人工智能原理与技术
5.马尔科夫决策MDP

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Question
问题

How would you get groceries on a Saturday afternoon in the least amount of time?

order grocery delivery
bike to the store
drive to the store
Uber/Lyft to the store
fly to the store

Course Plan
课程安排

Search problems

Constraint satisfaction problems

Markov decision processes

Markov networks

Adversarial games

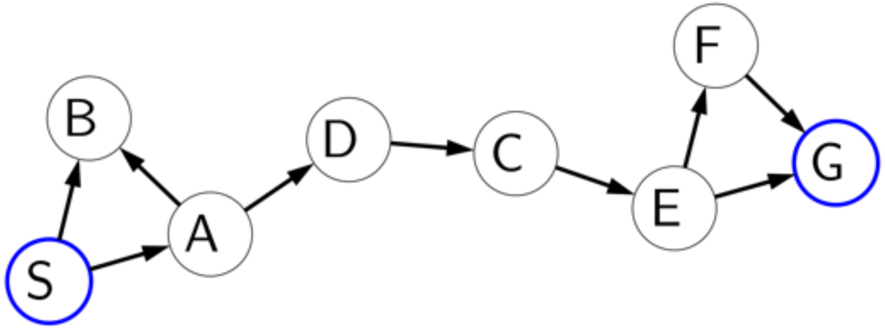
Bayesian networks

Reflex | **States** | **Variables** | **Logic**

Low-level → High-level

Machine learning

So far: search problems
搜索问题回顾

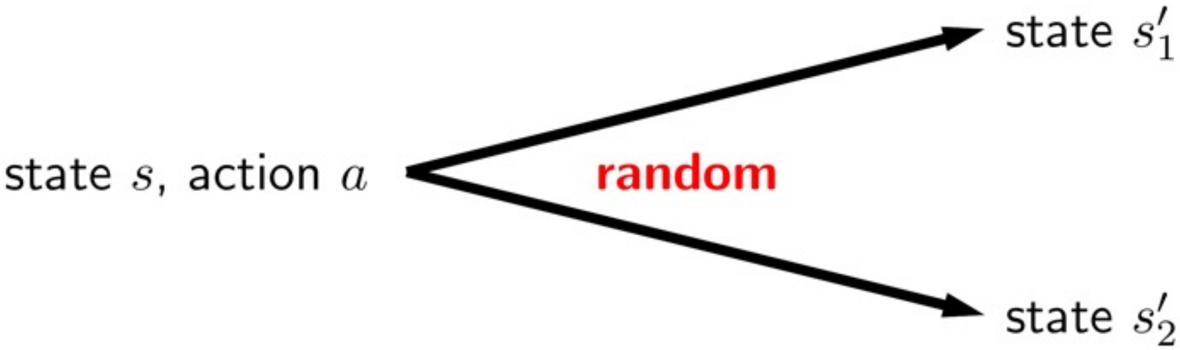


确定性的

deterministic

state s, action a ⟶ state $\mathrm{Succ}(s, a)$

Uncertainty in the real world
现实世界中的不确定性



History
历史

- MDPs: Mathematical Model for decision making under uncertainty.
  不确定性下决策的数学模型
- MDPs were first introduces in 1950s-60s.
- Ronald Howard's book on Dynamic Programming and Markov Processes
- The term 'Markov' refers to Andrey Makov as MDPs are extensions of Markov Chains, and they allow making decisions (taking actions or having choice).
  mdp是马尔可夫链的延伸，它们允许做出决定（采取行动或有选择余地）

Applications
应用

Robotics: decide where to move, but actuators can fail, hit unseen obstacles, etc.
机器人：决定移动到哪里，但执行器可能会失败，碰到看不见的障碍等。

Resource allocation: decide what to produce, don't know the customer demand for various products
资源配置：决定生产什么，不知道客户对各种产品的需求

Agriculture: decide what to plant, but don't know weather and thus crop yield
农业：决定种植什么，但不知道天气和作物产量

Volcano crossing
穿越火山

```
// Model
moveReward = 0  // For every action you take
passReward = 20  // If get to far green
volcanoReward = -50  // If fall into volcano
slipProb = 0.3  // If slip, go in random direction
discount = 1  // How much to value the future

// CHANGE THIS to 10
numIters = 10  // # iterations of value iteration
```

Run (or press ctrl-enter)

1.4	←2.9	-50	20
1.9	1.1	-50	13.8
2	6.5→	7.5→	13.2

Value: 1.86

- 向东(E)/ 西(W)/ 南(S)/ 北(N) 移动一格
- 移动到红色或绿色游戏结束
- -50: 火山　　2: 安全区　　20: 美丽海岛
- 怎么移动使收益最大?

Roadmap
路线图

Modeling

Modeling MDP Problems

Algorithms

Policy Evaluation

Value Iteration

Learning

Intro to Reinforcement Learning

Model-Based Monte Carlo

Model-Free Monte Carlo

SARSA

Q-learning

Epsilon Greedy

Function Approximation

MDPs: modeling
马尔可夫决策过程：建模

Modeling	Learning
$\boxed{\text{Modeling MDP Problems}}$	Intro to Reinforcement Learning
Algorithms	Model-Based Monte Carlo
Policy Evaluation	Model-Free Monte Carlo
Value Iteration	SARSA
	Q-learning
	Epsilon Greedy
	Function Approximation

Dice game
骰子游戏

Example: dice game

在每一轮 $r = 1,2,\cdots$

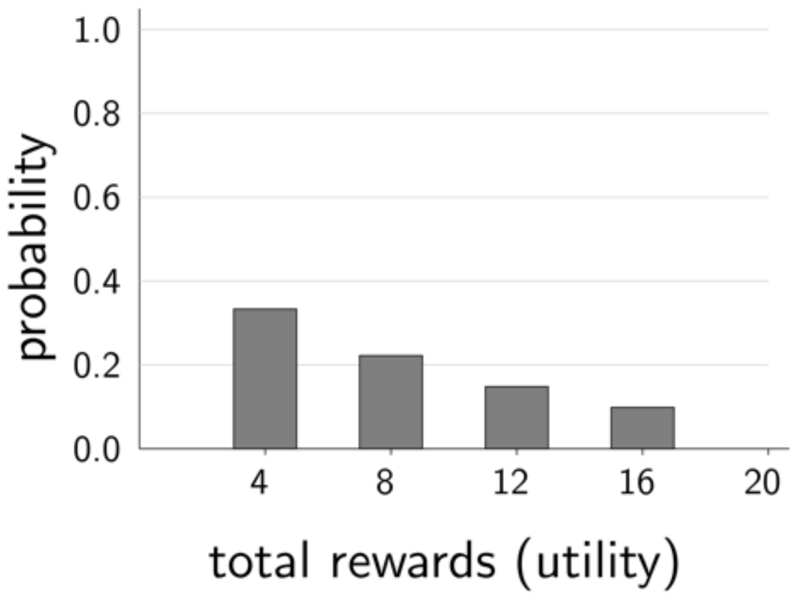
- 你选择留下或退出。
- 如果你退出，你会得到10美元，我们结束游戏。
- 如果留下，你得到4美元然后我掷一个六面骰子。
  - 如果骰子的结果是1或2，我们就结束游戏。
  - 否则，继续进行下一轮。

Start Stay Quit

Dice: Rewards: 0

Rewards
奖励

If follow policy "stay":

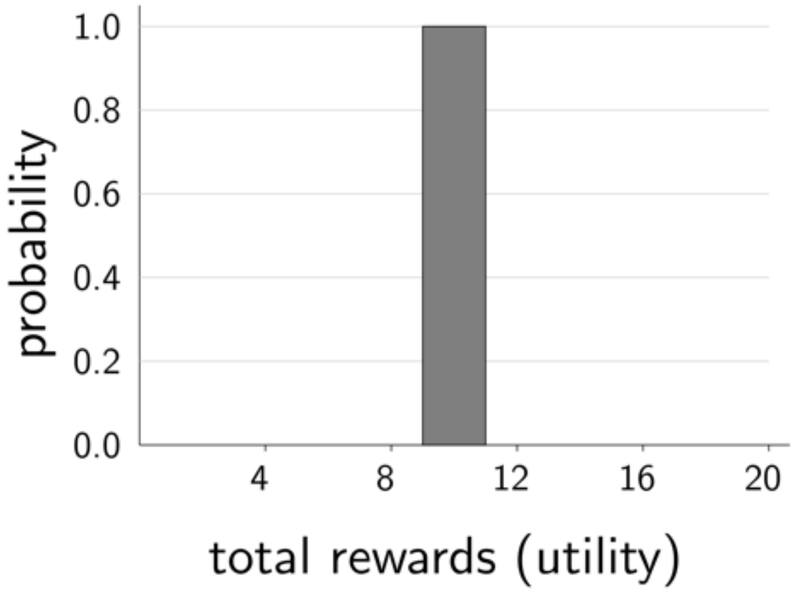


Expected utility:

$$\frac{1}{3}(4) + \frac{2}{3} \cdot \frac{1}{3}(8) + \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}(12) + \cdots = 12$$

Rewards
奖励

If follow policy "quit":



Expected utility:

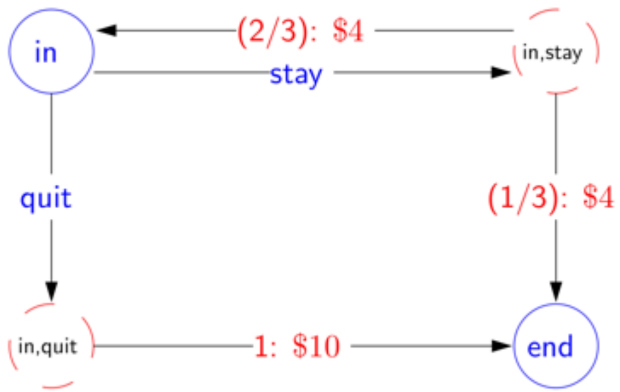
$$1(10) = 10$$

MDP for dice game
骰子游戏中的MDP

Example: dice game

在每一轮 r = 1,2,⋯

- 你选择留下或退出。
- 如果你退出，你会得到10美元，我们结束游戏。
- 如果留下，你得到4美元然后我掷一个六面骰子。
  - 如果骰子的结果是1或2，我们就结束游戏。
  - 否则，继续进行下一轮。



Markov decision process
马尔可夫决策过程

Definition: Markov decision process

States: the set of states

$s_{\text{start}} \in$ States: starting state

Actions(s): possible actions from state s

$T(s, a, s')$: probability of s' if take action a in state s

Reward(s, a, s'): reward for the transition (s, a, s')

IsEnd(s): whether at end of game

$0 \leq \gamma \leq 1$: discount factor (default: 1)

Search problems
对比：搜索问题

Definition: search problem

States: the set of states

$s_{\text{start}} \in$ States: starting state

Actions(s): possible actions from state s

Succ(s, a): where we end up if take action a in state s

Cost(s, a): cost for taking action a in state s

IsEnd(s): whether at end

- Succ$(s, a) \Rightarrow T(s, a, s')$ — $T(s, a, s')$: probability of s' if take action a in state s
- Cost$(s, a) \Rightarrow$ Reward(s, a, s') — Reward(s, a, s'): reward for the transition (s, a, s')

Transitions
转移概率

Definition: transition probabilities

The **transition probabilities** $T(s, a, s')$ specify the probability of ending up in state s' if taken action a in state s. 从状态s采取行动a最终到s'的概率

Example: transition probabilities

s	a	s'	$T(s, a, s')$
in	quit	end	1
in	stay	in	$2/3$
in	stay	end	$1/3$

Probabilities sum to one
转移概率之和为1

Example: transition probabilities

s	a	s'	$T(s,a,s')$
in	quit	end	1
in	stay	in	$2/3$
in	stay	end	$1/3$

For each state s and action a:

$$\sum_{s' \in \text{States}} T(s,a,s') = 1$$

Successors: s' such that $T(s,a,s') > 0$

Transportation example
运输的例子

Example: transportation

街块编号1到n。
从s走到s+1需要1分钟。
从s到2s乘一辆神奇的电车需要2分钟。
如何在最短的时间内从1到n？
电车故障概率为0.5。

What is a solution?
解决方法是什么?

Search problem: path (sequence of actions)

MDP:

Definition: policy

从状态s到行动a的映射

A **policy** π is a mapping from each state $s \in$ States to an action $a \in$ Actions(s).

Example: volcano crossing

s	$\pi(s)$
(1,1)	S
(2,1)	E
(3,1)	N
...	...

MDPs: policy evaluation
马尔可夫决策过程：策略评估

Modeling	Learning
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Evaluating a policy
评估一项政策

Definition: utility

Following a policy yields a **random path**.

The **utility** of a policy is the (discounted) sum of the rewards on the path (this is a random variable). 针对一条path而言，是随机变量

Path	Utility
[in; stay, 4, end]	4
[in; stay, 4, in; stay, 4, in; stay, 4, end]	12
[in; stay, 4, in; stay, 4, end]	8
[in; stay, 4, in; stay, 4, in; stay, 4, in; stay, 4, end]	16
...	...

Definition: value (expected utility)

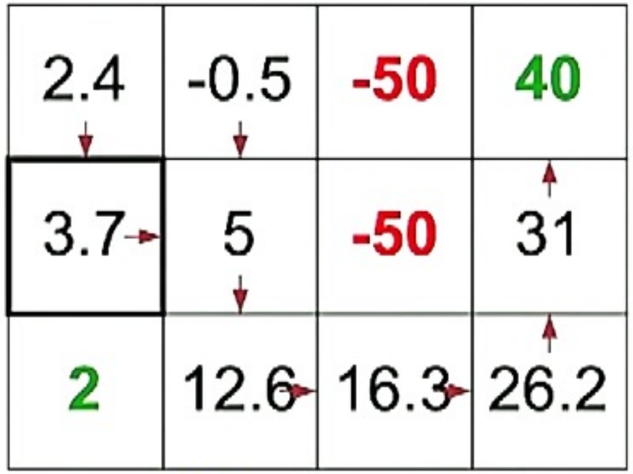
The **value** of a policy at a state is the **expected** utility.

Evaluating a policy: volcano crossing
策略评估：穿越火山

```
// Model
moveReward = -0.1  // For every action you take
passReward = 40  // If get to far green
volcanoReward = -50  // If fall into volcano
slipProb = 0.3  // If slip, go in random direction
discount = 0.9  // How much to value the future

// Run algorithms on model
numIters = 10  // # iterations of value iteration
numEpisodes = 1  // # simulations
```

Run (or press ctrl-enter)



a	r	s
		(2,1)
E	-0.1	(2,2)
S	-0.1	(2,1)
E	-0.1	(2,2)
S	-0.1	(3,2)
E	-0.1	(3,3)
E	-0.1	(3,4)
N	-0.1	(3,4)
N	-0.1	(3,3)
E	-0.1	(3,4)
N	-0.1	(2,4)
N	39.9	(1,4)

Value: 3.73
Utility: 13.26

- 通过程序模拟，随机生成路径，得到收敛后的value/utility

Discounting
“折扣”

Definition: utility

Path: $s_0, a_1 r_1 s_1, a_2 r_2 s_2, \ldots$ (action, reward, new state).

The **utility** with discount γ is

$$u_1 = r_1 + \gamma r_2 + \gamma^2 r_3 + \gamma^3 r_4 + \cdots$$

Discount $\gamma = 1$ (save for the future):

[stay, stay, stay, stay]: $4 + 4 + 4 + 4 = 16$

Discount $\gamma = 0$ (live in the moment):

[stay, stay, stay, stay]: $4 + 0 \cdot (4 + \cdots) = 4$

Discount $\gamma = 0.5$ (balanced life):

[stay, stay, stay, stay]: $4 + \frac{1}{2} \cdot 4 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 4 = 7.5$

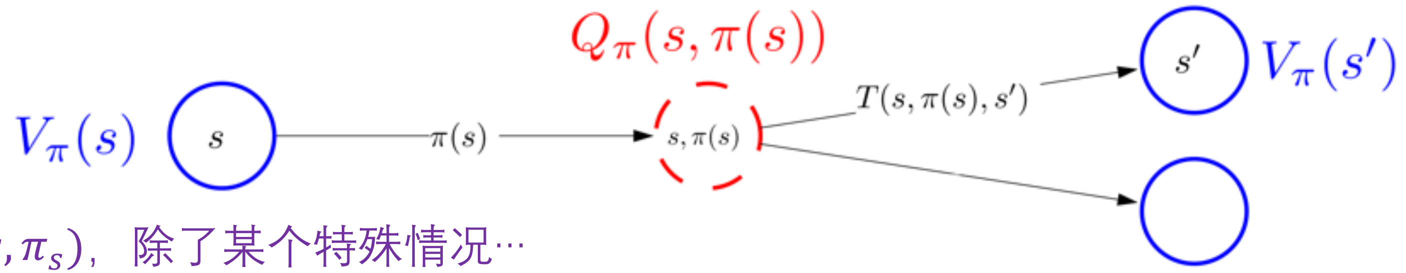
Policy evaluation
策略评估

Definition: value of a policy

Let $V_\pi(s)$ be the expected utility received by following policy π from state s.

Definition: Q-value of a policy

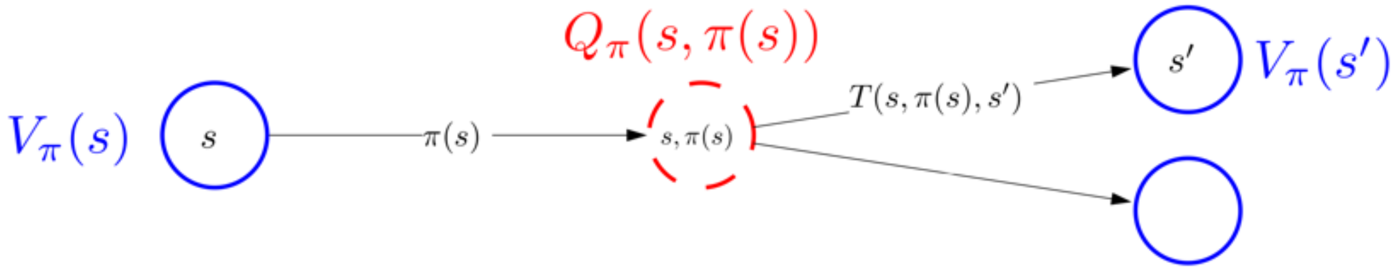
Let $Q_\pi(s, a)$ be the expected utility of taking action a from state s, and then following policy π.



- $V_\pi(s) = Q_\pi(s, \pi_s)$，除了某个特殊情况…

Policy evaluation
策略评估

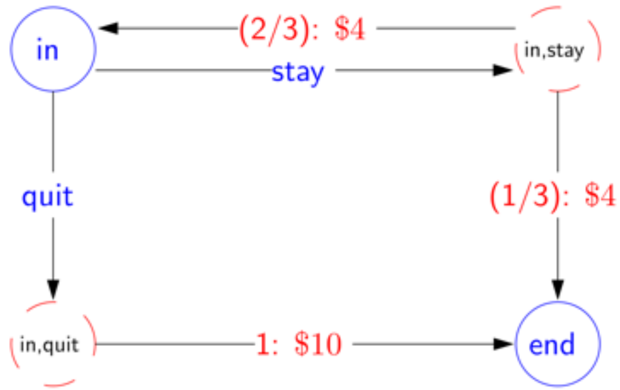
Plan: define recurrences relating value and Q-value



$$V_\pi(s) = \begin{cases} 0 & \text{if IsEnd}(s) \\ Q_\pi(s, \pi(s)) & \text{otherwise.} \end{cases}$$

$$Q_\pi(s, a) = \sum_{s'} T(s, a, s')[\text{Reward}(s, a, s') + \gamma V_\pi(s')]$$ 递归

Dice game
骰子游戏



(assume $\gamma = 1$)

Let π be the "stay" policy: $\pi(\text{in}) = \text{stay}$.

$$V_\pi(\text{end}) = 0$$

$$V_\pi(\text{in}) = \frac{1}{3}(4 + V_\pi(\text{end})) + \frac{2}{3}(4 + V_\pi(\text{in}))$$

In this case, can solve in closed form: 闭式解

$$V_\pi(\text{in}) = 12$$

Policy evaluation
策略评估

Key idea: iterative algorithm

Start with arbitrary policy values and repeatedly apply recurrences to converge to true values. 无闭式解时，通过迭代至收敛

Algorithm: policy evaluation

Initialize $V_\pi^{(0)}(s) \leftarrow 0$ for all states s.

For iteration $t = 1, \ldots, t_{\text{PE}}$:

For each state s:

$$V_\pi^{(t)}(s) \leftarrow \underbrace{\sum_{s'} T(s, \pi(s), s')[\text{Reward}(s, \pi(s), s') + \gamma V_\pi^{(t-1)}(s')]}_{Q^{(t-1)}(s,\pi(s))}$$

Policy evaluation implementation
策略评估的实现

How many iterations (t_{PE})? Repeat until values don't change much:

$$\max_{s \in \text{States}} |V_\pi^{(t)}(s) - V_\pi^{(t-1)}(s)| \leq \epsilon$$

Don't store $V_\pi^{(t)}$ for each iteration t, need only last two:

$$V_\pi^{(t)} \text{ and } V_\pi^{(t-1)}$$

Complexity
复杂度分析

Algorithm: policy evaluation

Initialize $V_\pi^{(0)}(s) \leftarrow 0$ for all states s.

For iteration $t = 1, \ldots, t_{\mathsf{PE}}$:

For each state s:

$$V_\pi^{(t)}(s) \leftarrow \underbrace{\sum_{s'} T(s, \pi(s), s')[\mathsf{Reward}(s, \pi(s), s') + \gamma V_\pi^{(t-1)}(s')]}_{Q^{(t-1)}(s,\pi(s))}$$

MDP complexity

S states

A actions per state

S' successors (number of s' with $T(s, a, s') > 0$)

Time: $O(t_{\mathsf{PE}} S S')$

Policy evaluation on dice game
骰子游戏的策略评估

Let π be the "stay" policy: $\pi(\text{in}) = \text{stay}$.

$$V_\pi^{(t)}(\text{end}) = 0$$

$$V_\pi^{(t)}(\text{in}) = \frac{1}{3}(4 + V_\pi^{(t-1)}(\text{end})) + \frac{2}{3}(4 + V_\pi^{(t-1)}(\text{in}))$$

s	end	in	
$V_\pi^{(t)}$	0.00	12.00	($t = 100$ iterations)

Converges to $V_\pi(\text{in}) = 12$.

- Try to implement it in Python.

Summary so far
总结

- MDP: graph with states, chance nodes, transition probabilities, rewards
  具有状态、机会节点、转移概率、奖励的图
- Policy: mapping from state to action (solution to MDP)
  从状态到动作的映射（MDP的解）
- Value of policy: expected utility over random paths
  随机路径上的期望效用
- Policy evaluation: iterative algorithm to compute value of policy
  计算策略价值的迭代算法

MDPs: value iteration
马尔可夫决策过程：价值迭代

Modeling	Learning
Modeling MDP Problems	Intro to Reinforcement Learning
Algorithms	
Policy Evaluation	Model-Based Monte Carlo
$\boxed{\text{Value Iteration}}$	Model-Free Monte Carlo
	SARSA
	Q-learning
	Epsilon Greedy
	Function Approximation

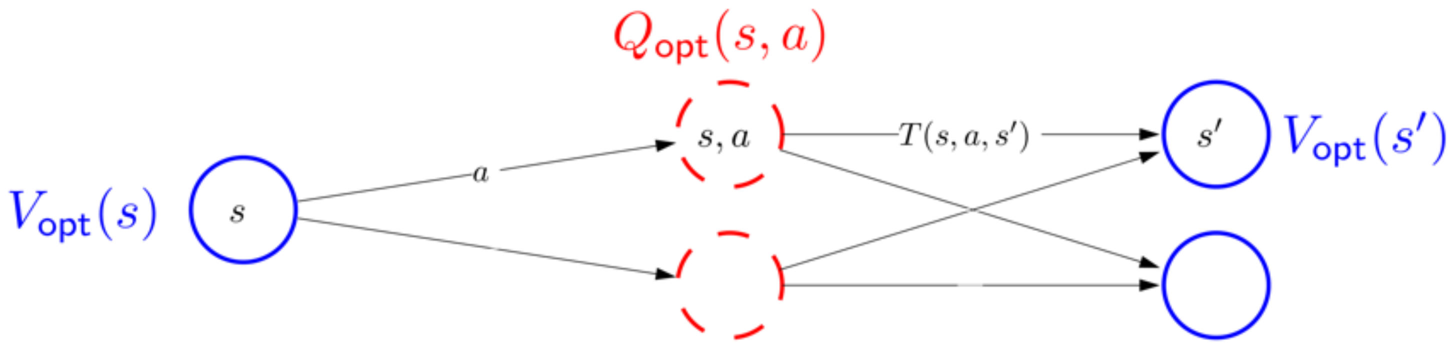
Optimal value and policy
最优价值与策略

Goal: try to get directly at maximum expected utility

Definition: optimal value

The **optimal value** $V_{\text{opt}}(s)$ is the maximum value attained by any policy.

Optimal values and Q-values
最优值和Q-value



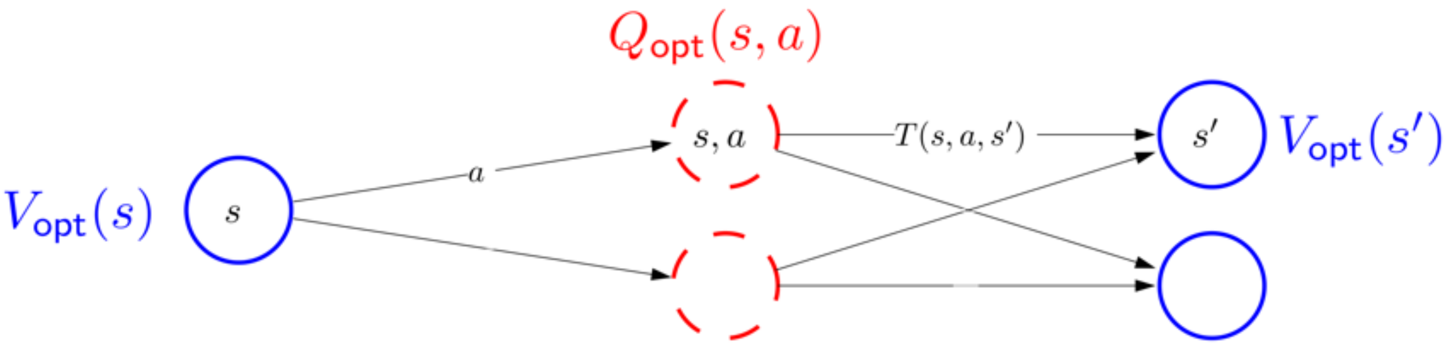
Optimal value if take action a in state s:

$$Q_{\text{opt}}(s,a) = \sum_{s'} T(s,a,s')[\text{Reward}(s,a,s') + \gamma V_{\text{opt}}(s')].$$

Optimal value from state s:

$$V_{\text{opt}}(s) = \begin{cases} 0 & \text{if IsEnd}(s) \\ \max_{a \in \text{Actions}(s)} Q_{\text{opt}}(s,a) & \text{otherwise.} \end{cases}$$

Optimal policies
最优策略



Given Q_{opt}, read off the optimal policy:

$$\pi_{\text{opt}}(s) = \arg\max_{a \in \text{Actions}(s)} Q_{\text{opt}}(s, a)$$

Value iteration
价值迭代

Algorithm: value iteration [Bellman, 1957]

Initialize $V_{\text{opt}}^{(0)}(s) \leftarrow 0$ for all states s.

For iteration $t = 1, \ldots, t_{\text{VI}}$:

For each state s:

$$V_{\text{opt}}^{(t)}(s) \leftarrow \max_{a \in \text{Actions}(s)} \underbrace{\sum_{s'} T(s,a,s')[\text{Reward}(s,a,s') + \gamma V_{\text{opt}}^{(t-1)}(s')]}_{Q_{\text{opt}}^{(t-1)}(s,a)}$$

Time: $O(t_{\text{VI}} S A S')$

[semi-live solution]

Value iteration: dice game
价值迭代：骰子游戏

s	end	in	
$V_{\text{opt}}^{(t)}$	0.00	12.00	($t = 100$ iterations)
$\pi_{\text{opt}}(s)$	-	stay	

Value iteration: volcano crossing
价值迭代：穿越火山

```
// Model
moveReward = 0  // For every action you take
passReward = 20  // If get to far green
volcanoReward = -50  // If fall into volcano
slipProb = 0.1  // If slip, go in random direction
discount = 1  // How much to value the future

// CHANGE THIS to 0, 1, 2, 3, ...
numIters = 10  // # iterations of value iteration
```

Run (or press ctrl-enter)

Convergence
收敛性

Theorem: convergence

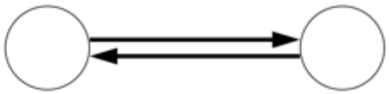
Suppose either

- discount $\gamma < 1$, or
- MDP graph is acyclic.

Then value iteration converges to the correct answer.

Example: non-convergence

discount $\gamma = 1$, zero rewards



Summary of algorithms
算法概述

- Policy evaluation: $(\text{MDP}, \pi) \to V_\pi$

- Value iteration: $\text{MDP} \to (Q_{\text{opt}}, \pi_{\text{opt}})$

Unifying idea
观点的统一

Algorithms:

- Search DP computes FutureCost(s)
- Policy evaluation computes policy value $V_\pi(s)$
- Value iteration computes optimal value $V_{\text{opt}}(s)$

Recipe:

- Write down recurrence (e.g., $V_\pi(s) = \cdots V_\pi(s') \cdots$)
- Turn into iterative algorithm (replace mathematical equality with assignment operator)