Math Logic: Assignment 3

Oct 18, 2022

Attention: To get full credits, you *must provide explanations to your answers*! You will get at most $1/3$ of the points if you only present the final results.

1. (5pt) Let $A_0, A_1, \ldots, A_n, \ldots$ be a listing of *all* the effectively decidable subset of $\mathbb{N}$. Let B be the binary relation on $\mathbb{N}$ defined by
$$(m, n) \in B \Longleftrightarrow m \in A_n.$$
Prove that B is *not* effectively decidable (Hint: Use a diagonal argument).

2. (5pt) For any wff α, prove that the number of occurrences of sentences symbols in α (denoted by $s(\alpha)$) is 1 greater than the number of binary connectives $(\wedge, \vee, \rightarrow, \leftrightarrow)$ in α (denoted by $c(\alpha)$). For example, if $\alpha = ((\neg A_3) \vee (A_8 \leftrightarrow A_3))$, then $s(\alpha) = 3, c(\alpha) = 2$ and $s(\alpha) = c(\alpha) + 1$. (Hint: use the induction principle for wffs.)

3. (4pt) Apply the parsing alogrithm to construct the parse tree of
$$((A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C)))$$
You need to draw the resulting parse tree and explain how is it constructed by the algorithm.

4. (5pt) Determine whether or not
   - (2pt) $(P \wedge Q) \rightarrow R$ tautologically implies $(P \rightarrow R) \vee (Q \rightarrow R)$;
   - (3pt) $(P \wedge Q) \rightarrow R$ is tautologically equivalent to $(P \rightarrow R) \vee (Q \rightarrow R)$.

5. (5pt) Determine whether or not the following wffs are tautologies:
   - (2pt) $((P \rightarrow Q) \rightarrow P) \rightarrow P$;
   - (3pt) $(A \leftrightarrow B) \rightarrow \neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A))$.

6. (6pt) Prove the following holds:
   - (3pt) If $\Sigma \vDash \alpha$ then for any β, $\Sigma \vDash \beta \rightarrow \alpha$;
   - (3pt) $\Sigma, \beta \vDash \alpha$ iff $\Sigma \vDash \beta \rightarrow \alpha$.