MATH 2352 Solution Sheet 09

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[Problems] 7.1: 5, 6, 8; 7.4: 4, 9; 7.5: 15, 17, 19; 7.6: 7, 15(a, b);

7.1 - 5. Transform the given eqaution into a system of first order equations.

$$u'' + 2u' + 4u \;=\; 2\cos 3t,$$

$$u(0) = 1, \qquad u'(0) = -2.$$

Solution. Let

$$v(t) \;=\; u'(t),$$

we have

$$\begin{aligned} u'(t) &= v(t), \\ v'(t) &= -2v(t) - 4u(t) + 2\cos 3t. \end{aligned}$$

Or equivallently,

$$\begin{pmatrix} u \\ v \end{pmatrix}' \;=\; \begin{pmatrix} 0 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 2\cos 3t \end{pmatrix}.$$

7.1 - 6. Transform the given eqaution into a system of first order equations.

$$u'' + p(t)\, u' + q(t)\, u \;=\; g(t),$$

$$u(0) = u_0, \qquad u'(0) = u_0'.$$

Solution. Let $y = [u, u']^T$, then

$$y' \;=\; -A(t)\, y + B(t),$$

where

$$A(t) \;=\; \begin{bmatrix} 0 & -1 \\ p(t) & q(t) \end{bmatrix},$$

$$B(t) \;=\; \begin{bmatrix} 0 \\ g(t) \end{bmatrix}.$$

7.1 - 8. For

$$\begin{aligned} x_1' &= 3x_1 - 2x_2, \\ x_2' &= 2x_1 - 2x_2. \end{aligned}$$

$$\begin{aligned} x_1(0) &= 3, \\ x_2(0) &= 1. \end{aligned}$$

(a) Transform the given system into a single equaion of second order.

(b) Find x_1 and x_2 that also satisfy the given initial conditions.

(c) Sketch the graph of the solution in the x_1x_2-plane for $t \geqslant 0$.

Solution.

(a) From $x_1' = 3x_1 - 2x_2$ we have

$$x_1'' = 3x_1' - 2x_2',$$

plugging in the other equation,

$$\begin{aligned} x_1'' &= 3x_1' - 2(2x_1 - 2x_2) \\ &= 3x_1' - 4x_1 + 4x_2. \end{aligned}$$

Then x_2 can be eliminated using the first equation, yielding

$$\begin{aligned} x_1'' + 2x_1' &= 3x_1' + 2x_1, \\ &\text{or} \\ x_1'' &= x_1' + 2x_1. \end{aligned}$$

For initial condition, we have

$$\begin{aligned} x_1(0) &= 3, \\ x_1'(0) &= (3x_1 - 2x_2)(0) = 7. \end{aligned}$$

7.4 - 4. If $x_1 = y$ and $x_2 = y'$, then the second order equation

$$y'' + p(t)y' + q(t)y = 0 \tag{1}$$

corresponds to the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -p(t)x_2 - q(t)x_1. \end{aligned} \tag{2}$$

Show that if $\boldsymbol{x}^{(1)}$ and $x^{(2)}$ are fundamental set of solutions of Eqs.(2), and if $y^{(1)}$ and $y^{(2)}$ are a fundamental set of solutions of Eq.(1), then $W[y^{(1)}, y^{(2)}] = c\,W[\boldsymbol{x}^{(1)}, x^{(2)}]$, where c is a nonzero constant.

Hint: $y^{(1)}(t)$ and $y^{(2)}(t)$ must be linear combinations of $x_{11}(t)$ and $x_{12}(t)$.

Solution. By definition,

$$W[y^{(1)}, y^{(2)}] = \det\begin{pmatrix} y^{(1)}(t) & y^{(2)}(t) \\ y^{(1)\prime}(t) & y^{(2)\prime}(t) \end{pmatrix},$$

via Abel's identity,

$$W[y^{(1)}, y^{(2)}] = W_0\, e^{-\int p(t)\,\mathrm{d}t},$$

where W_0 is a constant of integration.

On the other hand,

$$W[x^{(1)}, x^{(2)}] = \det\begin{pmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) \end{pmatrix},$$

via (generalized) Abel's theorem,

$$\begin{aligned} W\big[x^{(1)}, x^{(2)}\big] &= Z_0\, e^{\int \text{trace}(A(t))\,\text{dt}} \\ &= Z_0\, e^{-\int p(t)\,\text{dt}}, \end{aligned}$$

where Z_0 is a constant of integration.

therefore, $\exists c = \frac{W_0}{Z_0}$, such that

$$W\big[y^{(1)}, y^{(2)}\big] = c\, W\big[\boldsymbol{x}^{(1)}, x^{(2)}\big].$$

7.4 - 9. Let $x^{(1)}, ..., x^{(n)}$ be linearly independent solution of $x' = P(t)\, x$, where P is continuous on $\alpha < t < \beta$.

(a) Show that any solution $x = z(t)$ can be written in the form

$$z(t) = c_1 x^{(1)}(t) + ... + c_n x^{(n)}(t)$$

for suitable constants $c_1, ..., c_n$.

(b) Show that the expression for the solution $z(t)$ in part (a) is unique.

Solution.

(a) First of all, it is obvious that the solution space is a vector space. And because P is continuous on $\alpha < t < \beta$, we have existence and uniqueness for the solution to the initial value problem.

Given $x^{(1)}, ..., x^{(n)}$ linearly independent solutions, there must be a t_0 where the Wronskian of them is nonzero. At t_0, $z(t_0)$ can be written in the form

$$z(t_0) = c_1 x^{(1)}(t_0) + ... + c_n x^{(n)}(t_0),$$

then by existence and uniqueness for the solution,

$$z(t) = c_1 x^{(1)}(t) + ... + c_n x^{(n)}(t).$$

(b) This is already shown above, given that the decomposition

$$z(t_0) = c_1 x^{(1)}(t_0) + ... + c_n x^{(n)}(t_0)$$

is unique (linear algebra).

7.5 - 15. For

$$\begin{aligned} x' &= \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} x, \\ x(0) &= (3, -1)^T. \end{aligned}$$

Solve the initial value problem. Describe the behavior of the solution as $t \to \infty$.

Solution. The general solution is

$$x(t) = \begin{bmatrix} c_1\, e^{2t}\, (3\, e^{2t} - 1) - c_2\, e^{2t}\, (e^{2t} - 1) \\ 3\, c_1\, e^{2t}\, (e^{2t} - 1) - c_2\, e^{2t}\, (e^{2t} - 3) \end{bmatrix}.$$

By initial condition,

$$
\begin{aligned}
2c_1 &= 3,\\
2c_2 &= -1.
\end{aligned}
$$

Therefore, the solution to the I.V.P. is

$$
\begin{aligned}
x(t) &= \begin{bmatrix} \frac{3}{2}e^{2t}(3e^{2t}-1)+\frac{1}{2}e^{2t}(e^{2t}-1) \\ \\ \frac{9}{2}e^{2t}(e^{2t}-1)+\frac{1}{2}e^{2t}(e^{2t}-3) \end{bmatrix}\\
&= \begin{bmatrix} 5e^{4t}-2e^{2t} \\ \\ 5e^{4t}-6e^{2t} \end{bmatrix}.
\end{aligned}
$$

When $t\to\infty$, $x(t)\to\infty$, $\tan(\arg(x(t)))\to 1$.

7.5 - 17. For

$$
\begin{aligned}
x' &= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} x,\\
x(0) &= (2,0,3)^T.
\end{aligned}
$$

Solve the initial value problem. Describe the behavior of the solution as $t\to\infty$.

Solution. The general solution is

$$
x(t) = c_1\begin{bmatrix} e^{2t}(e^t-2) \\ \\ e^t(e^t-1)^2 \\ \\ \frac{1}{2}e^t(e^{2t}-1) \end{bmatrix} + c_2\begin{bmatrix} e^{2t}(e^t-1) \\ \\ e^t[e^t(e^t-1)+1] \\ \\ \frac{1}{2}e^t(e^{2t}-1) \end{bmatrix} + c_3\begin{bmatrix} 2e^{2t}(e^t-1) \\ \\ 2e^{2t}(e^t-1) \\ \\ e^{3t} \end{bmatrix}.
$$

By initial condition,

$$
\begin{aligned}
2 &= -c_1\\
0 &= c_2\\
3 &= c_3.
\end{aligned}
$$

Therefore, the solution to the I.V.P. is

$$
\begin{aligned}
x(t) &= \begin{bmatrix} -2e^{2t}(e^t-2)+6e^{2t}(e^t-1) \\ \\ -2e^t(e^t-1)^2+6e^{2t}(e^t-1) \\ \\ -e^t(e^{2t}-1)+3e^{3t} \end{bmatrix}\\
&= \begin{bmatrix} 4e^{3t}-2e^{2t} \\ \\ 4e^{3t}-2e^{2t}-2e^t \\ \\ 2e^{3t}+e^t \end{bmatrix}.
\end{aligned}
$$

When $t\to\infty$, $x(t)\to\infty$.

7.5 - 19. The system $tx' = A\,x$ is analogous to the second order Euler equation. Assuming that $x = \xi\, t^r$, where ξ is a constant vector, show that ξ and r must satisfy $(A - rI)\xi = 0$ in order to obtain nontrivial solutions of the given differential equation.

Solution. By assumption $x = \xi\, t^r$,

$$x' = \xi r t^{r-1},$$

thus

$$t\xi r t^{r-1} = A\xi t^r,$$

which is quivalent to $(t \neq 0)$

$$(A - rI)\xi = 0.$$

7.6 - 7. For

$$x' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} x.$$

Express the general solution of the given system of equations in terms of real-valued functions.

Solution. The general solution is

$$x(t) = c_1 \begin{bmatrix} 2e^t \\ e^t\,[2\sin(2t) + 3\cos(2t) - 3] \\ e^t\,[-3\sin(2t) + 2\cos(2t) - 2] \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^t\cos(2t) \\ e^t\sin(2t) \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -e^t\sin(2t) \\ e^t\cos(2t) \end{bmatrix}.$$

7.6 - 15. For

$$x' = \begin{pmatrix} 2 & \alpha \\ -5 & -2 \end{pmatrix} x,$$

the coefficient matrix contains a parameter α.

(a) Determine the eigenvalues in terms of α.

(b) Find the critical value or values of α where the qualitative nature of the phase portrait for the system changes.

Solution.

(a) The eigenvalues of coefficient matrix are $\lambda = \pm\sqrt{4 - 5\alpha}$.

(b) The bifurcation points are:

1. Real/Imaginary eigenvalues: $\alpha = \frac{4}{5}$