Causal Inference

Lab 5: Propensity Score, Matching, and Weighting

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Propensity Score

Definition (Rosenbaum and Rubin (1983))

Propensity score is defined as the conditional probability of receiving a treatment given pre-treatment covariates X:

$$e(X) = \Pr(D = 1 \mid X) = \mathbb{E}[D \mid X],$$

where $X = (X_1, \ldots, X_p)$ is the collection of p covariates.

- Propensity score is a probability
  - Analogous to a summary statistic of the assignment mechanism.

Balancing Property of Propensity Score

- Property 1. The propensity score $e(X)$ balances the distribution of all X between the treatment groups:

$$D \perp\!\!\!\perp X \mid e(X)$$

- Conditional on the PS, treatment is randomized.
- Definition: A balancing score $b(X)$ is a function of the covariates such that

$$Z \perp\!\!\!\perp X \mid b(X)$$

- Propensity score is a balancing score.

Balancing Scores

- Property 2. Let $b(X)$ be a function of X, $b(X)$ is a balancing score if and only if $b(X)$ is finer than $e(X)$ in the sense that $e(X) = f(b(X))$ for some function f.

Proof sketch:

$\Leftarrow$
- Suppose $b(X)$ is finer than $e(X)$, prove $\Pr(D = 1 \mid X, b(X)) = \Pr(D = 1 \mid b(X))$.
- LHS: $\Pr(D = 1 \mid X, b(X)) = \Pr(D = 1 \mid X) = e(X)$.
- RHS: $\Pr(D = 1 \mid b(X)) = \mathbb{E}[\mathbb{E}[D \mid X, b(X)] \mid b(X)] = \mathbb{E}[\mathbb{E}[D \mid X] \mid b(X)] = \mathbb{E}[e(X) \mid b(X)] = e(X)$.

$\Rightarrow$
- Suppose $b(X)$ is a balancing score but not finer than $e(X)$. Then $\exists X_1, X_2$ such that $e(X_1) \neq e(X_2)$ but $b(X_1) = b(X_2)$
- $\Pr(D = 1 \mid X_1, b(X_1)) = \Pr(D = 1 \mid X_1) \neq \Pr(D = 1 \mid X_2) = \Pr(D = 1 \mid X_2, b(X_2))$: $D \not\perp\!\!\!\perp X \mid b(X)$
- $b(X)$ cannot be a balancing score, so proof by contradiction, b(X) must be finer than $e(X)$ to be a balancing score.

Balancing Scores and Propensity Score

- Rosenbaum and Rubin (1983) showed that $e(X)$ is the coarsest balancing score, X is the finest balancing score.
- Knowledge of the balancing score could create a conditionally randomized trial, which supports valid causal inference (creating an ideal randomized experiment)
- Mostly focus on the coarsest balancing score $e(X)$, but the general theory holds for other specification of $b(X)$.

Remarks on the Balancing Property

1. If a subclass of units or a matched treatment-control pair is homogeneous in $e(X)$, then the treatment and control units have the same distribution of X.
2. If a subclass of units or a matched treatment-control pair is homogeneous in both $e(X)$ and certain X, the other components of X within those refined class is also balanced – practical implication: estimating causal estimand in subpopulation, e.g., male of female group
3. The balancing property is a statement on the distribution of X, NOT on assignment mechanism or potential outcomes.

Propensity Score: Unconfoundedness

- Property 3. If D is unconfounded given X, then D is unconfounded given $b(X)$, i.e.,

$$(Y(1), Y(0)) \perp\!\!\!\perp D \mid X \implies (Y(1), Y(0)) \perp\!\!\!\perp D \mid b(X)$$

- Proof. Sufficient to show

$$\Pr(D = 1 \mid Y(1), Y(0), b(X)) = \Pr(D = 1 \mid b(X))$$

1. Recall from the previous proof, RHS $\Pr(D = 1 \mid b(X)) = e(X)$.
2. LHS by LIE,

$$\begin{aligned} &\mathbb{E}\left[\Pr(D = 1 \mid Y(1), Y(0), X) \mid Y(1), Y(0), b(X)\right] \\ &= \mathbb{E}\left[e(X) \mid Y(1), Y(0), b(X)\right] = e(X), \end{aligned}$$

because $b(X)$ is finer than $e(X)$

Propensity Score: Unconfoundedness

- Property 4. If D is unconfounded given X, then D is unconfounded given $e(X)$, i.e.,

$$(Y(1), Y(0)) \perp\!\!\!\perp D \mid X \implies (Y(1), Y(0)) \perp\!\!\!\perp D \mid e(X)$$

1. Given a vector of covariates that ensure unconfoundedness, adjustment for differences in propensity scores removes all biases associated with differences in the covariates
2. $e(X)$ can be viewed as a summary score of the observed covariates
3. Causal inference can be drawn through stratification, matching, weighting, etc. using the scalar $e(X)$ instead of high dimensional covariates (dimension reduction).

Propensity Score: Remarks

- Controlling for propensity score (or more generally, balancing score)
  - Create a conditionally randomized trial $(Y(1), Y(0)) \perp\!\!\!\perp D \mid e(X)$
  - Balances covariates between groups $D \perp\!\!\!\perp X \mid e(X)$
- Propensity score balances the observed covariates but does not generally balance unobserved covariates
- Adjusting for differences in $e(X)$ can remove bias due to X
- In most observational studies, the propensity score $e(X)$ is unknown and thus needs to be estimated

Propensity Score Analysis of Causal Effects

1. Estimate $e(X)$ by logistic regression of machine learning methods
2. Given $\hat{e}(X)$, estimate the causal effects through
   - Subclassification
   - Matching
   - Regression
   - Weighting
   - Mixed procedure of the above

Propensity Score Matching

- Special case of matching: the distance metric is the (estimated) propensity score
- 1-to-k nearest neighbor matching is common when the control group is large compared to treatment group
- Pros: robust, popular, vast literature, and intuitive to understand
- Cons: inefficient (check Abadie and Imbens (2006)) for closed form variance expression

Propensity Score Weighting

- Most common approach for causal inference with observational data
- Intuitive: reweighting to mimic randomized treatment assignment
- Flexible: extends to binary, multi-valued, and continuous treatments
- Transparent: balance diagnostics and effective sample size
- Limitations: instability with extreme weights

Inverse Probability Weighting (IPW)

- Easy to show

$$\mathbb{E}\left[\frac{DY}{e(X)} - \frac{(1-D)Y}{1-e(X)}\right] = \tau^{\text{ATE}}.$$

- Observe

$$\mathbb{E}\left[\frac{DY}{e(X)}\right] = \mathbb{E}\left[\frac{1}{e(X)}\mathbb{E}\left[DY(1) \mid X\right]\right] = \mathbb{E}\left[\frac{\mathbb{E}\left[D \mid X\right]}{e(X)}\mathbb{E}\left[Y(1) \mid X\right]\right]$$

Inverse Probability Weighting (IPW), cont.

- Define the inverse probability weights (IPW)

$$\begin{cases} w_1(X_i) = \frac{1}{e(X_i)}, & D_i = 1 \\ w_0(X_i) = \frac{1}{1-e(X_i)} & D_i = 0 \end{cases}$$

- An unbiased moment-type estimator of ATE: difference in the mean of the weighted outcomes between groups

$$\begin{aligned} \hat{\tau}_{\text{IPW},1} &= \frac{1}{N}\left(\sum_{i=1}^{N} \frac{Y_i D_i}{e(X_i)} - \sum_{i=1}^{N} \frac{Y_i(1-D_i)}{1-e(X_i)}\right) \\ &= \frac{1}{N}\sum_{i=1}^{N} \left(Y_i D_i w_1(X_i) - Y_i(1-D_i) w_0(X_i)\right) \end{aligned}$$

Normalize the Weights

- When using any weighting methods, a good practice is to normalize the weights - sum of the total weights within one group should be 1
- Divide each unit's weight by the sum of all weights in that group $\frac{w_i}{\sum_{i:D=d} w_i}$ for $d = 0, 1$, i.e., the Hajek estimator:

$$\hat{\tau}_{\text{IPW, 2}} = \frac{\sum_{i=1}^{N} Y_i D_i w_1(X_i)}{\sum_{i=1}^{N} D_i w_1(X_i)} - \frac{\sum_{i=1}^{N} Y_i (1 - D_i) w_0(X_i)}{\sum_{i=1}^{N} (1 - D_i) w_0(X_i)}.$$

- Reduce variance: $\text{Var}\left(\hat{\tau}_{\text{IPW, 2}}\right) \leq \text{Var}\left(\hat{\tau}_{\text{IPW, 1}}\right)$, and lead to more stable estimate (Hirano et al. 2003)