The Connectivity of Erdős–Rényi Random Graph

Lemma 1: Assume any edge of G is positive, then: Any cut of G is positive $\leftrightarrow$ G is connected

Set the weight of every edge of G as 1. Let C_1, C_2, ⋯, $C_{2^{N-1}-1}$ denote the $(2^{N-1}-1)$ cuts of G. Define:

$$I(C) = \begin{cases} 1, \text{if } C \text{ is positive} \\ 0, \text{if } C \text{ is zero} \end{cases}$$

According to Lemma1, the Necessary and Sufficient Condition for G to be connected is:

$$\lim_{N\to+\infty} E\left[\sum_{k=1}^{2^{N-1}-1} I(C_k)\right] = 2^{N-1}-1$$

In order words:

$$\lim_{N\to+\infty} \sum_{k=1}^{\left\lfloor\frac{N}{2}\right\rfloor} \binom{N}{k}(1-p)^{k(N-k)} = 0$$

Let $p = \frac{(1+\varepsilon)\ln N}{N}$, $a_k = \binom{N}{k}(1-p)^{k(N-k)}$, then $a_1 = N(1-p)^{N-1}$. To make sure the series converge:

$$\lim_{N\to+\infty} a_1 = \lim_{N\to+\infty} N\left(1-\frac{(1+\varepsilon)\ln N}{N}\right)^{\frac{N}{(1+\varepsilon)\ln N}\times(1+\varepsilon)\ln N\times\frac{N-1}{N}} = \lim_{N\to+\infty} O(N^{-\varepsilon}) = 0$$

Thus, $\varepsilon > 0$ is the necessary condition for G to be connected. Thus, we show that $\varepsilon > 0$ is also a sufficient condition.

Lemma 2: When $0 < \varepsilon < 1$: $$a_k = O(N^{-k\varepsilon})$$ Proof: It holds when $k = o(N)$. Here we consider the case of $k = cN$ $\left(0 < c \le \frac{1}{2}\right)$. $$a_k \le \left(\frac{eN}{k}\right)^k \left(1-\frac{(1+\varepsilon)\ln N}{N}\right)^{\frac{N}{(1+\varepsilon)\ln N}\times(1+\varepsilon)\ln N\times k\left(1-\frac{k}{N}\right)} \le \frac{e^k}{k^k} N^{(1+\varepsilon)\times\frac{k^2}{N}-\varepsilon k}$$ $$= \left(\frac{e}{c}N^{(1+\varepsilon)c-1}\right)^k N^{-k\varepsilon}$$ Because $\varepsilon < 1$ and $c \le \frac{1}{2}$, so $(1+\varepsilon)c - 1 < 0$ and $a_k = O(N^{-k\varepsilon})$.

According to Lemma 2, when $0 < \varepsilon < 1$:

$$\lim_{N\to+\infty} \sum_{k=1}^{\left\lfloor\frac{N}{2}\right\rfloor} a_k = \lim_{N\to+\infty} \sum_{k=1}^{\left\lfloor\frac{N}{2}\right\rfloor} O(N^{-k\varepsilon}) = O\left(N^{-\varepsilon}\times\frac{1-N^{-\left\lfloor\frac{N}{2}\right\rfloor\varepsilon}}{1-N^{-\varepsilon}}\right) = O(N^{-\varepsilon}) = 0$$

In order words, if $\frac{\ln N}{N} < p < \frac{2\ln N}{N}$, then G is connected. Thus, $p > \frac{\ln N}{N}$ is also a sufficient condition for G to be connected.