First-Order Logic: More Semantics

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What is the equivalent definition of "tautological implication" in first-order logic?

Logical Implication

Definition

Let Γ be a set of wffs and φ a wff. Γ logically implies φ, written as

$$\Gamma \models \varphi$$

if for every structure $\mathfrak{A}$ and every assignment $s : V \to |\mathfrak{A}|$,

if $\mathfrak{A}$ satisfies Γ with s, then $\mathfrak{A}$ satisfies φ with s.

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Remark

$\Gamma \vDash \varphi$ is also read as:

- φ is a logical consequence of Γ, or
- Γ semantically implies φ, or
- φ is a semantic consequence of Γ.

Logical Implication for Sentences

Theorem

For a set of sentences Σ and a sentence σ, $\Sigma \models \sigma$ iff for every model $\mathfrak{A}$ of Σ, $\mathfrak{A}$ is a model of σ.

Scorates Again

Question

Assume the following premises:

- All men are mortal.
- Socrates is a man.

We can derive the conclusion:

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and a constant symbol $\dot{c}$ denoting Socrates.
Let $\Sigma = \{\forall x(\dot{P}x \to \dot{Q}x), \dot{P}\dot{c}\}$. Then

$$\Sigma \vDash \dot{Q}\dot{c}.$$

Logical Equivalence

As before, we write $\alpha \models \beta$ for $\{\alpha\} \models \beta$.

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α and β are logically equivalent, written as $\alpha \models\!\!\!\!\dashv \beta$, if $\alpha \models \beta$ and $\beta \models \alpha$.

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Example

$$\forall x \forall y (\dot{P}x \to \neg \dot{Q}y) \models\!\!\!\!\!\!\dashv \forall x \forall y (\neg(\dot{P}x \wedge \dot{Q}y))$$

Valid Wffs

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φ is valid iff $\vDash_{\mathfrak{A}} \varphi[s]$ for every structure $\mathfrak{A}$ for $\mathbb{L}$ and every assignment function s for $\mathfrak{A}$.

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Corollary

A sentence σ is valid iff it is true in every structure.

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In other words, is there an algorithm that on input a wff φ will give an output of "yes" if φ is valid and output "no", otherwise?

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- The set Γ of wffs is satisfiable if there is some structure $\mathfrak{A}$ and some assignment $s : V \to |\mathfrak{A}|$ such that $\models_{\mathfrak{A}} \varphi[s]$ for every φ in Γ.

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φ is not satisfiable iff $\neg\varphi$ is

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Theorem

φ is not satisfiable iff $\neg\varphi$ is valid.

Is There a Compactness Theorem for First-Order Logic?

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Is the following statement true?

For every first-order language $\mathbb{L}$, and every set Γ of wffs of $\mathbb{L}$, if every finite subset of Γ is satisfiable then Γ is satisfiable.

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Answer

Yes! But we have to wait for a while to see the answer.

How can we characterize relations in *structures* by looking at wffs in first-order logic?

Relations Defined by Wffs

Definition

Let

- $\mathfrak{A}$ be a structure, and
- φ be a wff and n be such that the variables occurring free in φ are included among $v_1, \ldots, v_n$.

The n-ary relation defined by φ in $\mathfrak{A}$ is

$$\{(a_1, \ldots, a_n) \mid \models_{\mathfrak{A}} \varphi[\![a_1, \ldots, a_n]\!]\}$$

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- Let $\mathfrak{R} = (\mathbb{R}, <, +, \times, 0, 1)$. The 1-ary relation $\{a \in \mathbb{R} \mid 0 \leq a\}$ is defined by
$$\exists v_2, v_1 \doteq v_2 \times v_2$$
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in $\mathfrak{R}$;
- Let $\mathfrak{N} = (\mathbb{N}, <, +, \times, 0, 1)$. The 2-ary relation $\{(a, b) \mid a < b\}$ is defined by
$$\exists v_3 (v_1 \dot{+} (\dot{1} \dot{+} v_3) \dot{=} v_2)$$
in $\mathfrak{N}$.

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- Let f be a n-ary function f whose domain is a subset of $\underbrace{|\mathfrak{A}| \times \ldots \times |\mathfrak{A}|}_{n}$ and whose range is a subset of $|\mathfrak{A}|$. f is definable in $\mathfrak{A}$ if the (n+1)-ary relation

$$\{(a_1, \ldots, a_n, b) \mid f(a_1, \ldots, a_n) = b\}$$

is definable in $\mathfrak{A}$.

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$$f(a, b) = \begin{cases} b - a & \text{if } a \leq b \\ \text{Undefined} & \text{Otherwise} \end{cases}$$

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What happens if $\doteq$ is not in $\mathbb{L}$?

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- If the $n+1$-ary relation R is definable in $\mathfrak{A}$ then so are the n-ary relations

$$\{(a_1, \ldots, a_n) \mid \text{there exists } b \in |\mathfrak{A}|, (a_1, \ldots, a_n, b) \in R\}$$
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In particular, if R is a binary relation that is definable in $\mathfrak{A}$ then $dom(R)$ and $rng(R)$ is definable.

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So the formula $\neg\varphi \wedge \forall v_3 (v_3 \dot{<} v_1 \rightarrow \varphi(v_3))$ defines $\{1\}$.

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- $\{n\}$ for each $n \in \mathbb{N}$.

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How Many Relations are Definable?

Lemma

(1) Given a structure $\mathfrak{A}$, the set of definable relations is *enumerable*;

(2) Not every subset of $\mathbb{N}$ is definable.

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For (1), note that the set of wffs is enumerable, and every wff may define only one relation.

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Proof.

For (1), note that the set of wffs is enumerable, and every wff may define only one relation.
For (2), note that the set of all subsets of $\mathbb{N}$ is uncountable. Therefore, some subset may not match a wff. $\square$

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Question

More generally, given a first-order language $\mathbb{L}$ and a structure $\mathfrak{A}$ for $\mathbb{L}$, how do we figure out which relations in $\mathfrak{A}$ are definable?

Given any wff φ, how do we relate its satisfactions in different structures?

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Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathbb{L}$. A homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is a function $h : |\mathfrak{A}| \to |\mathfrak{B}|$ such that:

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- An automorphism of $\mathfrak{A}$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}$.

Examples

Example

Let $\mathfrak{A} = (\mathbb{N}, <^{\mathbb{N}}, +^{\mathbb{N}})$ and $\mathfrak{B} = (\mathbb{E}, <^{\mathbb{E}}, +^{\mathbb{E}})$.
Here $\mathbb{E}$ is the set of even non-negative integers, $<^{\mathbb{E}}$ is the "less than" relation on $\mathbb{E}$, etc.
Then h is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, where for all $n \in \mathbb{N}$,

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- In general, if h is an automorphism of $\mathfrak{N}$, $h(n) = n$
- Therefore, the identity function is the only automorphism of $\mathfrak{N}$.

Substructures

A special kind of isomorphisms:

Definition

Let $\mathfrak{A} = (A, \ldots)$ and $\mathfrak{B} = (B, \ldots)$ be structures for $\mathbb{L}$. $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ (written $\mathfrak{A} \subseteq \mathfrak{B}$) if:

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Examples

Example

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- $\mathfrak{E} = (\mathbb{E}, <^{\mathbb{E}}, +^{\mathbb{E}}, \times^{\mathbb{E}})$.

Then $\mathfrak{E}$ is a substructure of $\mathfrak{N}$.

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Then $\mathfrak{E}$ is a substructure of $\mathfrak{N}$.

Question

Let

- $\mathfrak{A} = (\{0,1,2,3\}, P^{\mathfrak{A}})$, where $P^{\mathfrak{A}} = \{0,1,2\}$;
- $\mathfrak{B} = (\{0,1\}, P^{\mathfrak{B}})$, where $P^{\mathfrak{B}} = \{0\}$.

Is $\mathfrak{B}$ is a substructure of $\mathfrak{A}$?

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Let

- $\mathfrak{N} = (\mathbb{N}, <^{\mathbb{N}}, +^{\mathbb{N}}, \times^{\mathbb{N}})$
- $\mathfrak{E} = (\mathbb{E}, <^{\mathbb{E}}, +^{\mathbb{E}}, \times^{\mathbb{E}})$.

Then $\mathfrak{E}$ is a substructure of $\mathfrak{N}$.

Question

Let

- $\mathfrak{A} = (\{0,1,2,3\}, P^{\mathfrak{A}})$, where $P^{\mathfrak{A}} = \{0,1,2\}$;
- $\mathfrak{B} = (\{0,1\}, P^{\mathfrak{B}})$, where $P^{\mathfrak{B}} = \{0\}$.

Is $\mathfrak{B}$ is a substructure of $\mathfrak{A}$?

Answer

No. Because $P^{\mathfrak{A}} \cap \{0,1\} = \{0,1\} \neq \{0\} = P^{\mathfrak{B}}$.

Notation: Function Composition

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If f and g are functions, then $f \circ g$ is the composition of f and g. That is,

$$f \circ g(a) = f(g(a)).$$

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Example

Suppose $s : V \to |\mathfrak{A}|$ is an assignment function for $\mathfrak{A}$, and h is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Then $h \circ s$ is an assignment function for $\mathfrak{B}$.

The Value of Terms Under a Homomorphism

Lemma

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for the language $\mathbb{L}$.
Let h be a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, and $s : V \to |\mathfrak{A}|$ be an assignment for $\mathfrak{A}$. Then for every term t of $\mathbb{L}$,

$$h(\overline{s}(t)) = \overline{h \circ s}(t).$$

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Proof.

By induction on t. $\square$

The Homomorphism Theorem

Theorem (The Homomorphism Theorem)

Let h be a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ and s be an assignment function for $\mathfrak{A}$. The statement

$$\vDash_{\mathfrak{A}} \varphi[s] \iff \vDash_{\mathfrak{B}} \varphi[h \circ s]$$

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By induction on φ. $\square$

Corollaries of the Homomorphism Theorem

Corollary

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Answer

No.

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Question

Do you think the converse is true?

Answer

No. Take $\mathfrak{R} = (\mathbb{R}, <)$ and $\mathfrak{Q} = (\mathbb{Q}, <)$ as an counter example.

Corollaries of the Homomorphism Theorem

Corollary (Automorphism Theorem)

Let h be an automorphism of $\mathfrak{A}$. Let R be an n-rary relation on $|\mathfrak{A}|$ that is definable in $\mathfrak{A}$. For every n-tuple $(a_1, \ldots, a_n)$ of elements of $\mathfrak{A}$:

$$(a_1, \ldots, a_n) \in R \Longleftrightarrow (h(a_1), \ldots, h(a_n)) \in R.$$

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Example

Let $\mathfrak{R} = (\mathbb{R}, <)$. Its subset $\mathbb{N}$ is not definable in $\mathfrak{R}$ because $h(a) = a^3$ is an automorphism of $\mathfrak{R}$.