Midterm Exam of MATH2352, Spring 2015

(Please return the problem sheet with your exam booklet)

Problems (The numbers in brackets are credits, totalled to 60.):

1. (8) Identify the domain for the following ODE over which there exists a unique solution:

$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}.$$

2. (8) A pond initially contains 100 million liters of water, already contaminated with 10 million grams of toxic waste. Water containing toxic waste flows into the pond at rate of 25 million liters/year, and exits the pond at same rate. Let the concentration of toxic waste be $c(t) = 2(1 + \cos 2t)$g/liter in incoming water. Assume toxic waste is neither created or destroyed in pond, and distribution of toxic waste in pond is uniform, write down the initial value problem for the amount of toxic waste in the pond.

3. (8) Convert the following ODE,

$$y + (2x - ye^y)y' = 0,$$

into an exact equation and then solve it.

4. Consider solving the **Euler equation**,

$$\alpha x^2 \frac{d^2y}{dx^2} + \beta x \frac{dy}{dx} + \delta y = 0, \quad x > 0. \tag{1}$$

   (a) (4) Find a proper change of variable, $u = u(x)$, such that the equation is converted into a 2nd-order linear homogeneous ODE with constant coefficients:

$$a\frac{d^2y}{du^2} + b\frac{dy}{du} + cy = 0. \tag{2}$$

   Express a, b and c in terms of α, β and δ.

   (b) Provide the general solution to Equation (2) in terms of **real fundamental solutions**, with proofs not required, for the cases of

      i. (4) $b^2 - 4ac > 0$;
      ii. (4) $b^2 - 4ac = 0$; and
      iii. (4) $b^2 - 4ac < 0$.

   (c) (4) Write down the corresponding solutions to Equation (1).

5. (8) Use *the method of variation of parameters* to solve the following equation

$$ty'' - (t+1)y' + y = te^{2t}, \quad t > 0; \quad y_1(t) = e^t; \quad y_2(t) = t + 1.$$ homo

6. (8) Explain how to obtain the general solution of the nth-order linear homogeneous ODE with constant coefficients,

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0, \quad a_n \neq 0.$$

7. (**Extra credits**) Let $y_1(t)$ and $y_2(t)$ be two solutions to the 2nd-order linear homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0,$$

with initial conditions

$$y_1(t_0) = 1, \quad y_1'(t_0) = 0,$$
$$y_2(t_0) = 0, \quad y_2'(t_0) = 1,$$

(a) (4) Define the Wrongskian of the two solutions.

(b) (8) Show that the Wronskian is given by

$$W(t) = \exp\left[\int_{t_0}^{t} p(s)ds\right].$$

====================== Good Luck! ======================