Math Logic: Assignment 5

Nov 28, 2022

Attention: To get full credits, you *must provide explanations to your answers*! You will get at most $1/3$ of the points if you only present the final results.

1. (6pt) In sentential logic, prove that
   - (3pt) $(A \wedge B) \vee (\neg A \vee \neg B)$ is provable;
   - (3pt) $(A \wedge B) \vee (\neg A \wedge \neg B)$ is not provable. (Hint: remember the soundness and completeness theorems)

2. (6pt) Translate the following sentences into wffs in First-Order Logic:
   - (3pt) There is no such a set that every set is its member. ($\forall$ denotes for every set, $\exists$ denotes there exists a set, $\in$ denotes "is a member of").
   - (3pt) Every farmer who owns a donkey needs hay, and every farmer who owns a donkey beats it. ($F\ x$ denotes x is a farmer; $D\ x$ denotes x is a donkey; $O\ x\ y$ denotes x owns y; $H\ x$ denotes x needs hay; $B\ x\ y$ denotes x beats y.)

3. (6pt) List the variables occurring free in the following wffs (where Q and R are 1-ary predicate symbols; P is a 2-ary predicate symbol; f is a 2-ary function symbol)
   - (2pt) $\forall y\ (P\ x\ y \to \forall x\ P\ x\ y)$;
   - (2pt) $\forall x\ (Q\ y \to \exists\ y\ P\ x\ z)$;
   - (2pt) $(\neg \exists\ y\ R\ (f\ y\ z)) \wedge (\forall x\ \forall y\ R\ (f\ y\ z))$

4. (10pt) Let $\mathfrak{N} = (\mathbb{N}, +, \times, 0, 1, <)$. Let s be an assignment for $\mathfrak{N}$ such that $s(v_n) = 2n$. Are the following statement true or not? Give explanations to your answers (Hint: remember that assignments can only affect free occurrences of variables).
   - (2pt) $\models_{\mathfrak{N}} \exists v_0, v_0 \dot{+} v_0 \doteq v_1[s]$;
   - (2pt) $\models_{\mathfrak{N}} \exists v_0, v_0 \dot{\times} v_0 \doteq v_1[s]$;
   - (2pt) $\models_{\mathfrak{N}} \forall v_0 \exists v_1\ v_0 \doteq v_1[s]$;
   - (2pt) $\models_{\mathfrak{N}} \forall v_0 \forall v_1\ v_0 \dot{+} \dot{1} \dot{<} v_1 \to \exists v_2\ v_0 \dot{<} v_2 \wedge v_2 \dot{<} v_1[s]$;
   - (2pt) $\models_{\mathfrak{N}} \forall v_0 \forall v_1\ v_0 \dot{<} v_2 \wedge v_2 \dot{<} v_1[s]$;

5. (3pt) Prove that if x does not occur free in α, then for any structure $\mathfrak{A}$ and assignment $s : V \to |\mathfrak{A}|$,

   $$\models_{\mathfrak{A}} (\alpha \to \forall x\ \alpha)[s].$$

   (Hint: remember the lemmas about occurrences of free variables.)

6. (4pt) A monoid is a set M with an element $e \in M$ and a binary operator (function) $\circ : M \times M \to M$ (we write $\circ$ in infix form, i.e., $a \circ b$ denotes $\circ(a, b)$) that satisfies the following properties

- e *is an identity element:* for any $a \in M$, $e \circ a = a \circ e = a$;
- $\circ$ *is associative:* for any $a, b, c \in M$, $(a \circ b) \circ c = a \circ (b \circ c)$.

For example, the set $\mathbb{N}$ with $e = 0$ and $\circ = +$ is a monoid. For another example, the set $\mathbb{N}$ with $e = 1$ and $\circ = *$ is also a monoid.

Let $\mathbb{L}$ be a language containing $\doteq$, a constant $\dot{e}$, a 2-ary function symbol $\dot{\circ}$. Write down a sentence σ such that for any structure $\mathfrak{A}$, $|\mathfrak{A}|$ is a monoid with $\dot{e}^{\mathfrak{A}}$ as identity and $\dot{\circ}^{\mathfrak{A}}$ as the associative operator iff $\vDash_{\mathfrak{A}} \sigma$.