1. Vectors and Matrices

1.1. Vectors

Two-dimensional Position Vector

For $P = (x_1, y_1)$, $Q = (x_2, y_2)$

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

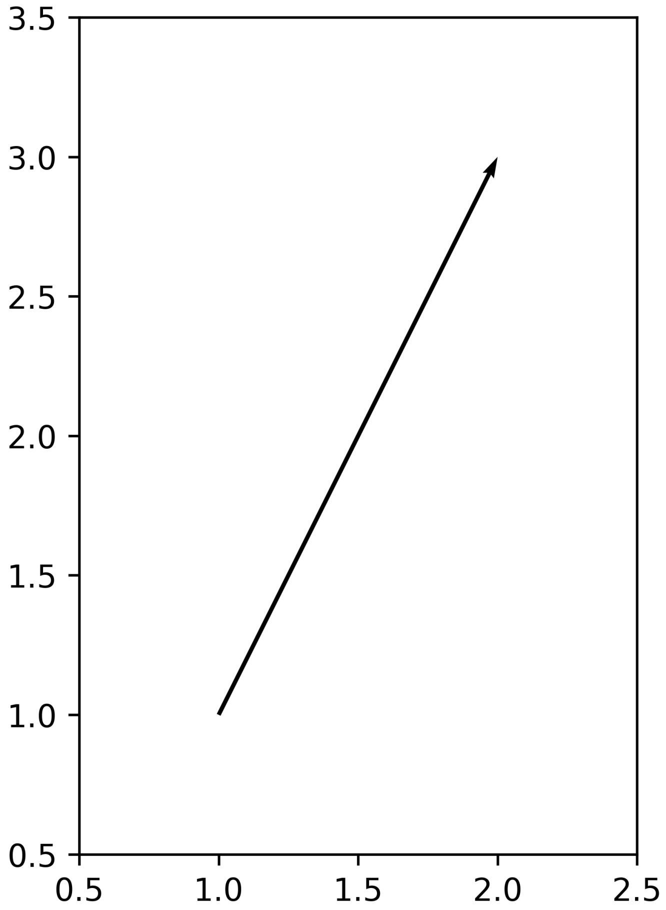
Three-dimensional Position Vector

For $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

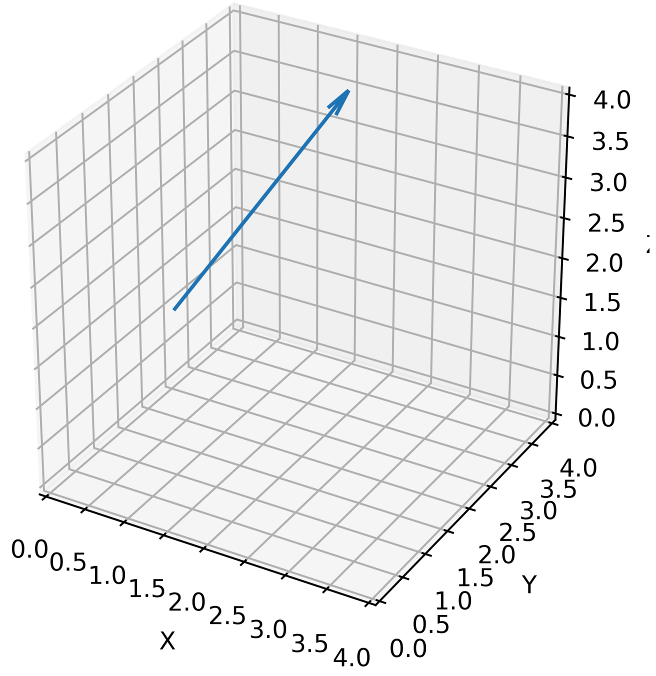
Drawing Two-dimensional Vectors

```
import matplotlib.pyplot as plt

quiver_params = {
    'angles': 'xy',
    'scale_units': 'xy',
    'scale': 1,
}

plt.quiver(1, 1, 1, 2, **quiver_params)
plt.xlim([.5, 2.5])
plt.ylim([.5, 3.5])
plt.gca().set_aspect('equal')
plt.show()
```



Drawing Three-dimensional Vectors

```
import matplotlib.pyplot as plt

ax = plt.figure().add_subplot(111, projection='3d')
ax.set_box_aspect([1,1,1])

ax.quiver(1, 1, 2, 1, 2, 2, arrow_length_ratio=0.1)

ax.set_xlim(0, 4)
ax.set_ylim(0, 4)
ax.set_zlim(0, 4)

ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')

plt.show()
```



Vector Additions

For $\overrightarrow{P_1Q_1} = \langle x_1, y_1 \rangle$, $\overrightarrow{P_2Q_2} = \langle x_2, y_2 \rangle$

$$\overrightarrow{P_1Q_1} + \overrightarrow{P_2Q_2} = \langle x_1 + x_2, y_1 + y_2 \rangle$$

For $\overrightarrow{P_1Q_1} = \langle x_1, y_1, z_1 \rangle$, $\overrightarrow{P_2Q_2} = \langle x_2, y_2, z_2 \rangle$

$$\overrightarrow{P_1Q_1} + \overrightarrow{P_2Q_2} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

Scalar Multiplication

For $\overrightarrow{PQ} = \langle x, y \rangle$ and $k \in \mathbb{R}$ (called a **scalar**),

$$k \cdot \overrightarrow{PQ} = \langle kx, ky \rangle$$

For $\overrightarrow{PQ} = \langle x, y, z \rangle$ and $k \in \mathbb{R}$,

$$k \cdot \overrightarrow{PQ} = \langle kx, ky, kz \rangle$$

Example

$P = (1, 1)\,, Q = (2, 3)$

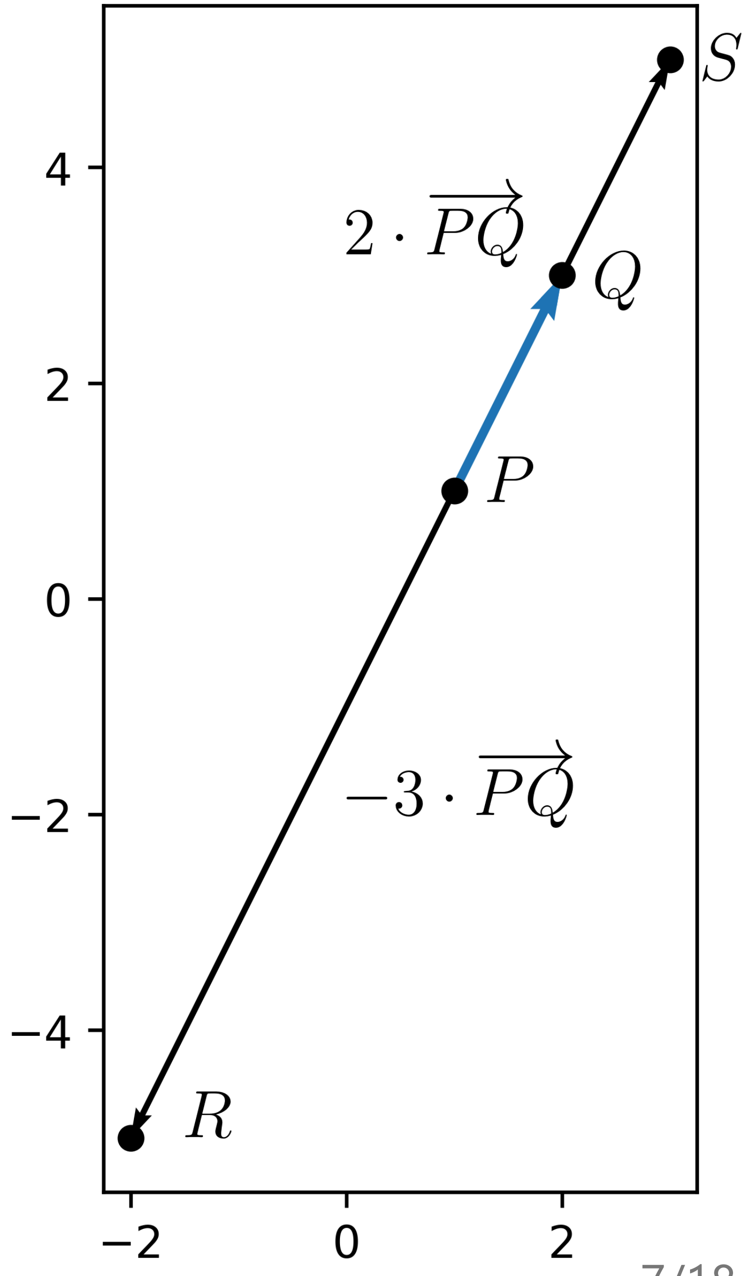
$$-3 \cdot \overrightarrow{PQ} = \langle -3, -6 \rangle$$

$$2 \cdot \overrightarrow{PQ} = \langle 2, 4 \rangle$$

Inverse and Zero

$$(-1) \cdot \overrightarrow{PQ} = -\overrightarrow{PQ}$$

$$0 \cdot \overrightarrow{PQ} = \mathbf{0}$$



n-dimensional Vector

- Vector Addition: $\mathbf{v} = \langle x_1, \cdots, x_n \rangle$, $\mathbf{w} = \langle y_1, \cdots, y_n \rangle$

$$\mathbf{v} + \mathbf{w} = \langle x_1 + y_1, \cdots, x_n + y_n \rangle$$

- Scalar Multiplication: $\mathbf{v} = \langle x_1, \cdots, x_n \rangle$

$$k \cdot \mathbf{v} = \langle kx_1, \cdots, kx_n \rangle$$

Linear Combination

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

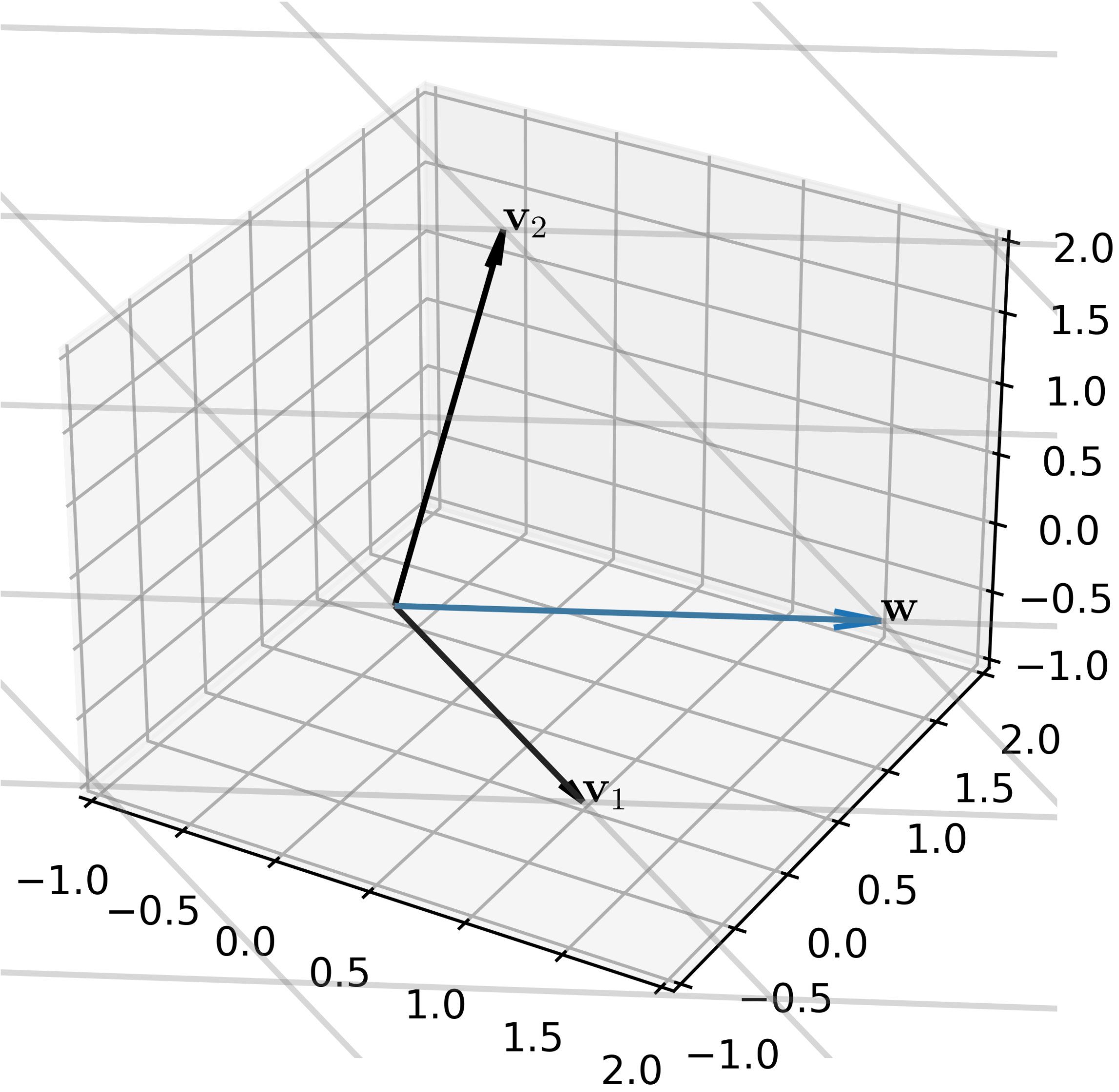
Vector Decomposition

$\mathbf{v}_1 = \langle 1, 0, -1 \rangle$

$\mathbf{v}_2 = \langle 0, 1, 2 \rangle$

$\mathbf{w} = \langle 2, 1, 0 \rangle$

$$\mathbf{w} = 2\mathbf{v}_1 + \mathbf{v}_2$$



Vector Space

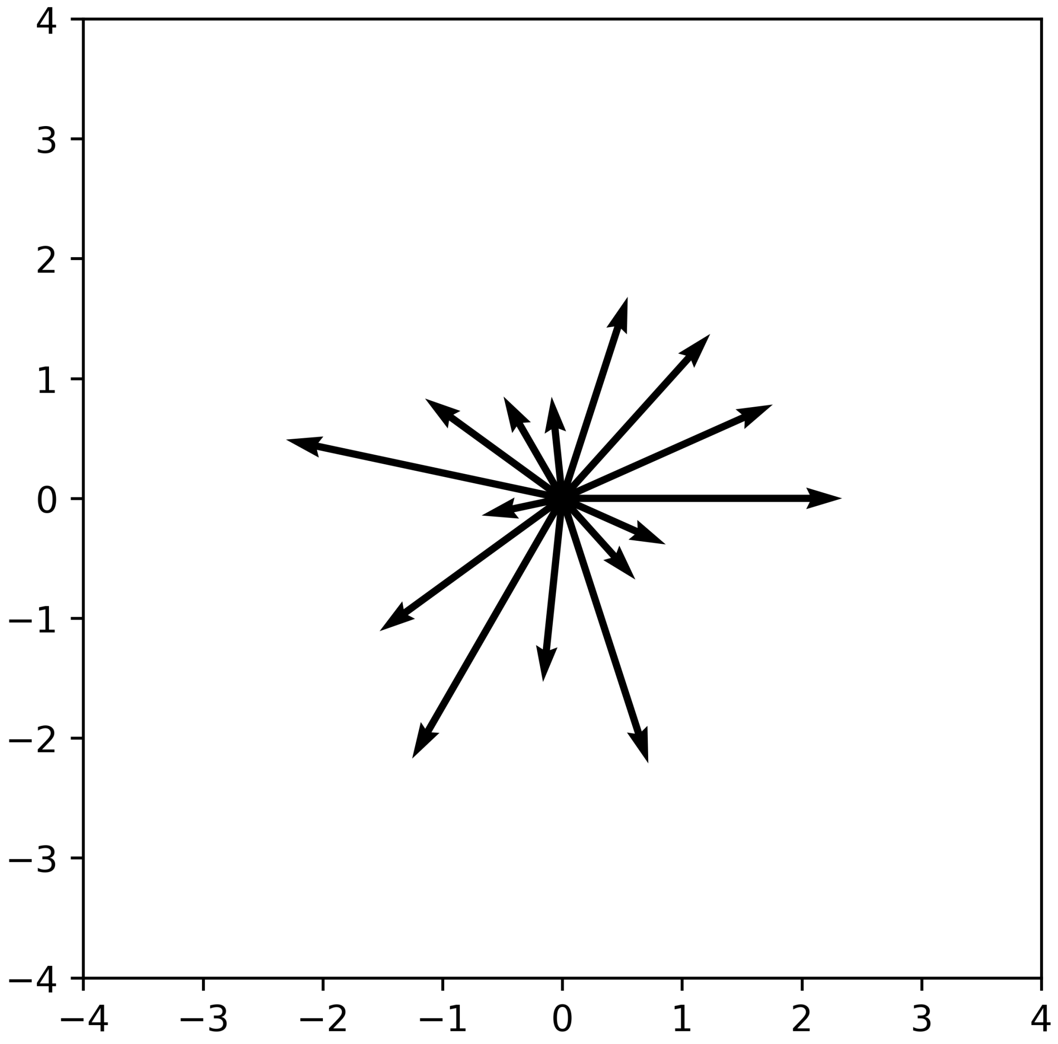
A set V of vectors $\mathbf{v}, \mathbf{w}$

- $\mathbf{v} + \mathbf{w} \in V$
- $k\mathbf{v} \in V$

n-dimensional Vector Spaces

$$\mathbb{R}^2 = \{\langle x, y \rangle \mid x, y \in \mathbb{R}\}$$

$$\mathbb{R}^3 = \{\langle x, y, z \rangle \mid x, y, z \in \mathbb{R}\}$$



Inner Product

$\mathbf{v} = \langle x_1, y_1 \rangle$, $\mathbf{w} = \langle x_2, y_2 \rangle$

$$\mathbf{v} \circ \mathbf{w} = x_1 x_2 + y_1 y_2$$

$\mathbf{v} = \langle x_1, y_1, z_1 \rangle$, $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$

$$\mathbf{v} \circ \mathbf{w} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Norm

$\mathbf{v} = \langle x, y \rangle$

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2}$$

$\mathbf{v} = \langle x, y, z \rangle$

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$$

Unit Vector

$$\|\mathbf{v}\| = 1$$

Normalization

$$\mathbf{v}^1 = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Inner Product and Relative Angle

If $\|\overrightarrow{OP}\| = 1$, $\|\overrightarrow{OQ}\| = 1$ and θ is the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then

$$\overrightarrow{OP} \circ \overrightarrow{OQ} = \cos\theta$$

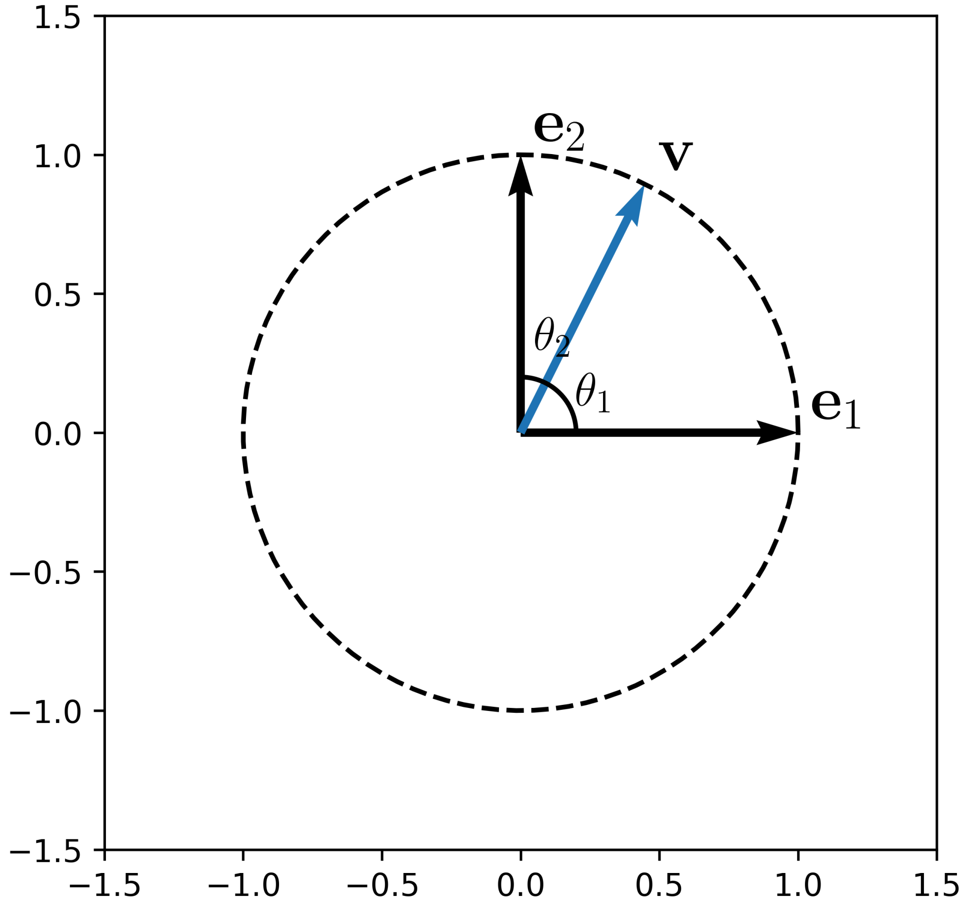
Direction of Vector

$$\mathbf{e}_1 = \langle 1, 0 \rangle, \quad \mathbf{e}_2 = \langle 0, 1 \rangle$$

$$\mathbf{v} \circ \mathbf{e}_1 = \frac{1}{\sqrt{5}} = \cos(\theta_1)$$

$$\mathbf{v} \circ \mathbf{e}_2 = \frac{2}{\sqrt{5}} = \cos(\theta_2)$$

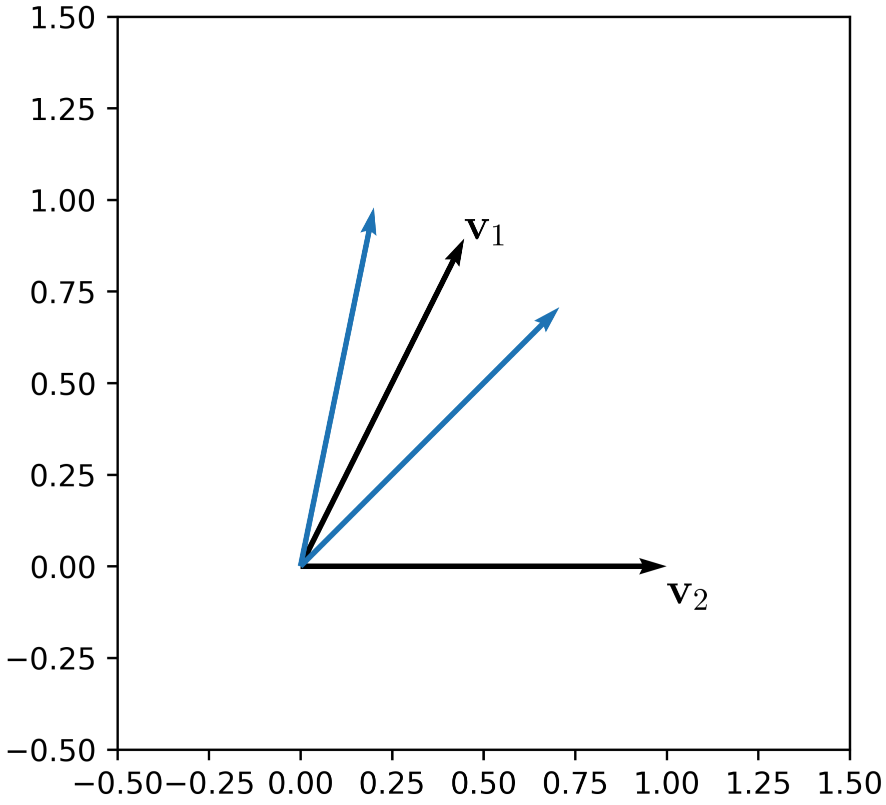
$$\theta_1 \approx 68.2^\circ, \quad \theta_2 \approx 21.8^\circ$$

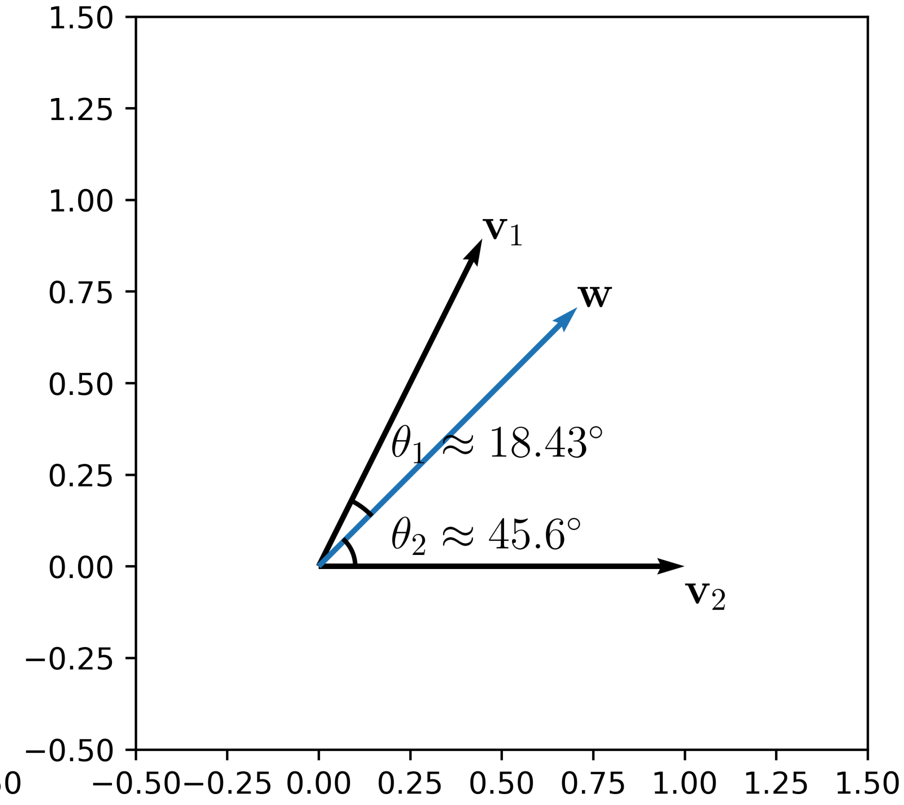


Example

Determin the direction of the vector $\mathbf{w}$ satisfying

$$\mathbf{v}_1 = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle, \quad \mathbf{v}_2 = \langle 1, 0 \rangle, \quad \mathbf{v}_1 \circ \mathbf{w} = \frac{3}{\sqrt{10}}, \quad \mathbf{v}_2 \circ \mathbf{w} = \frac{1}{\sqrt{2}}$$



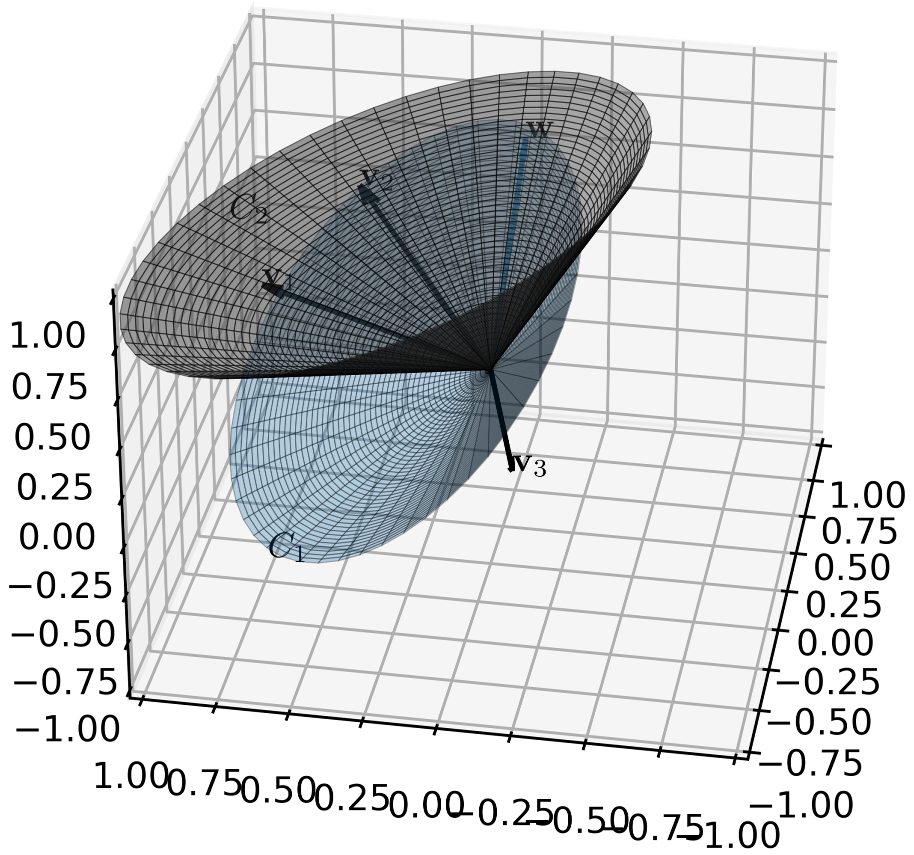


Example

Determine the direction of the vector $\mathbf{w}$ satisfying

$$\mathbf{v}_1 = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \rangle, \quad \mathbf{v}_2 = \langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle, \quad \mathbf{v}_3 = \langle \frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}} \rangle$$

$$\mathbf{v}_1 \circ \mathbf{w} = \frac{1}{\sqrt{10}}, \quad \mathbf{v}_2 \circ \mathbf{w} = \frac{2}{\sqrt{10}}, \quad \mathbf{v}_3 \circ \mathbf{w} = -\frac{3}{\sqrt{10}}$$



Orthogonality

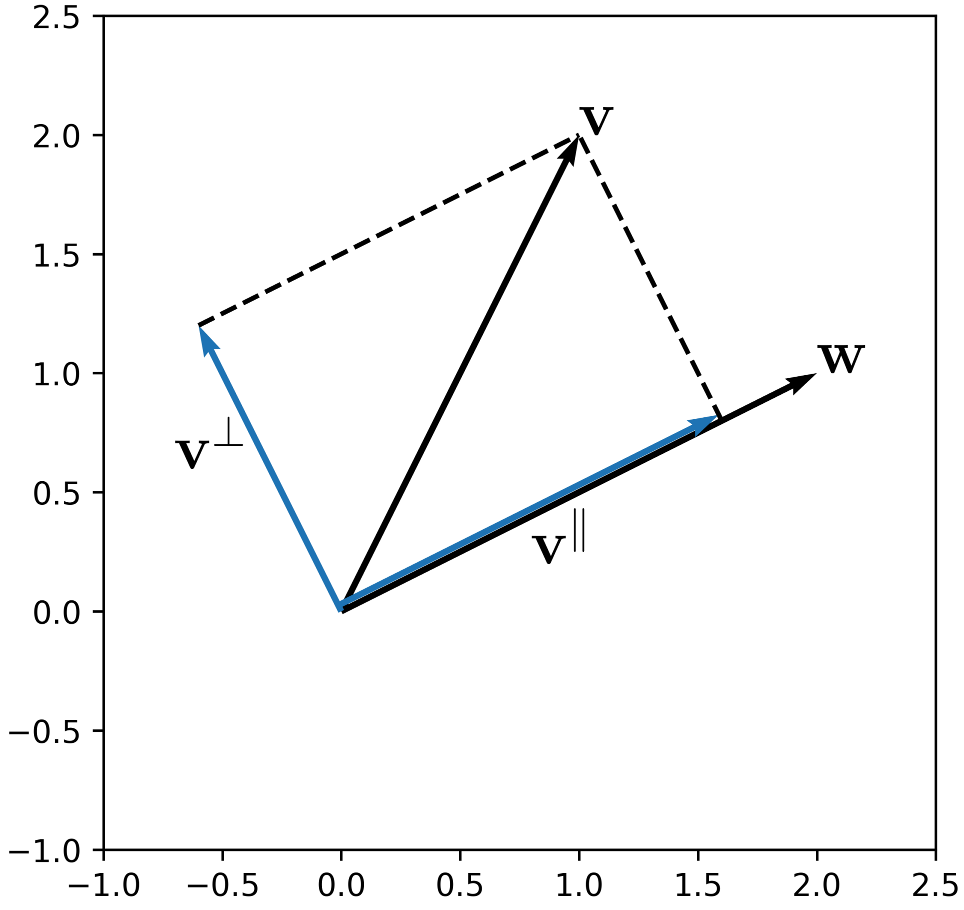
$$\mathbf{v} \circ \mathbf{w} = 0 \iff \mathbf{v} \perp \mathbf{w}$$

Parallelism

$$\mathbf{v} = k\mathbf{w} \iff \mathbf{v} \parallel \mathbf{w}$$

Orthogonal Decomposition

$$\mathbf{v} = \mathbf{v}^{\perp} + \mathbf{v}^{\parallel}$$

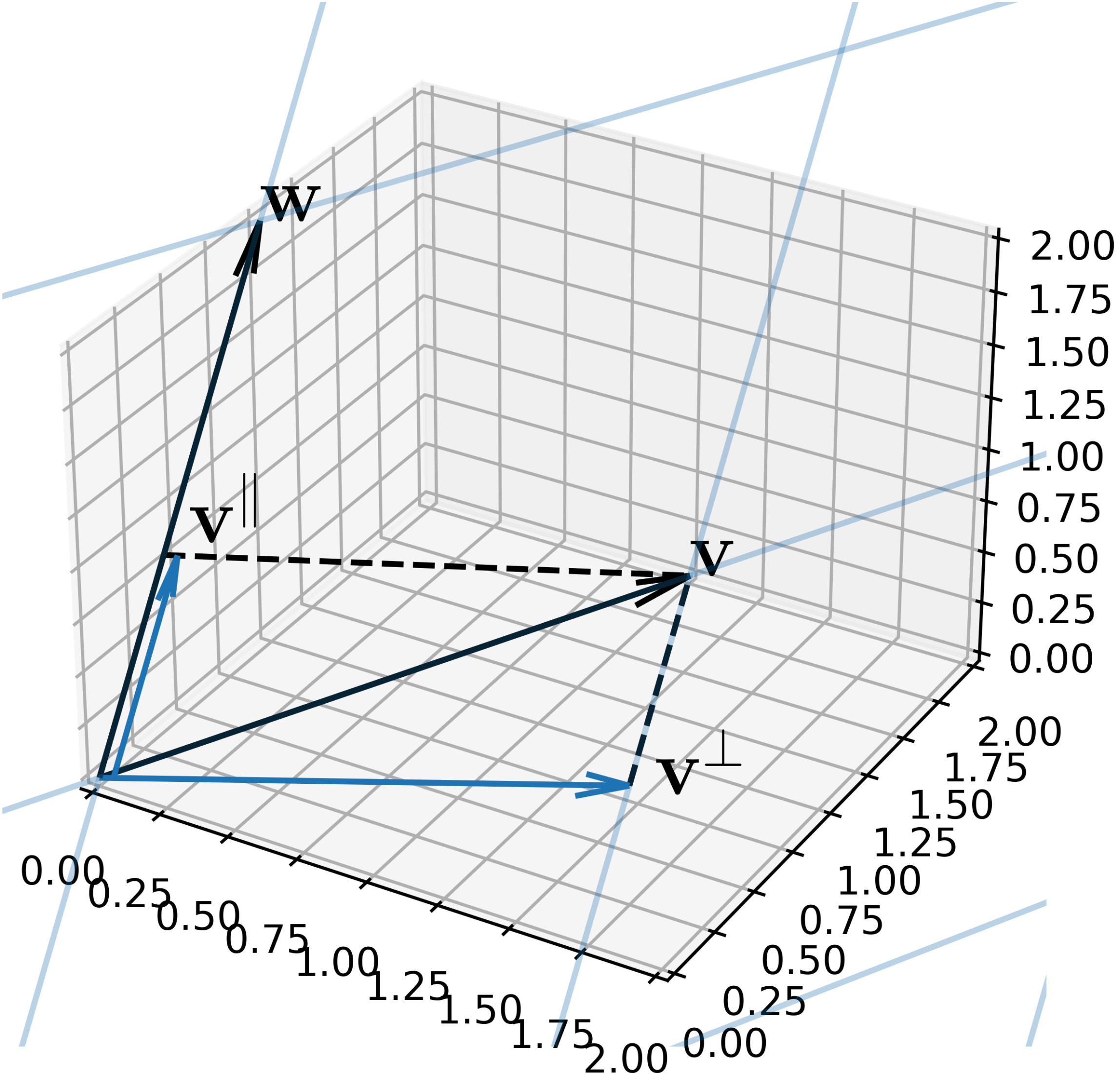


Example: Orthogonal Decomposition

Find the linear combination $\mathbf{v} = \mathbf{v}^{\perp} + \mathbf{v}^{\parallel}$ for

$$\mathbf{v} = \langle 1, 2, 0 \rangle, \quad \mathbf{w} = \langle 0, 1, 2 \rangle$$

$$\begin{aligned} \mathbf{v} &= \mathbf{v}^{\perp} + \mathbf{v}^{\parallel} \\ &= \langle 1, \frac{8}{5}, -\frac{4}{5} \rangle + \langle 0, \frac{2}{5}, \frac{4}{5} \rangle \end{aligned}$$



Projection

$$\mathrm{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \circ \mathbf{w}}{\mathbf{w} \circ \mathbf{w}}\mathbf{w}$$

Example

Find the $\mathrm{proj}_{\mathbf{w}}\mathbf{v}$ for $\mathbf{v} = \langle 1, 2 \rangle$, $\mathbf{w} = \langle 2, 1 \rangle$

$$\mathrm{proj}_{\mathbf{w}}\mathbf{v} = \langle \frac{8}{5}, \frac{4}{5} \rangle$$