Math Logic Assignment #5 💯

tags: `Math Logic` 作业

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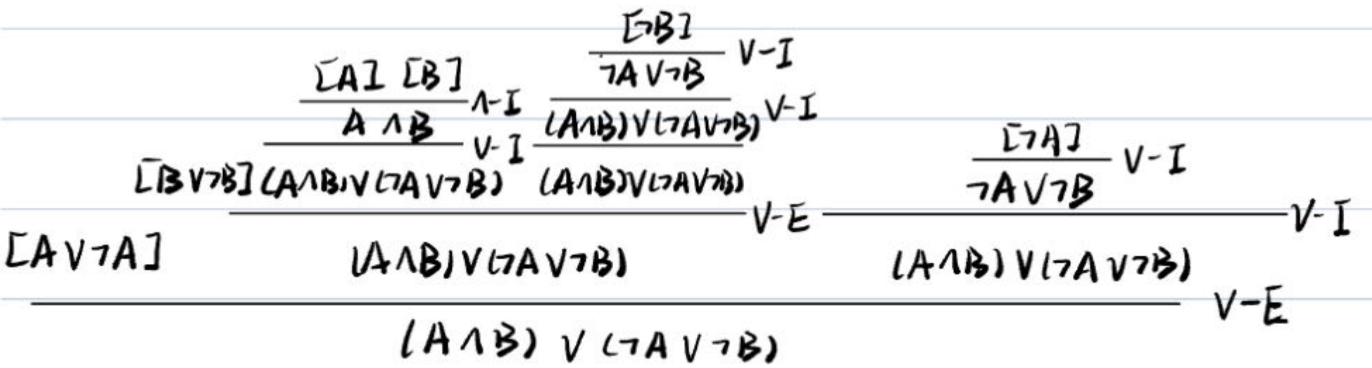
1.

(1) Solution:



Therefore, $\vdash (A \wedge B) \vee (\neg A \vee \neg B)$, $(A \wedge B) \vee (\neg A \wedge \neg B)$ is provable.

(2) Solution:

First let's make a truth value table:

A	B	$(A \wedge B) \vee (\neg A \wedge \neg B)$
T	T	T
T	F	F
F	T	F
F	F	T

From the truth value table we know that exists truth assignment V that $\bar{V}((A \wedge B) \vee (\neg A \wedge \neg B)) = False$. Therefore, $(A \wedge B) \vee (\neg A \wedge \neg B)$ is not a tautology. According to the soundness and completeness theorems, we can know that if $\models \alpha$ then $\vdash \alpha$

and if $\vdash \alpha$ then $\models \alpha$. That is, $\models \alpha \iff \vdash \alpha$, and $\not\models \alpha \iff \nvdash \alpha$. From our conclusion we know that $\not\models (A \wedge B) \vee (\neg A \wedge \neg B)$ is True. Therefore, $\nvdash (A \wedge B) \vee (\neg A \wedge \neg B)$ is True. Therefore, $(A \wedge B) \vee (\neg A \wedge \neg B)$ is not provable

2.

(1) Solution:

There is no such a set that every set is its member.
$\rightarrow$($\neg$ [There is such a set that every set is its member])
$\rightarrow$($\neg$ $\exists v_1$ [every set is v_1's member])
$\rightarrow$($\neg$ $\exists v_1 \forall v_2 [v_2$ is v_1's member])
$\rightarrow$($\neg$ $\exists v_1 \forall v_2$,$v_2 \in v_1$)

(2) Solution:

Every farmer who owns a donkey needs hay, and every farmer who owns a donkey beats it.
$\rightarrow$[Every farmer who owns a donkey needs hay]$\wedge$[every farmer who owns a donkey beats it]
$\rightarrow\forall x$[if x is a farmer and owns a donkey, x needs hay]$\wedge\forall x$[if x is a farmer and owns a donkey, x beats it]
$\rightarrow\forall x Fx$[if x owns a donkey, x needs hay]$\wedge\forall x Fx$ [if x owns a donkey, x beats it]
$\rightarrow\forall x \exists y Fx \vee Dy \vee Oxy \rightarrow$[x needs hay]$\wedge\forall x \exists y Fx \vee Dy \vee Oxy \rightarrow$ [x beats it]
$\rightarrow\forall x \exists y Fx Dy Oxy \rightarrow Hxy \wedge \forall x \exists y Fx Dy Oxy \rightarrow Bxy$

3.

(1) Solution:

$[x]$.

(2) Solution:

$[y, z]$.

(3) Solution:

$[z, x]$.

4.

(1). Solution:

$\models_{\mathfrak{N}} \exists v_0, v_0 \dot{+} v_0 \doteq v_1[s]$

$\Leftrightarrow$There is c $\in \mathbb{N}$,$\models_{\mathfrak{N}} v_0 \dot{+} v_0 \doteq v_1[s(v_0|c)]$(mark $s(v_0|c)$ as s_1)

$\Leftrightarrow$There is c $\in \mathbb{N}$,$\models_{\mathfrak{N}} v_0 \dot{+} v_0 \doteq v_1[s_1]$

$\Leftrightarrow$There is c $\in \mathbb{N}$,$s_1(v_0) + s_1(v_0) = s_1(v_1)$

$\Leftrightarrow$There is c $\in \mathbb{N}$,$c + c = 2c = s(v_1)$

From the start we can know that $s(v_1) = 2 * 1 = 2$. Therefore,$\models_{\mathfrak{N}} \exists v_0, v_0 \dot{+} v_0 \doteq v_1[s]$ holds.

(2). Solution:

$\models_{\mathfrak{N}} \exists v_0, v_0 \dot{\times} v_0 \doteq v_1[s]$

$\Leftrightarrow$There is c $\in \mathbb{N}$,$\models_{\mathfrak{N}} v_0 \dot{\times} v_0 \doteq v_1[s(v_0|c)]$(mark $s(v_0|c)$ as s_1)

$\Leftrightarrow$There is c $\in \mathbb{N}$,$\models_{\mathfrak{N}} v_0 \dot{\times} v_0 \doteq v_1[s_1]$

$\Leftrightarrow$There is c $\in \mathbb{N}$,$s_1(v_0) \times s_1(v_0) = s_1(v_1)$

$\Leftrightarrow$There is c $\in \mathbb{N}$,$c \times c = c^2 = s(v_1)$

We can easily know that $s(v_1) = 2 * 1 = 2$. It can not be reached by square of natural number. Therefore, $\models_{\mathfrak{N}} \exists v_0, v_0 \dot{\times} v_0 \doteq v_1[s]$ doesn't hold.

(3) Solution:

$\models_{\mathfrak{N}} \forall v_0 \exists v_1, v_0 \doteq v_1[s]$

$\Leftrightarrow$for all $a \in \mathbb{N}$, there is $b \in \mathbb{N}$, $\models_{\mathfrak{N}} v_0 \doteq v_1[s(v_0|a)(v_1|b)]$(mark $s(v_0|a)(v_1|b)$ as s_2)

$\Leftrightarrow$for all $a \in \mathbb{N}$, there is $b \in \mathbb{N}$, $s_2(v_0) = s_2(v_1)$

$\Leftrightarrow$for all $a \in \mathbb{N}$, there is $b \in \mathbb{N}$, $a = b$, and it obviously holds.

(4) Solution:

$\models_{\mathfrak{N}} \forall v_0 \forall v_1 \quad v_0 \dot{+} \dot{1} \dot{<} v_1 \rightarrow \exists v_2 v_0 \dot{<} v_2 \wedge v_2 \dot{<} v_1[s]$

$\Leftrightarrow$ for all $a, b \in \mathbb{N}$, $\models_{\mathfrak{N}} v_0 \dot{+} \dot{1} \dot{<} v_1 \rightarrow \exists v_2 v_0 \dot{<} v_2 \wedge v_2 \dot{<} v_1[s(v_0|a)(v_1|b)]$

(mark $s(v_0|a)(v_1|b)$ as s_2)

$\Leftrightarrow$ for all $a, b \in \mathbb{N}$, $\models_{\mathfrak{N}} v_0 \dot{+} \dot{1} \dot{<} v_1[s_2] \Rightarrow \models_{\mathfrak{N}} \exists v_2 v_0 \dot{<} v_2 \wedge v_2 \dot{<} v_1[s_2]$

$\Leftrightarrow$ for all $a, b \in \mathbb{N}$, $\models_{\mathfrak{N}} v_0 \dot{+} \dot{1} \dot{<} v_1[s_2] \Rightarrow$there is $c \in \mathbb{N}$,$\models_{\mathfrak{N}} v_0 \dot{<} v_2 \wedge v_2 \dot{<} v_1[s_2(v_2|c)]$

(mark $s_2(v_2|c)$ as s_3)

$\Leftrightarrow$ for all $a, b \in \mathbb{N}$, $s_2(v_0) + 1 < s_2(v_1) \Rightarrow$ there is $c \in \mathbb{N}$, $s_3(v_0) < s_3(v_2) \wedge s_3(v_2) < s_3(v_1)$

$\Leftrightarrow$ for all $a, b \in \mathbb{N}$, $a + 1 < b \Rightarrow$ there is $c \in \mathbb{N}$, $a < c \wedge c < b$

First since $a, b \in \mathbb{N}$,there is not element $\in \mathbb{N}$ that $\in [a, a+1]$. But if $a + 1 < b$, then there is at least $a + 1 \in [a, b]$ and $a + 1 \neq a, a + 1 \neq b$.Therefore,
$\models_{\mathfrak{N}} \forall v_0 \forall v_1 \quad v_0 \dot{+} \dot{1} \dot{<} v_1 \rightarrow \exists v_2 v_0 \dot{<} v_2 \wedge v_2 \dot{<} v_1[s]$ holds.

(5). Solution:

$\models_{\mathfrak{N}} \forall v_0 \forall v_1 v_0 \dot{<} v_2 \wedge v_2 \dot{<} v_1 [s]$
$\Leftrightarrow$ for all $a, b \in \mathbb{N}$,$\models_{\mathfrak{N}} v_0 \dot{<} v_2 \wedge v_2 \dot{<} v_1 [s(v_0|a)(v_1|b)]$ (mark $s(v_0|a)(v_1|b)$ as s_1)
$\Leftrightarrow$ for all $a, b \in \mathbb{N}$,$s_1(v_0) < s_1(v_2)$ and $s_1(v_2) < s_1(v_1)$
$\Leftrightarrow$ for all $a, b \in \mathbb{N}$,$a < 4$ and $4 < b$.

Obviously, if a=4 or b=4, $a < 4$ and $4 < b$ doesn't hold. Therefore, $\models_{\mathfrak{N}} \forall v_0 \forall v_1 v_0 \dot{<} v_2 \wedge v_2 \dot{<} v_1 [s]$ doesn't hold.

5. Solution:

$\models_{\mathfrak{A}} (\alpha \to \forall x \alpha)[s]$
$\Leftrightarrow \models_{\mathfrak{A}} \alpha[s] \Rightarrow \models_{\mathfrak{A}} \forall x \alpha[s]$

If $\models_{\mathfrak{A}} \alpha[s]$, that is, for any assignment, it can't be satisfied. Because x occurs not free in α, so x doesn't affect satisfaction of α. That is $s(y) = s(x|\alpha)(y)$. And obviously, x occurs no free in $\forall x \alpha$ either, and other variable in α that occurs free wasn't been affected by it and they will occur free in $\forall x \alpha$. If s can satisfy α, it can s.t $\forall x \alpha$ according to theorem in slices page 58/77.

In the same way, If s cannot satisfy α, it cannot s.t $\forall x \alpha$ according to theorem in slices page 58/77. Therefore, $\models_{\mathfrak{A}} (\alpha \to \forall x \alpha)[s]$ holds.

6. Solution:

$\forall x (x \dot{\circ} \dot{e} \dot{=} \dot{x} \wedge \dot{e} \dot{\circ} x \dot{=} \dot{x} \wedge \forall a \forall b \forall c (a \dot{\circ} b) \dot{\circ} c \dot{=} a \dot{\circ} (b \dot{\circ} c))$