A hierarchy bounds of bilevel mixed 0-1 programs

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Abstract

We consider general bilevel programs in which the decision variables of the lower-level problem are binary. A sequence of mixed-integer programming formulations are developed to obtain a hierarchy of bounds for the optimal objective function value of the bilevel programs. The proposed bounds are stronger to those provided by single-level relaxations. We then propose a family of valid inequalities based on mixing-set constraints to compute these bounds. Extended formulations are also explored to strengthen the formulations. Computational study indicates the efficiency of our bounds.

1 Introduction

The bilevel mixed-0,1 problem is given as follows:

$$[\mathrm{BP}] \quad \eta^* = \max_{x\in\mathcal{X}} \alpha_1^T x + \alpha_2^T y$$
$$\text{s.t.} \quad y \in \arg\max_{\hat{y}\in\{0,1\}^n}\{\beta^T \hat{y} : \ Ax + B\hat{y} \le d\}$$

where the feasible region of x is a polyhedron $\mathcal{X} \subseteq \mathbb{R}^{p_1} \times \mathbb{Z}^{p_2}$ and $A = (a_{i,j})_{m\times(p_1+p_2)}, B = (b_{i,j})_{m\times n}$. Let $\mathcal{P} = \{(x, y) \in \mathcal{X} \times \{0,1\}^n : \ Ax + By \le d\}$, and denote $\mathcal{P}(x) = \{y \in \{0,1\}^n : \ (x, y) \in \mathcal{P}\}$ as the feasible region of the follower with respect to the leader's decision x.

However, in some situations, the follower might choose a local optimal solution and sacrifice his profit to benefit or damage the leader's objectives. In this note, we develop a hierarchy bounds for BP by considering the k-optimal solution of the lower-level problem. We say $y \in \mathcal{P}(x)$ is a k-optimal solution of the lower-level problem $LL(x)$, if $\beta^T y$ is less than the objective of all neighborhood feasible solutions within k distances, i.e., $\beta^T y \ge \beta^T \hat{y}$ for all $\hat{y} \in \mathcal{P}(x)$ such that $||y - \hat{y}||_1 \le k$. Without loss of generality, we make the following assumptions throughout the paper.

Assumption 1. *The entries in A, B, d are integers.*

Assumption 2. *The coefficients $\beta \ge 0$.*

We note that the coefficients in A, B, d can be transformed into integers as long as they are rational. As for Assumption 2, let $\hat{y}_j = 1 - \hat{y}_j$ for any $\beta_j < 0$ and $j \in [n] := \{1, 2, \ldots, n\}$. The bilevel program with k-optimality for any positive integer k is formalized as follows.

$$[\mathrm{BP}_k] \quad \eta_k^* = \max_{x,y} \alpha_1^T x + \alpha_2^T y$$
$$\text{s.t.} \quad (x, y) \in \mathcal{P},$$
$$d^T y \ge d^T \hat{y} \quad \forall\, \hat{y} \in \mathcal{N}_k^x(y),$$

where $\mathcal{N}_k^x(y) = \{\hat{y} \in \mathcal{P}(x) : ||\hat{y} - y||_1 \leq k\}$ denotes as the neighborhood of y with distance k. Define the k-optimal reaction set of the follower with respect to the leader's decision x as

$$\mathcal{S}_k(x) = \{y \in \mathcal{P}(x) : \beta^T y \geq \beta^T \hat{y} \quad \forall\, \hat{y} \in \mathcal{N}_k^x(y)\}.$$

Note that $\mathcal{S}_0(x) = \mathcal{P}(x)$ for $k = 0$, it follows that η_0^* is the optimal objective function value of the single-level relaxation for the bilevel program.

Thus, we can establish progressively tighter upper bounds for the bilevel programs by increasing the neighborhood searching distance k.

Proposition 1. $\eta_0^* \geq \eta_1^* \geq \eta_2^* \geq \cdots \geq \eta_n^* = \eta^*$.

Proof. Note that $\eta_k^* = \max_{x,y}\ \{\alpha_1^T x + \alpha_2^T y : (x, y) \in \mathcal{P}, y \in \mathcal{S}_k(x)\}$. To prove $\eta_k^* \geq \eta_{k+1}^*$ for $k = 0, 1, \ldots, n-1$, it suffices to show that $\mathcal{S}_k(x) \supseteq \mathcal{S}_{k+1}(x)$. For any solution $y \in \mathcal{S}_{k+1}(x)$, then $\beta^T y \geq \beta^T \hat{y}$ for all $\hat{y} \in \mathcal{N}_{k+1}^x(y)$. Observe that $\mathcal{N}_{k+1}^x(y) \supseteq \mathcal{N}_k^x(y)$, thus $\beta^T y \geq \beta^T \hat{y}$ for all $\hat{y} \in \mathcal{N}_k^x(y)$, which implies that $y \in \mathcal{S}_k(x)$.

If $k = n$, then $\mathcal{N}_n^x(y) = \mathcal{P}(x)$. For any solution $y \in \mathcal{S}_n(x)$, then $\beta^T y \geq \beta^T \hat{y}$ for all $\hat{y} \in \mathcal{P}(x)$. It follows that $\mathcal{S}_n(x) = \arg\max_{\hat{y}}\{\beta^T \hat{y} :\ \hat{y} \in \mathcal{P}(x)\}$ and $\eta_n^* = \eta^*$. ■

Theorem 1. *BP_k is NP-hard for $k \geq 2$.*

Proof. Consider the minimum spanning tree interdiction problem [9] to find the most vital arcs whose removal increase the weight of minimum spanning tree as large as possible. Formally, given a graph $G = (V, E)$ with edge weight vector $w \in \mathbb{R}_+^{|E|}$. Then the minimum spanning tree interdiction problem is given as

$$\begin{aligned} \max_{x \in \{0,1\}^E}\ \min_{y \in \{0,1\}^E}\ & w^T y \\ & \sum_i x_i \leq k, \\ & y \in \mathcal{T}(G), \\ & y \leq 1 - x, \end{aligned}$$

where $\mathcal{T}(G)$ is the set of incidence vectors for the spanning trees in G. Frederickson and Solis-Oba [9] show the NP-hardness of the above problem. Note that for the minimum spanning tree problem, any 2-optimal solution is a global minimum due to the matroid property. Therefore, the minimum spanning tree interdiction problem when the follower takes k-optimal solutions for $k \geq 2$ is equivalent to the minimum spanning tree interdiction problem. It follows that BLKOP is NP-hard in general. ■

The bilevel programming is known as the NP-hard problem. The common method for the bilevel linear programming is to reformulate into a single-level mixed-integer problem by using KKT conditions. However, there do not have efficient solvers for the bilevel mixed-integer problem. The branch-and-cut approach has been investigated broadly to solve the bilevel mixed-integer problem. In this paper, we formulate the bilevel problem with k-optimality as a single-level problem, and investigate the general framework to obtain tighter upper bounds for the bilevel mixed 0-1 problems. Furthermore, the proposed framework can be used in the following manners:

- By solving bilevel problem with k-optimality, we can obtain a near-optimal leader's decisions. By increasing the number of k, the tighter upper and lower bounds can be obtained but generally more difficult to compute.

- The single-level relaxation is mostly used formulation in the branch-and-cut approach. The bilevel problem with k-optimality provides tighter relaxation methods that can be embedded into the general branch-and-cut framework.
- For the bilevel problem in which any k-optimal solution is a global optima for the lower-level problem, then the proposed framework provides a single-level reformulation for these bilevel problems. We would show that the minimum spanning tree interdiction problem can be reformulated into a single-level mixed-integer problem.
- Practically, it is difficult for the followers to choose their global optimal solution due to the computational complexity. The reaction solution set of the follower might be the local optima of the lower-level problem. The bilevel problem with k-optimality actually considers that the k-optimal solution is selected for the follower.

Notation. Denote set $[n] = \{1, 2, \ldots, n\}$ for any positive integer n.

2 The bilevel problem with k-optimality

In this section, we propose to formulate BP_k into a single-level mixed-integer problem. A general branch-and-cut approach is then developed to solve the problem.

2.1 $k = 1$ case

When $k = 1$, then recall the 1-optimal reaction set of the follower is:

$$\mathcal{S}_1(x) = \{y \in \mathcal{P}(x) : \beta^T y \geq \beta^T \hat{y} \quad \forall\, \hat{y} \in \mathcal{N}_1^x(y)\},$$

where $\mathcal{N}_1^x(y) = \{\hat{y} \in \mathcal{P}(x) : ||\hat{y} - y||_1 \leq 1\}$. An equivalent condition of $y \in \mathcal{S}_1(x)$ is given:

Lemma 1. *Given a solution $(x, y) \in \mathcal{P}$, then $y \in \mathcal{S}_1(x)$ if and only if either $y_j = 1$, or $y_j = 0$ and $y + e_j \notin \mathcal{P}(x)$ for $j \in I_\beta^+$, where e_j is the jth unit vector, and $I_\beta^+ = \{j \in [n] : \beta_j > 0\}$.*

Follows from Lemma 1, a sufficient condition is given that the bilevel problem with 1-optimality is equivalent to the single-level relaxation.

Proposition 2. *If $\alpha_{2j} > 0$ for all $j \in I_\beta^+$, then $\eta_0^* = \eta_1^*$.*

Proof. Based on Proposition 1, we have $\eta_0^* \geq \eta_1^*$. Thus, we only need to prove that $\eta_0^* \leq \eta_1^*$. Suppose (x^*, y^*) is an optimal solution of single-level relaxation problem, it suffices to show that (x^*, y^*) is also a feasible solution for BP$_1$, that is $y^* \in \mathcal{S}_1(x^*)$. We first note that $y^* \in \mathcal{P}(x^*)$.

Suppose $y^* \notin \mathcal{S}_1(x^*)$, then based on Lemma 1, there exists some $j \in I_\beta^+$ such that $y_j^* = 0$ and $y^* + e_j \in \mathcal{P}(x)$. Let $y' = y^* + e_j$, then (x^*, y') is a feasible solution of the single-level relaxation problem. Since $\alpha_{2j} > 0$ for $j \in I_\beta^+$, then $\alpha_1 x^* + \alpha_2 y' = \alpha_1 x^* + \alpha_2 y^* + \alpha_{2j} > \alpha_1 x^* + \alpha_2 y^*$, which contradicts with the fact that (x^*, y^*) is an optima. Therefore, $y_1^* \in \mathcal{S}_1(x^*)$, and the result follows. ■

Through Proposition 2, we can establish the NP-hardness of BP$_k$ for $k = 1$.

Theorem 2. *BP_k is NP-hard for $k = 1$.*

Proof. Consider the bilevel knapsack problem proposed in [14],

$$\max_{x\in\{0,1\}^m} \Big\{c^1x + c^2y : y \in \arg\max_{y\in\{0,1\}^n} \{dy : a^1x + a^2y \le b\}\Big\},$$

where c^i, d, a^i are nonnegative integral vectors, and $b \in \mathbb{Z}_+$. From Proposition 2, we have that BP_k with $k = 1$ is equivalent to the single-level relaxation problem, which is a knapsack problem given by

$$\max_{x,y} \Big\{c^1x + c^2y : a^1x + a^2y \le b, x \in \{0,1\}^m, y \in \{0,1\}^n\Big\}.$$

It is known the fact that the knapsack is NP-hard in general, then the result follows. ■

We next focus on the single-level formulation for the bilevel problem with 1-optimality. Observe that for any solution $y \in \{0,1\}^n$, if $y \notin \mathcal{P}(x)$, then there must exist some row $i \in [m]$ such that

$$\sum_j a_{i,j}x_j + \sum_j b_{i,j}y_j \ge d_i + 1,$$

which yields that

$$\Big\{y \in \{0,1\}^n : y \notin \mathcal{P}(x)\Big\} = \bigcup_{i=1}^m \Big\{y \in \{0,1\}^n : \sum_j b_{i,j}y_j \ge b_i + 1 - \sum_j a_{i,j}x_j\Big\}.$$

To formulate the constraints that $y + e_j \notin \mathcal{P}(x)$ for $j \in I_\beta^+$ and $y_j = 0$, we introduce binary variables that $z_{i,j} = 0, j \in I_\beta^+$ indicates the ith row is violated by $y + e_j$ for $j \in I_\beta^+$. Then we reformulate BP_1 as a mixed-integer linear problem:

$$\begin{aligned}
[\mathrm{BP}_1]\quad \eta_1^* = \max_{x,y,z}\ & \alpha_1^T x + \alpha_2^T y \\
\text{s.t. }\ & (x,y) \in \mathcal{P}, \\
& \sum_{j=1}^n a_{i,j}x_j + \sum_{j=1}^n b_{i,j}y_j + (h_{i,j} - \mu_i)z_{i,j} \ge h_{i,j} \quad i \in [m], j \in I_\beta^+, && (1)\\
& \sum_{i=1}^m z_{i,j} - y_j = m - 1 \quad j \in I_\beta^+, && (2)\\
& z_{i,j} \in \{0,1\} \quad i \in [m], j \in I_\beta^+,
\end{aligned}$$

where $h_{i,j} = d_i + 1 - b_{i,j}$, and μ_i is a sufficiently small constant. Given the bound of x that $\ell_j \le x_j \le u_j$ for any j, then we have for ith constraint:

$$\sum_{j=1}^n a_{i,j}x_j + \sum_{j=1}^n b_{i,j}y_j \ge \sum_{j:a_{i,j}>0} \ell_j + \sum_{j:a_{i,j}<0} u_j + \sum_{j:b_{i,j}<0} 1.$$

Then we can set $\mu_i = \sum_{j:a_{i,j}>0} \ell_j + \sum_{j:a_{i,j}<0} u_j + \sum_{j:b_{i,j}<0} 1$. Constraints (1) ensure that if $z_{i,j} = 0$, then $y + e_j$ violates ith constraint, otherwise if $z_{i,j} = 1$, then the ith constraint for $y + e_j$ is not considered. Constraints (2) guarantee that if $y_j = 0$, then there exits some i such that $z_{i,j} = 0$ which implies that the ith constraint is violated by $y + e_j$ and $y + e_j \notin \mathcal{P}(x)$. Otherwise if $y_j = 1$, then $z_{i,j} = 1$ for all $i \in [m]$ and constraints (1) are always satisfied. Next, we discuss how to reduce the number of variables by preprocessing.

Proposition 3. *If $\alpha_{2,j_0} > 0$ and $b_{i,j_0} \geq 0$ for some $j_0 \in I_\beta^+$ and all $i \in [m]$, then it is equivalent to compute η_1^* by removing variables z_{i,j_0} and the corresponding constraints.*

Proof. Denote BP_1' and η_1' as the problem by setting $z_{i_0,j_0} = 1$ and its optimal objective function value, respectively. It is clear that $\eta_1' \geq \eta_1^*$. Assume (x', y', z') is the optimal solution of BP_1', then we show that $y' \in \mathcal{S}_1(x')$, which implies that (x', y') is feasible for BP_1. It immediately results in $\eta_1' \leq \eta_1^*$. Note that to check $y' \in \mathcal{S}_1(x')$, it suffices to show that $y' + e_{j_0} \notin \mathcal{P}(x)$.

If $y'_{j_0} = 1$, then the statement holds trivially. If $y'_{j_0} = 0$, then suppose $y' + e_{j_0} \in \mathcal{P}(x)$. For those $j \in [n]$ and $y'_j = 0$, then we have $y' + e_{j_0} + e_j \notin \mathcal{P}(x)$ due to $b_{i,j_0} \geq 0$ for all i. Therefore, $y' + e_{j_0}$ is a feasible solution in BP_k'. Also, $\alpha_1^T x' + \alpha_2^T (y' + e_j) = \alpha_1^T x' + \alpha_2^T y' + \alpha_{2j} > \alpha_1^T x' + \alpha_2^T y'$, which contradicts with the assumption that (x', y', z') is an optima of BP_1'. Thus, $y' + e_j \notin \mathcal{P}(x)$ and the result follows. ■

Preprocessing:

- If $b_{i,j} \leq 0$ for some $i \in [m]$ and $j \in I_\beta^+$, then we can set $z_{i,j} = 1$.
- If $b_{\cdot,j_1} \leq b_{\cdot,j_2}$ for some $j_1, j_2 \in [n]$, then we can strengthen constraints (2) for j_2 as

$$\sum_i z_{i,j_2} - y_{j_1} y_{j_2} + y_{j_1} = m.$$

We next focus on the valid inequalities of BP_1. Let $\pi_i = \sum_{j=1}^n a_{i,j} x_j + \sum_{j=1}^n b_{i,j} y_j$ for $i \in [m]$, then constraints (1) become

$$\pi_i + (h_{i,j} - \mu_i) z_{i,j} \geq h_{i,j} \quad \forall i \in [m], j \in I_\beta^+.$$

Note that the above inequalities are called mixing set constraints, which have been extensively studied by [10, 4, 13, 12, 1, 20]. Furthermore, the convex hull of $\{(\pi, z) \in \mathbb{R} \times \{0,1\}^n : \pi + h_j z_j \geq h_j\}$ is sufficiently described by the star inequalities

$$\pi + \sum_{j=1}^{l} (h_{t_j} - h_{t_{j+1}}) z_{t_j} \geq h_{t_1}$$

where $h_1 \geq \cdots \geq h_n$ and $t_1 < \cdots < t_l, h_{t_{l+1}} = 0$. Similarly, we can propose a family of valid inequalities for BP_1.

Proposition 4. *Given some $i \in [m]$, we assume $I_\beta^+ = \{1, \ldots, |I_\beta^+|\}$ and $b_{i,1} \leq \cdots \leq b_{i,|I_\beta^+|}$, then the following inequalities*

$$\sum_{j=1}^n a_{i,j} x_j + \sum_{j=1}^n b_{i,j} y_j + \sum_{j=1}^{l} (h_{i,t_j} - h_{i,t_{j+1}}) z_{i,t_j} \geq h_{i,t_1} \quad \forall \{t_1, \ldots, t_l\} \subseteq I_\beta^+ \tag{3}$$

are valid for the feasible region of BP_1, where $h_{i,t_{l+1}} = \mu_i$. Furthermore, if $h_{i,t_1} < h_{i,1}$, we can strengthen the above inequality by containing 1 in the set.

Proof. The result follows directly from [10, 13]. However, since our formulations are somewhat different, we provide a self-contained proof. Let (x, y, z) be a feasible solution of BP_1. Remind that in constraint (1), if $z_{ij} = 0$ for some $j \in [n]$, then $\sum_{j=1}^n a_{i,j} x_j + \sum_{j=1}^n b_{i,j} y_j \geq h_{i,j}$.

For any $\{t_1, \dots, t_l\} \subseteq I_\beta^+$ and $i \in [m]$, if $z_{i,t_j} = 1$ for all $j \in \{1, \dots, l\}$, then the inequality (3) always holds as $\sum_{j=1}^n a_{i,j}x_j + \sum_{j=1}^n b_{i,j}y_j \geq \mu_i$. If there exists $z_{i,t_j} = 0$ for some $j \in \{1, \dots, l\}$, let $j^* = \min\{j \in \{1, \dots, l\} : \ z_{i,t_j} = 0\}$. Thus, $\sum_{j=1}^n a_{i,j}x_j + \sum_{j=1}^n b_{i,j}y_j \geq h_{i,t_{j^*}}$ and

$$\begin{aligned}
\sum_{j=1}^n a_{i,j}x_j + \sum_{j=1}^n b_{i,j}y_j + \sum_{j=1}^l (h_{i,t_j} - h_{i,t_{j+1}})z_{i,t_j} &\geq \sum_{j=1}^n a_{i,j}x_j + \sum_{j=1}^n b_{i,j}y_j + \sum_{j=1}^{j^*} (h_{i,t_j} - h_{i,t_{j+1}})z_{i,t_j} \\
&= \sum_{j=1}^n a_{i,j}x_j + \sum_{j=1}^n b_{i,j}y_j + \sum_{j=1}^{j^*-1} (h_{i,t_j} - h_{i,t_{j+1}}) \\
&= \sum_{j=1}^n a_{i,j}x_j + \sum_{j=1}^n b_{i,j}y_j + h_{i,t_1} - h_{i,t_{j^*}} \\
&\geq h_{i,t_1}.
\end{aligned}$$

If $h_{i,t_1} < h_{i,1}$, and we add 1 to the set, then firstly based on the above discussion, we have that the inequality

$$\sum_{j=1}^n a_{i,j}x_j + \sum_{j=1}^n b_{i,j}y_j + (h_{i,1} - h_{i,t_1})z_{i,1} + \sum_{j=1}^l (h_{i,t_j} - h_{i,t_{j+1}})z_{i,t_j} \geq h_{i,1} \tag{4}$$

is valid for BP_1. Since $h_{i,1} - (h_{i,1} - h_{i,t_1})z_{i,1} = h_{i,1}(1 - z_{i,1}) + h_{i,t_1}z_{i,1} > h_{i,t_1}$, the inequality (4) is stronger than (3).

■

2.2 General k

Let $\mathcal{T}_k = \{(T_1, T_2) \subseteq [n] \times [n] : T_1 \cap T_2 = \emptyset, \beta(T_1) > \beta(T_2), |T_1 \cup T_2| = k\}$. We first provide an equivalent condition for $\mathcal{S}_k(x)$.

Lemma 2. *For any $x \in X$, then $y \in \mathcal{S}_k(x)$ if and only if the following conditions are satisfied:*

(i) *$y \in \mathcal{S}_{k-1}(x)$;*

(ii) *For any $(T_1, T_2) \in \mathcal{T}_k$, if $y_j = 0$ for $j \in T_1$ and $y_j = 1$ for $j \in T_2$, then $y + \sum_{j \in T_1} e_j - \sum_{j \in T_2} e_j \notin \mathcal{P}(x)$.*

Proposition 5. *If $\alpha_2(T_1) > \alpha_2(T_2)$ for any $(T_1, T_2) \in \cup_{j=1}^k \mathcal{T}_j$, then $\eta_k^* = \eta_{k-1}^* = \dots = \eta_0^*$.*

Proof. Based on Proposition 1, we have $\eta_0^* \geq \dots \geq \eta_{k-1}^* \geq \eta_k^*$. Thus, we only need to prove that $\eta_0^* \leq \eta_k^*$. Suppose (x^*, y^*) is an optimal solution of BP_0, it suffices to show that (x^*, y^*) is also a feasible solution for BP_k, that is $y^* \in \mathcal{S}_k(x^*)$. We first note that $y^* \in \mathcal{P}(x^*)$ and $\alpha_2 y^* \geq \alpha_2 y$ for any $y \in \mathcal{P}(x^*)$.

Suppose $y^* \notin \mathcal{S}_k(x^*)$, then based on Lemma 2, there exists $(T_1, T_2) \in \cup_{j=1}^k \mathcal{T}_j$ such that $y_j^* = 0$ for $j \in T_1$, $y_j^* = 1$ for $j \in T_2$ and $y + \sum_{j \in T_1} e_j - \sum_{j \in T_2} e_j \in \mathcal{P}(x)$. Since $\alpha_2(T_1) > \alpha_2(T_2)$, then $\alpha_2 y + \alpha_2(T_1) - \alpha_2(T_2) > \alpha_2 y$, which contradicts with the assumption that (x^*, y^*) is optima for BP_0. Therefore, $y \in \mathcal{S}_k(x^*)$ and the result follows. ■

To interpret the conditions in Lemma 2 into the constraints, we introduce binary variables $z_{i,t}$ for $i \in [m]$ and $t = (T_1, T_2) \in \mathcal{T}^k = \cup_{j=1}^k \mathcal{T}_k$. Then the BP_k is reformulated as

$$
\begin{aligned}
[BP_k] \quad \eta_k^* = \max_{x,y,\pi,z} \ & \alpha_1^T x + \alpha_2^T y \\
\text{s.t.} \ & (x, y) \in \mathcal{P}, \\
& \pi = Ax + By, && (5)\\
& \pi_i + (h_{i,t} - \mu_i) z_{i,t} \geq h_{i,t} \ \forall i \in [m], t \in \mathcal{T}^k, && (6)\\
& \sum_{i=1}^m z_{i,t} + \sum_{j \in T_2} (y_j - 1) - \sum_{j \in T_1} y_j \leq m - 1 \ \forall t = (T_1, T_2) \in \mathcal{T}^k, && (7)\\
& \sum_{i=1}^m z_{i,t} \geq m - 1 \quad \forall t \in \mathcal{T}^k, && (8)\\
& z_{i,t} \in \{0, 1\} \quad \forall i \in [m], t \in \mathcal{T}^k, && (9)
\end{aligned}
$$

where $h_{i,t} = b_i + 1 - (\sum_{j \in T_1} b_{i,j} - \sum_{j \in T_2} b_{i,j})$ for any $t = (T_1, T_2) \in \mathcal{T}^k$. Note that for the fixed k, the cardinality of $\mathcal{T}^k$ is $O(n^k)$, which implies that the reformulation is polynomial-time reducible from BP_k. Furthermore, we can decrease the included variables and constraints through preprocessing process.

Proposition 6. (i) *If $\sum_{j \in T_1} b_{i,j} - \sum_{j \in T_2} b_{i,j} \leq 0$ for some $t = (T_1, T_2) \in \mathcal{T}^k$ and $i \in [m]$, then we can set $z_{i,t} = 1$.*

(ii) *Given some $t_1 = (T_1, T_2) \in \mathcal{T}^{k-1}$ and $j \in [n] \setminus (T_1 \cup T_2)$. Let $t_2 = (T_1 \cup \{\ell\}, T_2) \in \mathcal{T}^k$, if $b_{i,\ell} \geq 0$ for all $i \in [m]$, then we can remove variable z_{i,t_2} and its corresponding constraints.*

(iii) *Given some $t_1 = (T_1, T_2) \in \mathcal{T}^{k-1}$ and $j \in [n] \setminus (T_1 \cup T_2)$. Let $t_2 = (T_1, T_2 \cup \{\ell\}) \in \mathcal{T}^k$, if $b_{i,\ell} \leq 0$ for all $i \in [m]$, then we can remove variable z_{i,t_2} and its corresponding constraints.*

(iv) *If $h_{i,t_1} = h_{i,t_2}$ for some $i \in [m]$ and $t_1, t_2 \in \mathcal{T}^k$, then replace z_{i,t_2} as z_{i,t_1} in the model.*

(v) *If $\alpha_2(T_1) > \alpha_2(T_2)$ for some $t = (T_1, T_2) \in \mathcal{T}^k$, and $\sum_{j \in T_1} a_{i,j} \geq \sum_{j \in T_2} a_{i,j}$ for all $i \in [m]$, then we can remove $z_{i,t}$ for all i and its corresponding constraints.*

Proof. (i) It is sufficient to show that $z_{i,t} = 1$ for any feasible solution (x, y, z) for BP_k. Since $(x, y) \in \mathcal{P}$, then $\pi_i \leq b_i$. If $z_{i,t} = 0$, then according to constraint (6), we have

$$\pi_i \geq b_i + 1 - (\sum_{j \in T_1} b_{i,j} - \sum_{j \in T_2} b_{i,j}) \geq b_i + 1,$$

where the second inequality follows from the assumption that $\sum_{j \in T_1} b_{i,j} - \sum_{j \in T_2} b_{i,j} \leq 0$. Thus, we obtain a contradiction. It is immediate that $z_{i,t}$ should equal to 1.

(ii) Denote BP'_k and η'_k as the problem by removing variables z_{i,t_2} for all $i \in [m]$ and its optimal objective function value, respectively. It is clear that $\eta'_k \geq \eta_k$. Assume (x', y', π', z') is the optimal solution of BP'_k, then we show that there exists z^* such that (x', y', π', z^*) is a feasible solution of BP_k, which yields that $\eta'_k \leq \eta_k^*$.

If there exists either $y'_j = 0$ for some $j \in T_2$ or $y'_j = 1$ for some $j \in T_1 \cup \{\ell\}$, then let $z^*_{i,t_2} = 1$ for all $i \in [m]$, and $z^*_{i,t} = z'_{i,t}$ for other $t \in \mathcal{T}^k$. Then (x', y', π', z^*) is feasible for BP_k. On the other hand, if $y'_j = 0$ for all $T_1 \cup \{\ell\}$ and $y'_j = 1$ for all T_2, then based on constraint (12b) for t_1 and t_2, we need $\sum_{i=1}^m z'_{i,t_1} = m - 1$ and $\sum_{i=1}^m z^*_{i,t_2} = 1$. Since $b_{i,\ell} \geq 0$, then $h_{i,t_2} = b_i + 1 - (\sum_{j \in T_1} b_{i,j} + b_{i,\ell} - \sum_{j \in T_2}) \leq h_{i,t_1}$ for all i. Let $z^*_{i,t_2} = z'_{i,t_1}$, and $z^*_{i,t} = z'_{i,t}$ for other $t \in \mathcal{T}^k$, then observe that (x', y', π', z^*) is feasible for BP_k. This completes the proof.

(iii) The proof is similar with (ii), thus we omit it.

(iv) Denote BP'_k and η'_k as the problem by replacing variables z_{i,t_2} as z_{i,t_1} and its optimal objective function value, respectively. Suppose (x', y', π', z') is the optimal solution of BP'_k, then it suffices to show that there exists z^* such that (x', y', π', z^*) is a feasible solution of BP_k.

Assume $t_2 = (T_1, T_2)$. If $z'_{i,t_1} = 1$, then let $z^*_{i,t_2} = 1$ and $z^*_{i,t} = z'_{i,t}$ for all $i \in [m]$ and $t \in \mathcal{T}^k$. Then it is clear that (x', y', π', z^*) satisfies constraint (12b) as $z^*_{i,t_2} = z'_{i,t_1}$. On the other hand, if $z'_{i,t_1} = 0$, then there have two possible cases:

- if $y'_j = 0$ for all $j \in T_1$ and $y'_j = 1$ for all $j \in T_2$, then let $z{*}_{i,t_2} = 0$ and $z^*_{i,t} = z'_{i,t}$ for all $i \in [m]$ and $t \in \mathcal{T}^k$. Then (x', y', π', z^*) is feasible for BP_k.
- if there exists either $y'_j = 0$ for $j \in T_2$ or $y'_j = 1$ for some $j \in T_1$, then let $z{*}_{i,t_2} = 1$ and $z^*_{i,t} = z'_{i,t}$ for all $i \in [m]$ and $t \in \mathcal{T}^k$. Then the constructed solution (x', y', π', z^*) is feasible for BP_k.

Therefore, the proof is completed.

(v) Denote BP'_k and η'_k as the problem by removing variables $z_{i,t}$ and its optimal objective function value, respectively. Suppose (x', y', π', z') is the optimal solution of BP'_k, then we would show that $y' \in \mathcal{S}_k(x)$, which implies that $\eta'_k \leq \eta^*_k$. To verify $y' \in \mathcal{S}_k(x)$, it suffices to show that $y' + \sum_{j \in T_1} e_j - \sum_{j \in T_2} e_j \notin \mathcal{P}(x)$.

If there exists either $y'_j = 1$ for some $j \in T_1$ or $y'_j = 0$ for some $j \in T_2$, then the statement holds trivially. Otherwise, suppose

$$y' + \sum_{j \in T_1} e_j - \sum_{j \in T_2} e_j \in \mathcal{P}(x)$$

. Since $\sum_{j \in T_1} a_{i,j} \geq \sum_{j \in T_2} a_{i,j}$, then observe that $(x', y' + \sum_{j \in T_1} e_j - \sum_{j \in T_2} e_j)$ is a feasible solution for BP'_k. Also, $\alpha_1 x' + \alpha_2 (y' + \sum_{j \in T_1} e_j - \sum_{j \in T_2} e_j) > \alpha_1 x' + \alpha_2 y'$, which contradicts with the assumption that (x', y', π', z') is an optimal solution for BP'_k. Thus, $y' + \sum_{j \in T_1} e_j - \sum_{j \in T_2} e_j \notin \mathcal{P}(x)$ and result follows.

■

Proposition 7. *Given some $i \in [m]$, we re-number $\mathcal{T}^k = \cup_{j=1}^k \mathcal{T}_j$ as $\{1, \ldots \ell\}$ and sort $h_{i,t}$ into a decreasing order such that $h_{i,1} \geq h_{i,2} \geq \cdots \geq h_{i,\ell}$, then the star inequalities*

$$\pi_i + \sum_{j=1}^{l} (h_{i,t_j} - h_{i,t_{j+1}}) z_{i,t_j} \geq h_{i,t_1} \quad \forall \{t_1, \ldots, t_l\} \subseteq \mathcal{T}^k$$

are valid for feasible region of BP_k, where $h_{i,t_l+1} = \mu_i$. Furthermore, if $h_{i,t_1} < h_{i,1}$, we can strengthen the above inequality by containing 1 into the set.

Proof. We omit the proof as it is similar with the proof in Proposition 4. ■

Separation. Shortest path problem.

2.3 The hierarchy lower bounds

3 Extended formulation

In this section, we provide the strong extended formulations for k-optimality bilevel programs. For each $i \in [m]$, we sort $h_{i,t}$ into a nonincreasing order such that $h_{i,(1)} \geq h_{i,(2)} \geq \cdots \geq h_{i,(\ell)}$. Then for each $i \in [m]$ and $t \in \mathcal{T}^k$, we introduce binary variables $v_{i,t}$. Then we have the extended formulation for BP_k as follows:

$$
\begin{aligned}
[EBP_k] \quad \tilde{\eta}_k^* = \max_{x,y} \ & \alpha_1^T x + \alpha_2^T y \\
\text{s.t.} \quad & (x, y) \in \mathcal{P}, (5), (7), (8), \\
& \pi_i + \sum_{t=1}^{\ell} (h_{i,(t)} - h_{i,(t+1)}) v_{i,(j)} \geq h_{i,(1)} \quad \forall i \in [m], && \text{(10a)} \\
& v_{i,(t)} \geq v_{i,(t+1)} \quad \forall i \in [m], t \in \mathcal{T}^k, && \text{(10b)} \\
& v_{i,(t)} \leq z_{i,(t)} \quad \forall i \in [m], t \in \mathcal{T}^k, && \text{(10c)} \\
& v_{i,t}, z_{i,t} \in \{0, 1\} \quad \forall i \in [m], t \in \mathcal{T}^k,
\end{aligned}
$$

where $v_{i,(\ell+1)} = 0$ and $h_{i,(\ell+1)} = \mu_i$ for all $i \in [m]$. The motivation by adding variable $v_{i,t}$ is that the bound of π_i is controlled by the largest order whose $z_{i,(t)}$ is equal to zero.

Theorem 3. *The bilevel problem with k-optimality is equivalent to the extended formulation EBP_k, that is $\eta_k^* = \tilde{\eta}_k^*$.*

4 Case study: knapsack interdiction problem

We consider the knapsack interdiction problem [7, 8] as follows:

$$
\begin{aligned}
[\text{DNeg}] \ \min_x \ \max_y \ & \sum_{j=1}^{n} p_j y_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_j^1 x_j \leq C_u, && \text{(11a)} \\
& \sum_{j=1}^{n} a_j^2 y_j \leq C_l, && \text{(11b)} \\
& x_j + y_j \leq 1 \quad \forall j \in [n], && \text{(11c)} \\
& x \in \{0, 1\}^n, y \in \{0, 1\}^n,
\end{aligned}
$$

where $p_j, a_j^1, a_j^2, C_u, C_l$ are positive integers for all $j \in [n]$. Then we can formulate DNeg k-optimality problem as a single-level mixed-integer program:

$$
\begin{aligned}
[\text{DNeg}_k] \ \min \ & \sum_{i=1}^{n} p_i y_i \\
\text{s.t.} \ & (11\text{a}) - (11\text{c}), \\
& \sum_{i=1}^{n} a_i^2 y_i + h_t z_t \geq h_t \quad \forall t \in \mathcal{T}^k, && (12\text{a}) \\
& z_t - \sum_{j \in T_2} (1 - y_j + x_j) - \sum_{j \in T_1} (y_j + x_j) \leq 0 \quad \forall t = (T_1, T_2) \in \mathcal{T}^k, && (12\text{b}) \\
& x \in \{0,1\}^n, y \in \{0,1\}^n, z \in \{0,1\}^{|\mathcal{T}^k|}, && (12\text{c})
\end{aligned}
$$

where $h_t = C_l + 1 - (\sum_{j \in T_1} a_j^2 - \sum_{j \in T_2} a_j^2)$ for any $t = (T_1, T_2) \in \mathcal{T}^k$ and $\mathcal{T}^k = \cup_{l=1}^k \mathcal{T}_k$. The binary variables z_t for $t = (T_1, T_2) \in \mathcal{T}^k$ identify whether or not the condition $y + \sum_{j \in T_1} e_j - \sum_{j \in T_2} e_j$ should be considered. When the current solution (x, y) satisfies $y_j = 0, x_j = 0$ for $j \in T_1$, and $x_j = 0, y_j = 1$ for $j \in T_2$, then constraints (12a)(12b) enforce that $z_t = 0$, and the solution $y + \sum_{j \in T_1} e_j - \sum_{j \in T_2} e_j$ violates the follower's budget constraint. Note that we strengthen constraints (12b) by adding variable x_j.

We next briefly discuss how to apply preprocessing procedure to significantly decrease the size of proposed mixed-integer program.

Preprocessing.

- For any $t = (T_1, T_2) \in \mathcal{T}^k$, if $\sum_{j \in T_1} a_j^2 - \sum_{j \in T_2} a_j^2 < 0$, then we set $z_t = 1$ and remove the corresponding constraint (12a).
- For any $(T_1, T_2) \in T^k$, if $\sum_{j \in T_1} a_j^2 - \sum_{j \in T_2} a_j^2 \geq 0$, then we can remove (T_1', T_2) from $\mathcal{T}^k$ for any T such that $T_1 \subseteq T_1'$.
- If $h_t = h_{t'}$ for some $t, t' \in \mathcal{T}^k$, then we can set $z_t = z_{t'}$ to decrease the random variables and constraints.

Extended Formulation: Without loss of generality, assume $h_1 \geq h_2 \geq \cdots \geq h_{|\mathcal{T}^k|}$. By introducing binary variables v_t, we have

$$
\begin{aligned}
[\text{EDNeg}_1] \ \min \ & \sum_{i=1}^{n} p_i y_i \\
\text{s.t.} \ & (11\text{a}) - (11\text{c}), (12\text{b}), \\
& \sum_{i=1}^{n} a_i^2 y_i + \sum_{t=1}^{|\mathcal{T}^k|} (h_t - h_{t+1}) v_t \geq h_1, \\
& v_t \geq v_{t+1} \quad t \in \mathcal{T}^k, \\
& v_t \leq z_t \quad t \in \mathcal{T}^k, v_{|\mathcal{T}^k|+1} = 0, \\
& x \in \{0,1\}^n, y \in \{0,1\}^n, z \in \{0,1\}^n, v \in \{0,1\}^n
\end{aligned}
$$

where $h_{|\mathcal{T}^k|} = 0$.

4.1 Experiment

Test Setup. For generating the instance data, we adopt the knapsack generator described in [15, 7]. The costs p_i and weights a_i^1, a_i^2 are generated randomly and independently from interval $[0, 100]$. For each $n \in \{10, 20, 30, 40, 50\}$ and $r \in \{1, 2, \ldots, 10\}$, C_l is set to $\lceil \frac{r}{11} \sum_{i=1}^{n} a_i^2 \rceil$, and C_u is chosen uniformly from interval $[C_l - 10, C_l + 10]$. For each pair of n, r, we generate 10 instances, and report their average performance.

We examine the performances of bilevel k-optimality programs by checking the results of the general bilevel solver, Mibs [17] and the specific algorithm, CCLW [7], for the knapsack interdiction problem. Our numerical experiments are conducted using CPLEX 12.80 [11] on a Ubuntu 16.04 system with a 3.2GHz CPU and 19 GB of RAM. We set time limit for Mibs as 10 mins to avoid out of memory.

Test Results. Our first preliminary experiment shows the mixed-integer formulation size after the preprocessing procedure. The preprocessing can reduce the number of constraints (12a) in a great extent.

Table 1. The number of constraints for the proposed formulation after the preprocessing procedure.

	$k = 1$		$k = 2$		$k = 3$	
n	Cons (12b)	Cons (12a)	Cons (12b)	Cons (12a)	Cons (12b)	Cons (12a)
10	9	9	53	25	163	57.01
20	20	18	2×10^2	61	1.3×10^3	136
30	30	26	4.6×10^2	80	4.5×10^3	163
40	40	32	8.2×10^2	88	1.1×10^4	174
50	50	39	1.3×10^3	92	2.1×10^4	180

In the second preliminary experiment, we test the convergence of the hierarchy bounds for different k. Figure 1 and Figure 2 demonstrate the results for $n = 10$ and $n = 15$, respectively.

Table 2 and Table 3 report the average performance of CCLW, Mibs, and $k = 1, 2, 3$ for extended formulations. For the CCLW algorithm, the bilevel linear programming relaxation is used to obtain valid upper bound. The average number of runtime (in seconds) are also reported. For the solver Mibs, we report the average runtime for the instances that can be solved within the time limit 10 mins. For the instances that are not able to be solved within 10 mins, we report ratios of between the best lower bound and the optimal value as the column "ObjL". Also, the ratio between the best upper bound and the optimal value is reported as the column "ObjU". For the bilevel k-optimality problem, the BP_k provides a valid lower bound and a feasible leader's decision x_k^*. We report the ration between the optimal value of BP_k and the primal problem in the column "ObjL". Meantime, we can solve the follower's problem with respect to x_k^* to obtain a valid upper bound for the bilevel interdiction problem, whose ration with the optimal value of the primal problem is also reported in the column "ObjU".

From Table 2-3, we have the following observations:

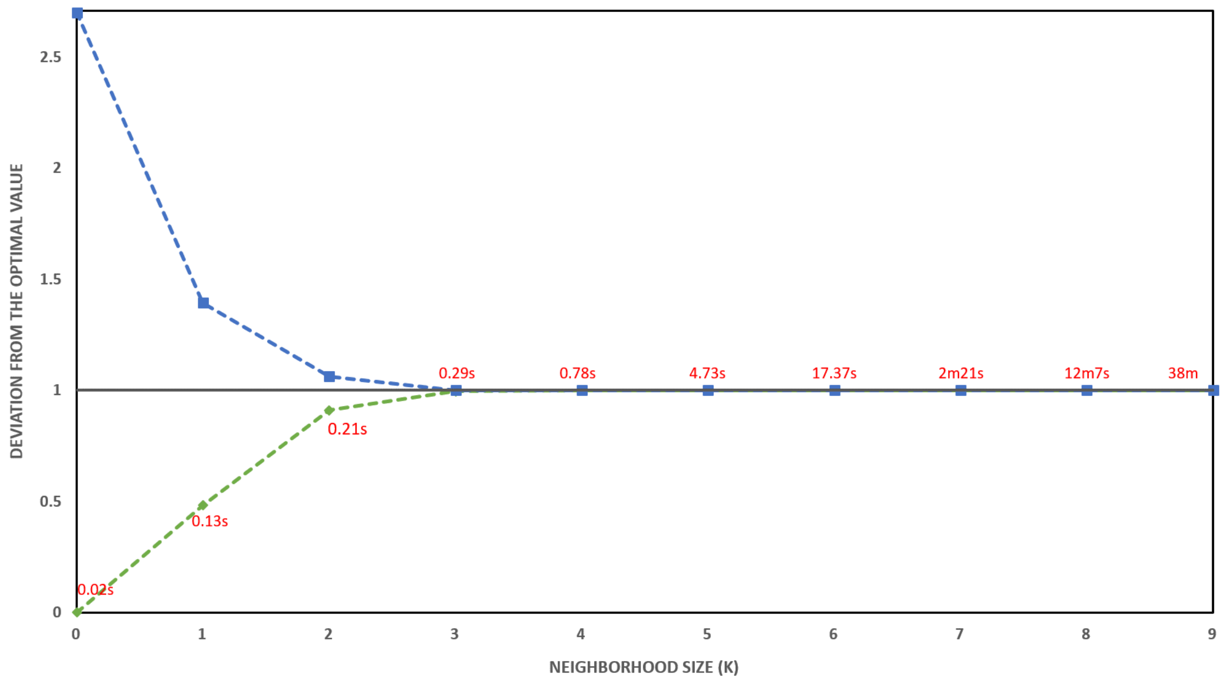

Figure 1. The average results for $n = 10, r = 4$.

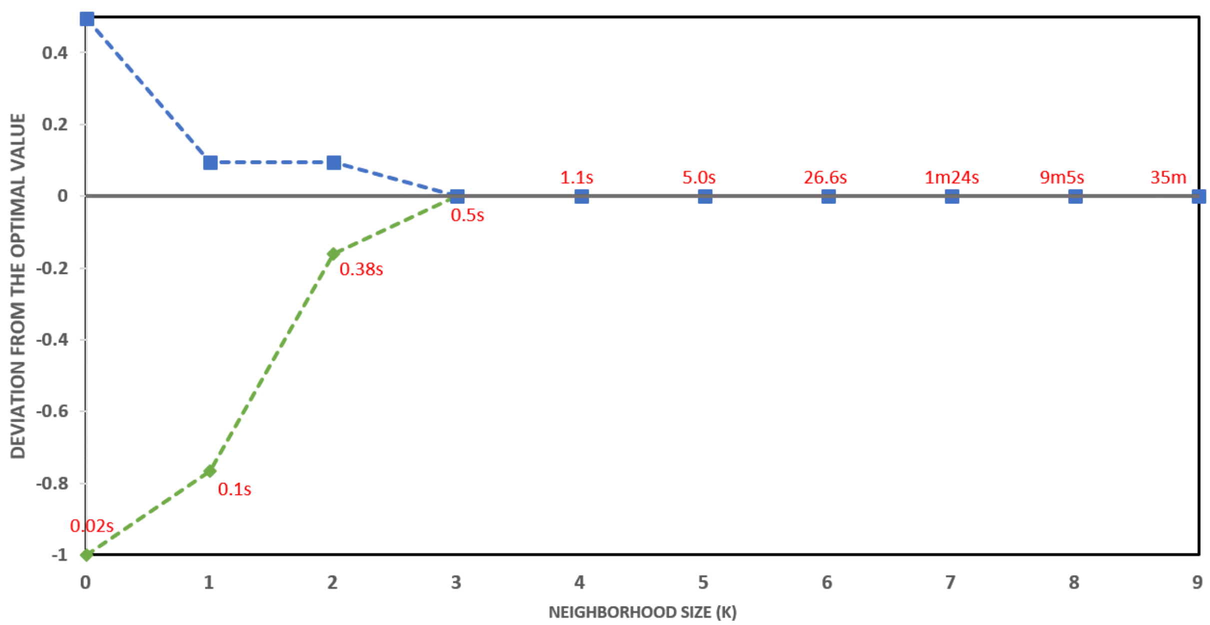

Figure 2. The results for $n = 15, r = 4$.

		CCLW [7]	Mibs [17]				Single-level relaxation ($k = 0$)			$k = 1$				$k = 2$				$k = 3$			
n	r	Time	Unsolved	ObjL	ObjU	Time	ObjL	ObjU	Time	ObjL	ObjU	Time	Ext Time	ObjL	ObjU	Time	Ext Time	ObjL	ObjU	Time	Ext Time
10	0	0.13				0.01	0	1.8	0.02	0.34	1.8	0.04	0.14	0.99	1.01	0.02	0.03	1	1	0.03	0.03
10	1	0.17				0.02	0	1.8	0.01	0.23	1.41	0.04	0.07	0.78	1.17	0.07	0.05	0.99	1	0.06	0.09
10	2	0.26				0.03	0	1.93	0.01	0.41	1.47	0.04	0.08	0.94	1.11	0.14	0.12	1	1	0.19	0.23
10	3	0.37				0.05	0	2.73	0.01	0.34	1.36	0.05	0.06	0.94	1.05	0.13	0.1	1	1.01	0.19	0.17
10	4	0.33				0.03	0	12.09	0.02	0.52	1.51	0.08	0.1	0.98	1.12	0.11	0.08	1.03	1.03	0.14	0.15
20	0	0.15				0.14	0	1.59	0.01	0.14	1.43	0.08	0.07	0.75	1.12	0.12	0.09	0.97	1	0.19	0.13
20	1	1.26				1.91	0	1.8	0.01	0.14	1.3	0.07	0.1	0.71	1.15	0.13	0.15	0.99	1.03	0.23	0.23
20	2	2.15				9.7	0	1.98	0.01	0.19	1.37	0.12	0.09	0.78	1.11	0.18	0.16	0.98	1.03	0.33	0.25
20	3	0.93				15.12	0	2.62	0.01	0.34	1.7	0.15	0.09	0.84	1.13	0.15	0.15	1	1.01	0.25	0.2
20	4	0.97				7.9	0	3.45	0.01	0.63	1.43	0.11	0.1	0.89	1.03	0.16	0.15	1	1.01	0.23	0.2
30	0	1				2.98	0	1.54	0.01	0.06	1.38	0.1	0.07	0.7	1.12	0.18	0.2	1	1.01	0.42	0.35
30	1	8.17	6	0.95	1	212.99	0	1.82	0.01	0.1	1.45	0.11	0.08	0.73	1.13	0.22	0.15	0.97	1.01	1.01	0.92
30	2	19.99	8	0.81	1	249.20	0	1.94	0.01	0.19	1.46	0.17	0.09	0.73	1.09	0.19	0.16	0.97	1.02	1.29	0.89
30	3	15.46	10	0.81	1.03		0	2.54	0.01	0.31	1.62	0.16	0.12	0.8	1.16	0.27	0.21	0.99	1.01	1.16	0.64
30	4	0.33	5	0.82	1	203.42	0	3.82	0.01	0.67	1.52	0.18	0.16	1	1.03	0.28	0.21	1.01	1.01	0.62	0.38
40	0	2.21	2	0.86	1.00	73.03	0	1.66	0.01	0.07	1.35	0.09	0.07	0.57	1.21	0.24	0.19	1	1.05	2.22	1.52
40	1	27.7	10	0.70	1		0	1.8	0.01	0.11	1.46	0.14	0.09	0.65	1.12	0.27	0.24	0.96	1.03	3.84	2.81
40	2	378.93	10	0.57	1		0	2	0.01	0.2	1.4	0.19	0.09	0.69	1.11	0.39	0.32	0.97	1.02	5.28	3.14
40	3	82.22	10	0.54	1.00		0	2.5	0.01	0.34	1.48	0.21	0.13	0.82	1.17	0.56	0.33	0.99	1	5.63	2.64
40	4	78.1	10	0.60	1		0	3.22	0.01	0.62	1.49	0.21	0.16	0.92	1.08	0.47	0.24	1	1	3.01	1.78
50	0	6.51	6	0.86	1	166.22	0	1.66	0.01	0.06	1.45	0.08	0.09	0.6	1.29	0.39	0.29	1.04	1.07	10.41	8.78
50	1	632.64	10	0.69	1		0	1.79	0.01	0.09	1.42	0.15	0.1	0.69	1.09	0.64	0.46	0.98	1.01	21.81	10.44
50	2	1422.86	10	0.49	1		0	2.05	0.01	0.18	1.42	0.23	0.09	0.64	1.07	0.64	0.45	0.98	1.01	22.14	11.69
50	3	1312.54	10	0.41	1.03		0	2.61	0.01	0.37	1.49	0.25	0.16	0.79	1.16	0.84	0.52	0.99	1.01	13.3	8.39
50	4	10.28	10	0.46	1.01		0	3.35	0.02	0.65	1.52	0.31	0.21	0.93	1.06	0.9	0.48	1	1	7.75	3.97

Table 2. Results of hard instances for knapsack interdiction problem.

		CCLW [7]	Mibs [17]				Single-level relaxation ($k=0$)			$k=1$				$k=2$				$k=3$			
n	r	Time	Unsolved	ObjL	ObjU	Time	ObjL	ObjU	Time	ObjL	ObjU	Time	Ext Time	ObjL	ObjU	Time	Ext Time	ObjL	ObjU	Time	Ext Time
10	5	0.09				0.02	0	11.14	0.01	0.72	1.94	0.05	0.06	1	1	0.08	0.09	1	1	0.07	0.12
10	6	0.09				0.02	0	10.6	0.01	0.93	1.05	0.07	0.04	1	1	0.06	0.06	1	1	0.08	0.15
10	7	0.03				0.02	0	9.61	0.01	1	1	0.05	0.06	1	1	0.08	0.07	1	1	0.08	0.08
10	8	0.03				0.01	0	18.88	0.01	1	1	0.07	0.06	1	1	0.07	0.05	1	1	0.07	0.04
10	9	0.03				0.01	0	49.58	0.01	1	1	0.09	0.05	1	1	0.05	0.04	1	1	0.05	0.07
20	5	0.07				2.02	0	5.87	0.01	0.97	1.15	0.11	0.09	1	1	0.16	0.13	1	1	0.19	0.18
20	6	0.05				0.98	0	10.24	0.01	0.99	1.06	0.11	0.13	1	1	0.18	0.09	1	1	0.16	0.16
20	7	0.06				0.32	0	32.24	0.01	1	1	0.11	0.1	1	1	0.09	0.09	1	1	0.14	0.16
20	8	0.04				0.04	0	92.07	0.01	1	1	0.08	0.09	1	1	0.08	0.08	1	1	0.11	0.13
20	9	0.03				0.02	0	257	0.01	1	1	0.06	0.04	1	1	0.05	0.07	1	1	0.07	0.1
30	5	0.08	2	0.97	1	203.42	0	6.44	0.01	0.99	1.23	0.21	0.11	1	1	0.21	0.21	1	1	0.45	0.27
30	6	0.05				19.62	0	9.09	0.01	0.99	1.08	0.19	0.13	1	1	0.25	0.14	1	1	0.29	0.25
30	7	0.06				3.19	0	35.47	0.01	1	1	0.17	0.07	1	1	0.17	0.1	1	1	0.22	0.22
30	8	0.03				0.24	0	88.24	0.01	1	1	0.1	0.08	1	1	0.07	0.1	1	1	0.14	0.15
30	9	0.08				0.02	0	239.54	0.01	1	1	0.07	0.07	1	1	0.06	0.07	1	1	0.16	0.15
40	5	0.15	9	0.75	1.03	24.52	0	5.36	0.01	0.99	1.19	0.29	0.14	1.02	1.02	0.27	0.19	1.02	1.02	1.4	0.82
40	6	0.05	1	0.83	1	62.85	0	14.73	0.01	1	1	0.21	0.12	1	1	0.22	0.16	1	1	0.46	0.61
40	7	0.07	2	0.91	1	11.29	0	21.12	0.01	1	1	0.21	0.15	1	1	0.18	0.13	1	1	0.3	0.6
40	8	0.06				2.33	0	35.59	0.01	1	1	0.13	0.1	1	1	0.1	0.09	1	1	0.21	0.47
40	9	0.06				0.20	0	324.56	0.01	1	1	0.08	0.06	1	1	0.05	0.09	1	1	0.12	0.32
50	5	0.37	10	0.58	1.01		0	5.43	0.01	0.97	1.07	0.59	0.19	1	1	0.68	0.25	1	1	3.37	1.98
50	6	0.07	8	0.72	1.01	326.93	0	11.86	0.02	1	1	0.41	0.18	1	1	0.31	0.21	1	1	1.34	1.77
50	7	0.15	5	0.94	1	164.01	0	17.11	0.01	1	1	0.26	0.13	1	1	0.22	0.21	1	1	0.65	1.4
50	8	0.05				5.96	0	86.23	0.02	1	1	0.2	0.08	1	1	0.15	0.24	1	1	0.45	0.57
50	9	0.04				0.07	0	90.79	0.01	1	1	0.06	0.16	1	1	0.14	0.11	1	1	0.21	0.41

Table 3. Results of easy instances for knapsack interdiction problem.

5 Case study

5.1 vertex cover

Given a graph $G = (N, E)$, the vertex cover problem is to find a subset of vertices whose total weight is as small as possible such that each vertex in the graph is either in this subset or connected to at least one vertex in this subset. There have three possible versions about bilevel vertex cover problem. The first version is that the leader destroys the vertices and then the follower finishes the vertex cover problem [6]. Also, we assume that the weights of two decision-makers are different.

$$
\begin{aligned}
\text{[BVC]} \ \max_x \ & \sum_j w_j^1 y_j \\
\text{s.t.} \ & \sum_j x_j \leq K, && \text{(13a)} \\
& x_j \in \{0,1\} \quad \forall j \in N, && \text{(13b)} \\
& \min_y \sum_j w_j^2 y_j \\
& \text{s.t. } x_j + y_j \leq 1 \quad \forall j \in N, && \text{(13c)} \\
& \quad \sum_{j \in N_i} y_j \geq 1 \quad \forall i \in N, && \text{(13d)} \\
& \quad y_j \in \{0,1\} \quad \forall i \in N, && \text{(13e)}
\end{aligned}
$$

where N_j is the neighborhood of vertex $j \in N$, that includes j itself. Let $\mathcal{T}^k = \{(T_1, T_2) : w^2(T_1) < w^2(T_2), |T_1 \cup T_2| \leq k, T_1 \cap T_2 = \emptyset, T_i \subseteq N \ \ i = 1,2\}$. To formulate the corresponding bilevel k-optimality problem, we introduce logistic variables $z_{i,t} \in \{0,1\}$ for $i \in N, t \in \mathcal{T}^k$.

$$
\begin{aligned}
[\text{BVC}_k] \ \max_x \ & \sum_j w_j^1 y_j \\
\text{s.t. } & \text{(13a)} - \text{(13e)}, \\
& \sum_{j \in N_i} y_j + (h_{i,t} - \mu_i) z_{i,t} \leq h_{i,t} \quad \forall i \in N, t \in \mathcal{T}^k && \text{(14a)} \\
& \sum_{i=1}^n z_{i,t} - \sum_{j \in T_2} (1 - y_j + x_j) - \sum_{j \in T_1} (x_j + y_j) \leq n - 1 \forall t = (T_1, T_2) \in \mathcal{T}^k, && \text{(14b)} \\
& \sum_{i=1}^n z_{i,t} \geq n - 1 \quad \forall t \in \mathcal{T}^k, && \text{(14c)} \\
& z_{i,t} \in \{0,1\} \ \forall i \in N, t \in \mathcal{T}^k, && \text{(14d)}
\end{aligned}
$$

where $h_{i,t} = |N_i \cap T_2| - |N_i \cap T_1|$ for $t = (T_1, T_2) \in \mathcal{T}^k$, and $\mu_i = |N_i|$ for all $i \in N$. We next discuss how to reduce the size $\mathcal{T}^k$ and the formulation size.

Preprocessing.

- For any $t = (T_1, T_2) \in \mathcal{T}^k$, if there exists $i \in N$ such that either $N_i \subseteq T_1$ or $N_i \subseteq T_2$, then we can remove t from $\mathcal{T}^k$.
- If $|N_i \cap T_1| - |N_i \cap T_2| \geq 0$ for some $i \in N$ and $t = (T_1, T_2) \in \mathcal{T}^k$, then we can set $z_{i,t} = 1$.

- For any $t = (T_1, T_2) \in T^k$, then we can remove (T_1, T') in $\mathcal{T}^k$ such that $T_2 \subseteq T'$.
- If $h_{i,t_1} = h_{i,t_2}$ for some $t_1, t_2 \in \mathcal{T}^k$, then we set $z_{i,t_1} = z_{i,t_2}$.

Extended formulation. For each $i \in [n]$, we sort $h_{i,t}$ into a decreasing order such that $h_{i,(1)} \leq h_{i,(2)} \leq \cdots \leq h_{i,(\ell)}$, where $\ell = |\mathcal{T}^k|$. Then the extended formulation for BVC is formalized as

$$
\begin{aligned}
[\mathrm{EBVC}_k]\ \max_x\ & \sum_j w_j^1 y_j \\
\text{s.t. } & (13\mathrm{a}) - (13\mathrm{e}), (14\mathrm{b}), (14\mathrm{c}), \\
& \sum_{j \in N_i} y_j + \sum_{t=1}^{\ell} (h_{i,(t)} - h_{i,(t+1)}) v_{i,(t)} \leq h_{i,(1)} \quad \forall i \in N, && (15\mathrm{a}) \\
& v_{i,(t)} \geq v_{i,(t+1)} \quad \forall i \in N, t \in \mathcal{T}^k, && (15\mathrm{b}) \\
& v_{i,(t)} \leq z_{i,(t)} \quad \forall i \in N, t \in \mathcal{T}^k, && (15\mathrm{c}) \\
& z_{i,t} \in \{0,1\}, v_{i,t} \in \{0,1\}\ \forall i \in N, t \in \mathcal{T}^k,
\end{aligned}
$$

where $h_{i,(\ell+1)} = \mu_i$ for $i \in N$.

5.2 Facility location with fortification

Data:

- $N = \{1, \ldots, n\}$: the set of customers
- F: the set of possible facility locations
- $a_{ij}, i \in N, j \in F$: distances from customer i to facility j. The customer will consume at the nearest opening facility. At the beginning, all facilities open.
- $d_i, i \in N$: demand of customer i

Decision Variables:

Leader's decision variables:

- $x_j \in \{0, 1\}, j \in F$: whether or not the facility j is protected

Follower's decision variables:

- $y_j \in \{0, 1\}, j \in F$: whether or not the facility is destroyed
- $\pi_{ij} \in \{0, 1\}$: whether or not customer i decides to consume at facility j

Thus, the r-interdiction median problem with fortification [16, 3, 19] is formalized as follows:

$$
\begin{aligned}
\min_{x} \ & \sum_{i,j} d_i a_{ij} \pi_{ij} \\
\text{s.t.} \ & \sum_{j} x_j \leq q, \\
& x_j \in \{0, 1\} \quad \forall j \in F, \\
& \max_{y,\pi} \sum_{i,j} d_i a_{ij} \pi_{ij} \\
& \text{s.t.} \sum_{j \in F} x_{ij} = 1 \quad \forall i \in N, \\
& \quad \sum_{j \in F} y_j \leq r, \\
& \quad \sum_{h \in T_{ij}} \pi_{ih} \leq y_j \quad \forall i \in N, j \in F, \\
& \quad x_j + y_j \leq 1 \quad \forall j \in F, \\
& \quad y_i \in \{0, 1\}, \pi_{i,j} \in \{0, 1\},
\end{aligned}
$$

where $T_{ij} = \{h \in F : d_{ih} > d_{ij}\}$ denotes the set of facility whose distances with customer i is longer than facility j.

6 Minimum spanning tree interdiction problem

We consider the most vital arcs in the minimum spanning tree problem [9, 18]: given a graph $G = (N, E)$, the leader interdicts a number of edges to increase the minimum spanning tree as large as possible. In this section, we show that the k-optimality idea can be applied to minimum spanning tree interdiction problem to get a single-level mixed-integer reformulation, which can be directly handled by commercial solver. Firstly, the minimum spanning tree is give by

$$\begin{aligned} \max_x \min_y \; & c^T y \\ \text{s.t.} \; & \sum_{(u,v)\in E} x_{u,v} \le K, && \text{(16a)} \\ & x_{u,v} + y_{u,v} \le 1 \quad \forall (u,v) \in E, && \text{(16b)} \\ & G[y] \text{ is a spanning tree of graph } \; G, && \text{(16c)} \\ & x_{u,v} \in \{0,1\}, y_{u,v} \in \{0,1\}, \end{aligned}$$

where $G[y]$ is the subgraph induced by the edges whose entries in y is 1 (i.e., $y_{u,v} = 1$). Let $\mathcal{P} = \{(x, y) \in \{0,1\}^{|E|} \times \{0,1\}^{|E|} : \text{(16a)} - \text{(16c)}\}$. It is worthwhile to notice that the single-level minimum spanning tree problem can be solved through greed algorithm [2], in which iteratively adds the smallest edges to the solution and avoids the cycles. It implies that the optimal solution of the minimum spanning tree problem is a 2-optimal solution. Based on this observation, we can identify the reaction solution of the follower.

Lemma 3. *For a given leader's decision, y is a rational reaction solution of the follower ($y \in \mathcal{S}_2(x)$) if and only if*

(i) $y \in \mathcal{P}(x)$*;*

(ii) *For any pairs of edges $(u_1, v_1), (u_2, v_2) \in E$ and $c_{u_1,v_1} < c_{u_2,v_2}$, if $y_{u_1,v_1} = 0, x_{u_1,v_1} = 0$ and $y_{u_2,v_2} = 1, x_{u_2,v_2} = 0$, then $G[y + e_{u_1,v_1} - e_{u_2,v_2}]$ is not a spanning tree.*

To formulate the condition that $G[y + e_{u_1,v_1} - e_{u_2,v_2}]$ is not a spanning tree as constraints, we consider a shortest path problem. Denote the directed graphs $\mathcal{G}[\mathcal{A}_{u_1v_1}(y)] = (N, \mathcal{A}_{u_1v_1}(y))$, where $\mathcal{A}_{u_1v_1}(y) = \{(u, v), (v, u) : y_{u,v} = 1, (u, v) \neq (u_1, v_1)\}$. The shortest path problem from u_1 to v_1 in graph $\mathcal{G}[\mathcal{A}_{u_1v_1}(y)]$ and its dual as the following linear programs:

$$\begin{aligned} \min \; & \sum_{(u,v)\in\mathcal{A}_f^t} x^t_{uv} + n x^t_{u_2v_2} \\ \text{s.t.} \; & A_{u_1v_1}(y) x^t = \begin{cases} 1, & \text{for vertex } u_2 \\ 0, & \text{for other verticies }, \\ -1, & \text{for vertex } \; v_2 \end{cases} \\ & x^t_{uv} \ge 0 \quad \forall \, (u,v) \in \mathcal{A}_{u_1v_1}(y) \cup (u_2, v_2). \end{aligned} \qquad \begin{aligned} \max \; & \pi^t_{u_2} - \pi^t_{v_2} \\ \text{s.t.} \; & \pi^t_u - \pi^t_v \le 1 \quad \forall \, (u,v) \in \mathcal{A}_{u_1v_1}(y), \\ & \pi^t_{u_2} - \pi^t_{v_2} \le n. \end{aligned}$$

Matrix $A_{u_1v_1}(y)$ is the node-arc matrix of graph $\mathcal{G}[\mathcal{A}_{u_1v_1}(y) \cup (u_2, v_2)]$. Note that $G[y + e_{u_1,v_1} - e_{u_2,v_2}]$ is not a spanning tree if and only if the above shortest path problem has optimal value n.

References

[1] Ahmad Abdi and Ricardo Fukasawa. On the mixing set with a knapsack constraint. *Mathematical Programming*, 157(1):191–217, 2016.

[2] Ravindra K Ahuja, Thomas L Magnanti, and James B Orlin. Network flows. 1988.

[3] Deniz Aksen and Necati Aras. A bilevel fixed charge location model for facilities under imminent attack. *Computers & Operations Research*, 39(7):1364–1381, 2012.

[4] Alper Atamtürk, George L Nemhauser, and Martin WP Savelsbergh. The mixed vertex packing problem. *Mathematical Programming*, 89(1):35–53, 2000.

[5] Mourad Baïou and Francisco Barahona. Stackelberg bipartite vertex cover and the preflow algorithm. *Algorithmica*, 74(3):1174–1183, 2016.

[6] Cristina Bazgan, Sonia Toubaline, and Zsolt Tuza. The most vital nodes with respect to independent set and vertex cover. *Discrete Applied Mathematics*, 159(17):1933–1946, 2011.

[7] Alberto Caprara, Margarida Carvalho, Andrea Lodi, and Gerhard J Woeginger. Bilevel knapsack with interdiction constraints. *INFORMS Journal on Computing*, 28(2):319–333, 2016.

[8] Scott DeNegre. Interdiction and discrete bilevel linear programming. 2011.

[9] Greg N Frederickson and Roberto Solis-Oba. Increasing the weight of minimum spanning trees. *Journal of Algorithms*, 33(2):244–266, 1999.

[10] Oktay Günlük and Yves Pochet. Mixing mixed-integer inequalities. *Mathematical Programming*, 90(3):429–457, 2001.

[11] IBM. ILOG CPLEX optimization studio (12.8), 2017. https://www.ibm.com/products/ilog-cplex-optimization-studio (Accessed January 06, 2019).

[12] Simge Küçükyavuz. On mixing sets arising in chance-constrained programming. *Mathematical programming*, 132(1-2):31–56, 2012.

[13] James Luedtke, Shabbir Ahmed, and George L Nemhauser. An integer programming approach for linear programs with probabilistic constraints. *Mathematical programming*, 122(2):247–272, 2010.

[14] Raid Mansi, Cláudio Alves, JM Valério de Carvalho, and Saïd Hanafi. An exact algorithm for bilevel 0-1 knapsack problems. *Mathematical Problems in Engineering*, 2012, 2012.

[15] Silvano Martello, David Pisinger, and Paolo Toth. Dynamic programming and strong bounds for the 0-1 knapsack problem. *Management Science*, 45(3):414–424, 1999.

[16] Maria P Scaparra and Richard L Church. A bilevel mixed-integer program for critical infrastructure protection planning. *Computers & Operations Research*, 35(6):1905–1923, 2008.

[17] Sahar Tahernejad, Ted K Ralphs, and Scott T DeNegre. A branch-and-cut algorithm for mixed integer bilevel linear optimization problems and its implementation. Technical report, Tech. rep., COR@ L Laboratory Technical Report 16T-015-R3, Lehigh University, 2016.

[18] Rico Zenklusen. An o (1)-approximation for minimum spanning tree interdiction. In *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*, pages 709–728. IEEE, 2015.

[19] Ying Zhang, Lawrence V Snyder, Ted K Ralphs, and Zhaojie Xue. The competitive facility location problem under disruption risks. *Transportation Research Part E: Logistics and Transportation Review*, 93:453–473, 2016.

[20] Ming Zhao, Kai Huang, and Bo Zeng. A polyhedral study on chance constrained program with random right-hand side. *Mathematical Programming*, 166(1-2):19–64, 2017.