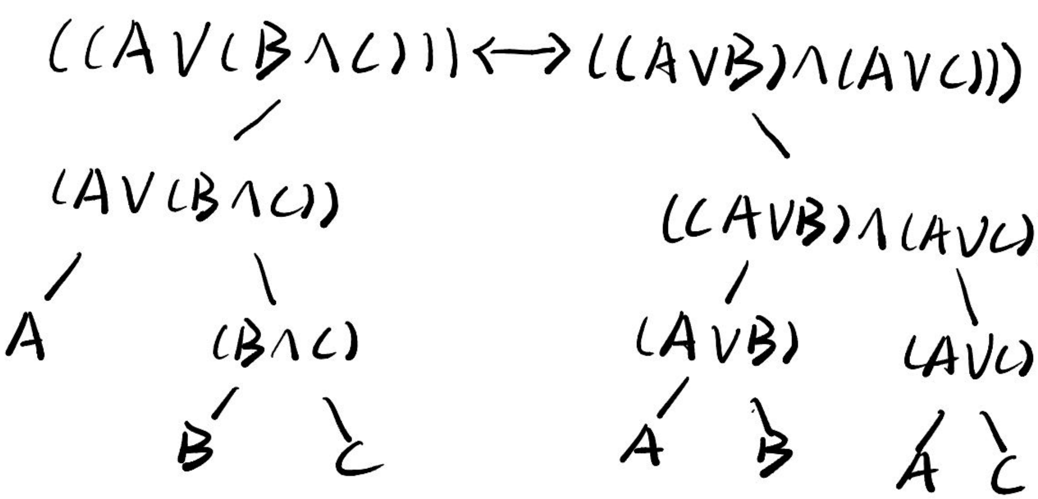
1. For proposition $\alpha \vee \beta$: $s(\alpha \vee \beta) = s(\alpha) + s(\beta) = c(\alpha) + c(\beta) + 1 + 1 = c(\alpha \vee \beta) + 1$, conclusion is true for this case.
2. For proposition $\alpha \wedge \beta$: $s(\alpha \wedge \beta) = s(\alpha) + s(\beta) = c(\alpha) + c(\beta) + 1 + 1 = c(\alpha \wedge \beta) + 1$, conclusion is true for this case.

3. For proposition $\alpha \to \beta$:
   $s(\alpha \to \beta) = s(\alpha) + s(\beta) = c(\alpha) + c(\beta) + 1 + 1 = c(\alpha \to \beta) + 1$, conclusion is true for this case.
4. For proposition $\alpha \leftrightarrow \beta$:
   $s(\alpha \leftrightarrow \beta) = s(\alpha) + s(\beta) = c(\alpha) + c(\beta) + 1 + 1 = c(\alpha \leftrightarrow \beta) + 1$, conclusion is true for this case.

Therefore, the conclusion is true by my induction.

3.

Solution:



This is the parse tree I construct. The construct is like below:
First we scan the whole proposition, we observe that there is no $\neg$ after (. So we continue scan for a complete proposition with balanced parentheses and followed by connectives. And we get $(A \vee (B \wedge C))$, and it is the first node in the first layer of parse tree. Then we continue scan to get a proposition with balanced parentheses and followed by), and we get $((A \vee B) \wedge (A \vee C))$, which is the second node of the first layer. And for nodes in the first layer, we do the same scanning, we get $A, (B \wedge C), (A \vee B), (A \vee C)$ as the second layer's nodes. And we get B, C, A, B, A, C

In a word, for nodes which have parentheses, we will first judge whether there is $\neg$. And there is no $\neg$ in this problem.Then we scan the node for a balanced proposition followed by) and a connective. That's the first node the parent node produces. Then we start to scan after the connective for a balanced proposition followed by). And that's the second node the parent node produces.

4.

Solution:

If $(P \wedge Q) \rightarrow R$ is false iff $(P \wedge Q)$ is true and R is false, that is, P, Q is true, R is false. Then $(P \rightarrow R) \vee (Q \rightarrow R)$ is false iff $(P \rightarrow R)$ and $(Q \rightarrow R)$ are false. Then we know P, Q is true and R is false. And we can easily know that for other truth assignments they are both true. So if a truth assignment satisties $(P \wedge Q) \rightarrow R$, then it also satisfies $(P \rightarrow R) \vee (Q \rightarrow R)$. That is $(P \wedge Q) \rightarrow R$ tautologically implies $(P \rightarrow R) \vee (Q \rightarrow R)$, and $(P \rightarrow R) \vee (Q \rightarrow R)$ tautologically implies $(P \wedge Q) \rightarrow R$.

Therefore, $(P \wedge Q) \rightarrow R$ tautologically implies $(P \rightarrow R) \vee (Q \rightarrow R)$, and they are tautologically equivalent.

5.

Solution:

1. Suppose $((P \rightarrow Q) \rightarrow P) \rightarrow P$ is false. $((P \rightarrow Q) \rightarrow P) \rightarrow P$ is false iff $(P \rightarrow Q) \rightarrow P$ is true and P is false. But we know if P is false then $(P \rightarrow Q)$ is true, then $(P \rightarrow Q) \rightarrow P$ is false, contradictory! **Therefore,** $((P \rightarrow Q) \rightarrow P) \rightarrow P$ **is a tautology.**
2. Suppose $(A \leftrightarrow B) \rightarrow \neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A))$ is false. $(A \leftrightarrow B) \rightarrow \neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A))$ is false iff $(A \leftrightarrow B)$ is true and $\neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A))$ is flase. From $(A \leftrightarrow B)$ is true we know $A = B$. Then if A is true:$(A \rightarrow B)$ is true and $(B \rightarrow A)$ is true.Then $\neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A))$ is true. And if A is true, then $(A \rightarrow B)$ is true and $(B \rightarrow A)$ is true. Then $\neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A))$ is true, contradictory! **Therefore,** $(A \leftrightarrow B) \rightarrow \neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A))$ **is a tautology.**

6.

Solution:

1. Proof:$\Sigma \models \alpha \leftrightarrow$ for every truth assignment satisfies Σ, it also satisfies α. If a truth assignment satisfies α, it also satisfies $\beta \rightarrow \alpha$ because $\beta \rightarrow \alpha$ is always true when α is true. Therefore, $\Sigma \models \beta \rightarrow \alpha$.
2. Proof:**(necessity):**$\Sigma, \beta \models \alpha$, that is $(\Sigma \wedge \beta) \models \alpha$. Then if a truth assignment satisfies Σ and β, then it also satisfies α. If a truth assignment satisfies Σ and β, it also satisfies α. Then $\beta \rightarrow \alpha$ will also be satisfied by the same truth assignment because β and α will both be satisfied by the same truth assignment which satisfied Σ and β. Therefore, if $\Sigma, \beta \models \alpha$ then $\Sigma \models \beta \rightarrow \alpha$.
   (sufficiency):$\Sigma \models \beta \rightarrow \alpha$, that is if a truth assignment satisfies Σ, then it also satisfies $\beta \rightarrow \alpha$. That means if a truth assignment satisfies Σ, it satisfies α when it satisfied β. Therefore, we can know that if a truth assignment satisfies Σ and β, it also satisfies α. That is $\Sigma, \beta \models \alpha$.