Math Logic: Assignment 4

Nov 7, 2022

Attention: To get full credits, you *must provide explanations to your answers*! You will get at most $1/3$ of the points if you only present the final results.

1. (4pt) Given the wff $(A \leftrightarrow B) \leftrightarrow C$ where A, B and C are sentence symbols,
   - (2pt) Compute its disjunctive normal form by using the truth table method;
   - (2pt) Compute its conjunctive normal form from its disjunctive normal form.
2. (3pt) Given a formula α and a set Σ of wffs, assume $\Sigma \cup \{\neg\alpha\}$ is not satisfiable. Show that there exists some finite set Δ such that $\Delta \subseteq \Sigma$ and $\Delta \vDash \alpha$.
3. (3pt) Let Σ be an effectively enumerable. Assume that for each wff α, either $\Sigma \vDash \alpha$ or $\Sigma \vDash \neg\alpha$. Show that the set of semantic consequences of Σ is effectively decidable.
4. (12pt) Construct proof trees for the following wffs (Hint: some of them may only be constructed by using the Law of Excluded Middle):

   Notice: you must clearly mark the name of each inference rule in your proof trees.
   - (4pt) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$;
   - (4pt) $\neg(\neg A \wedge \neg B) \to A \vee B$;
   - (4pt) $((P \to Q) \to P) \to P$;
5. (8pt) Prove the soundness of natural deduction for the inductive cases when the bottom rule is $\neg$-I and $\neg$-E. Remember the soundness theorem is stated as follows:

   If $\Sigma \vdash \alpha$ then $\Sigma \vDash \alpha$.

   The proof proceeds by induction on the height of the partial proof tree for $\Sigma \vdash \alpha$. You need to show the following:
   - (4pt) Assume $\alpha = \neg\beta$ and the tree looks like

     $$\dfrac{\begin{array}{c}\gamma_1 \quad \cdots \quad \gamma_n \quad [\beta] \\ \vdots \\ \delta \wedge \neg\delta\end{array}}{\neg\beta}\ \neg\text{-}I$$

     where $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Sigma$. Assume soundness holds for proof trees with a smaller height, prove $\Sigma \vDash \neg\beta$.
   - (4pt) Assume the tree looks like

     $$\dfrac{\begin{array}{c}\gamma_1 \quad \cdots \quad \gamma_n \\ \vdots \\ \beta\end{array} \qquad \begin{array}{c}\sigma_1 \quad \cdots \quad \sigma_m \\ \vdots \\ \neg\beta\end{array}}{\alpha}\ \neg\text{-}E$$

     where $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Sigma$ and $\{\sigma_1, \ldots, \sigma_m\} \subseteq \Sigma$. Assume soundness holds for proof trees with a smaller height, prove $\Sigma \vDash \alpha$.