

Causal Inference

Lab 5: Propensity Score, Matching, and Weighting

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Propensity Score

Definition (Rosenbaum and Rubin (1983))

Propensity score is defined as the conditional probability of receiving a treatment given pre-treatment covariates X :

$$e(X) = \Pr(D = 1 | X) = \mathbb{E}[D | X],$$

where $X = (X_1, \dots, X_p)$ is the collection of p covariates.

- Propensity score is a probability
 - ▶ Analogous to a summary statistic of the assignment mechanism.

Balancing Property of Propensity Score

- Property 1. The propensity score $e(X)$ balances the distribution of all X between the treatment groups:

$$D \perp\!\!\!\perp X \mid e(X)$$

- Conditional on the PS, treatment is randomized.
- Definition: A balancing score $b(X)$ is a function of the covariates such that

$$Z \perp\!\!\!\perp X \mid b(X)$$

- Propensity score is a balancing score.

Balancing Scores

- Property 2. Let $b(X)$ be a function of X , $b(X)$ is a balancing score if and only if $b(X)$ is finer than $e(X)$ in the sense that $e(X) = f(b(X))$ for some function f .

Proof sketch:

- \Leftarrow
- ▶ Suppose $b(X)$ is finer than $e(X)$, prove
 $\Pr(D = 1 | X, b(X)) = \Pr(D = 1 | b(X))$.
 - ▶ LHS: $\Pr(D = 1 | X, b(X)) = \Pr(D = 1 | X) = e(X)$.
 - ▶ RHS: $\Pr(D = 1 | b(X)) = \mathbb{E}[\mathbb{E}[D | X, b(X)] | b(X)] = \mathbb{E}[\mathbb{E}[D | X] | b(X)] = \mathbb{E}[e(X) | b(X)] = e(X)$.
- \Rightarrow
- ▶ Suppose $b(X)$ is a balancing score but not finer than $e(X)$. Then $\exists X_1, X_2$ such that $e(X_1) \neq e(X_2)$ but $b(X_1) = b(X_2)$
 - ▶ $\Pr(D = 1 | X_1, b(X_1)) = \Pr(D = 1 | X_1) \neq \Pr(D = 1 | X_2) = \Pr(D = 1 | X_2, b(X_2))$: $D \not\perp\!\!\!\perp X | b(X)$
 - ▶ $b(X)$ cannot be a balancing score, so proof by contradiction, $b(X)$ must be finer than $e(X)$ to be a balancing score.

Balancing Scores and Propensity Score

- Rosenbaum and Rubin (1983) showed that $e(X)$ is the coarsest balancing score, X is the finest balancing score.
- Knowledge of the balancing score could create a conditionally randomized trial, which supports valid causal inference (creating an ideal randomized experiment)
- Mostly focus on the coarsest balancing score $e(X)$, but the general theory holds for other specification of $b(X)$.

Remarks on the Balancing Property

1. If a subclass of units or a matched treatment-control pair is homogeneous in $e(X)$, then the treatment and control units have the same distribution of X .
2. If a subclass of units or a matched treatment-control pair is homogeneous in both $e(X)$ and certain X , the other components of X within those refined class is also balanced – practical implication: estimating causal estimand in subpopulation, e.g., male of female group
3. The balancing property is a statement on the distribution of X , NOT on assignment mechanism or potential outcomes.

Propensity Score: Unconfoundedness

- Property 3. If D is unconfounded given X , then D is unconfounded given $b(X)$, i.e.,

$$(Y(1), Y(0)) \perp\!\!\!\perp D \mid X \implies (Y(1), Y(0)) \perp\!\!\!\perp D \mid b(X)$$

- Proof. Sufficient to show

$$\Pr(D = 1 \mid Y(1), Y(0), b(X)) = \Pr(D = 1 \mid b(X))$$

1. Recall from the previous proof, RHS $\Pr(D = 1 \mid b(X)) = e(X)$.
2. LHS by LIE,

$$\begin{aligned}\mathbb{E} [\Pr(D = 1 \mid Y(1), Y(0), X) \mid Y(1), Y(0), b(X)] \\ = \mathbb{E}[e(X) \mid Y(1), Y(0), b(X)] = e(X),\end{aligned}$$

because $b(X)$ is finer than $e(X)$

Propensity Score: Unconfoundedness

- Property 4. If D is unconfounded given X , then D is unconfounded given $e(X)$, i.e.,

$$(Y(1), Y(0)) \perp\!\!\!\perp D \mid X \implies (Y(1), Y(0)) \perp\!\!\!\perp D \mid e(X)$$

1. Given a vector of covariates that ensure unconfoundedness, adjustment for differences in propensity scores removes all biases associated with differences in the covariates
2. $e(X)$ can be viewed as a summary score of the observed covariates
3. Causal inference can be drawn through stratification, matching, weighting, etc. using the scalar $e(X)$ instead of high dimensional covariates (dimension reduction).

Propensity Score: Remarks

- Controlling for propensity score (or more generally, balancing score)
 - ▶ Create a conditionally randomized trial ($Y(1), Y(0)$) $\perp\!\!\!\perp D \mid e(X)$
 - ▶ Balances covariates between groups $D \perp\!\!\!\perp X \mid e(X)$
- Propensity score balances the observed covariates but does not generally balance unobserved covariates
- Adjusting for differences in $e(X)$ can remove bias due to X
- In most observational studies, the propensity score $e(X)$ is unknown and thus needs to be estimated

Propensity Score Analysis of Causal Effects

1. Estimate $e(X)$ by logistic regression or machine learning methods
2. Given $\hat{e}(X)$, estimate the causal effects through
 - ▶ Subclassification
 - ▶ Matching
 - ▶ Regression
 - ▶ Weighting
 - ▶ Mixed procedure of the above

Propensity Score Matching

- Special case of matching: the distance metric is the (estimated) propensity score
- 1-to- k nearest neighbor matching is common when the control group is large compared to treatment group
- Pros: robust, popular, vast literature, and intuitive to understand
- Cons: inefficient (check Abadie and Imbens (2006) for closed form variance expression)

Propensity Score Weighting

- Most common approach for causal inference with observational data
- Intuitive: reweighting to mimic randomized treatment assignment
- Flexible: extends to binary, multi-valued, and continuous treatments
- Transparent: balance diagnostics and effective sample size
- Limitations: instability with extreme weights

Inverse Probability Weighting (IPW)

- Easy to show

$$\mathbb{E} \left[\frac{DY}{e(X)} - \frac{(1-D)Y}{1-e(X)} \right] = \tau^{\text{ATE}}.$$

- Observe

$$\mathbb{E} \left[\frac{DY}{e(X)} \right] = \mathbb{E} \left[\frac{1}{e(X)} \mathbb{E} [DY(1) | X] \right] = \mathbb{E} \left[\frac{\mathbb{E} [D | X]}{e(X)} \mathbb{E} [Y(1) | X] \right]$$

Inverse Probability Weighting (IPW), cont.

- Define the inverse probability weights (IPW)

$$\begin{cases} w_1(X_i) = \frac{1}{e(X_i)}, & D_i = 1 \\ w_0(X_i) = \frac{1}{1-e(X_i)} & D_i = 0 \end{cases}$$

- An unbiased moment-type estimator of ATE: difference in the mean of the weighted outcomes between groups

$$\begin{aligned}\hat{\tau}_{\text{IPW}, 1} &= \frac{1}{N} \left(\sum_{i=1}^N \frac{Y_i D_i}{e(X_i)} - \sum_{i=1}^N \frac{Y_i (1 - D_i)}{1 - e(X_i)} \right) \\ &= \frac{1}{N} \sum_{i=1}^N (Y_i D_i w_1(X_i) - Y_i (1 - D_i) w_0(X_i))\end{aligned}$$

Normalize the Weights

- When using any weighting methods, a good practice is to normalize the weights - sum of the total weights within one group should be 1
- Divide each unit's weight by the sum of all weights in that group $\frac{w_i}{\sum_{i:D=d} w_i}$ for $d = 0, 1$, i.e., the Hajek estimator:

$$\hat{\tau}_{IPW, 2} = \frac{\sum_{i=1}^N Y_i D_i w_1(X_i)}{\sum_{i=1}^N D_i w_1(X_i)} - \frac{\sum_{i=1}^N Y_i (1 - D_i) w_0(X_i)}{\sum_{i=1}^N (1 - D_i) w_0(X_i)}.$$

- Reduce variance: $\text{Var}(\hat{\tau}_{IPW, 2}) \leq \text{Var}(\hat{\tau}_{IPW, 1})$, and lead to more stable estimate (Hirano et al. 2003)