Homework #04 Answers and Hints
(MATH4052 Partial Differential Equations)

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Problem 1. *(Page 52, Q3). Use (8) to solve the diffusion equation if* $\phi(x) = e^{3x}$.

Solution. Using equation (8),

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y) dy && (1)\\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} e^{3y} dy && (2)\\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-\frac{(y-x)^2-12kty}{4kt}} dy && (3)\\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-\frac{(y-x-6kt)^2-12ktx-36k^2t^2}{4kt}} dy && (4)\\
&= \frac{1}{\sqrt{4\pi kt}} e^{3x+9kt} \int_{-\infty}^{+\infty} e^{-\frac{(y-x-6kt)^2}{4kt}} dy && (5)\\
&= \frac{1}{\sqrt{\pi}} e^{3x+9kt} \int_{-\infty}^{+\infty} e^{-\left(\frac{y-x-6kt}{\sqrt{4kt}}\right)^2} d\left(\frac{y-x-6kt}{\sqrt{4kt}}\right) && (6)\\
&= \frac{1}{\sqrt{\pi}} e^{3x+9kt} \int_{-\infty}^{+\infty} e^{-s^2} ds. && (7)
\end{aligned}
$$

To compute the integral above, consider

$$
\begin{aligned}
\left(\int_{-\infty}^{+\infty} e^{-s^2} ds\right)^2 &= \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy && (8)\\
&= \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy. && (9)
\end{aligned}
$$

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Transform to polar coordinate, letting $x = r\cos\theta$, $y = r\sin\theta$, namely $x^2 + y^2 = r^2$, and $dxdy = (\cos\theta dr - r\sin\theta d\theta)(\sin\theta dr + r\cos\theta d\theta) = rdrd\theta$,

$$\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy = \int_{-\pi}^{\pi}\int_{0}^{+\infty} e^{-r^2} rdrd\theta \tag{10}$$

$$= \int_{-\pi}^{\pi} \frac{1}{2} d\theta \tag{11}$$

$$= \pi. \tag{12}$$

Substitute into the solution, and we have

$$u(x,t) = \frac{1}{\sqrt{\pi}} e^{3x+9kt}\sqrt{\pi} \tag{13}$$

$$= e^{3x+9kt}. \tag{14}$$

□

Problem 2. *(Page 52, Q4). Solve the diffusion equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.*

Solution. Using the solution formula, we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y) dy \tag{15}$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} e^{-y} dy \tag{16}$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{+\infty} e^{-\frac{(y-x)^2+4kty}{4kt}} dy \tag{17}$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{+\infty} e^{-\frac{(y-x+2kt)^2+4ktx-4k^2t^2}{4kt}} dy \tag{18}$$

$$= \frac{1}{\sqrt{4\pi kt}} e^{-x+kt} \int_{0}^{+\infty} e^{-\left(\frac{y-x+2kt}{\sqrt{4kt}}\right)^2} dy \tag{19}$$

$$= \frac{1}{\sqrt{\pi}} e^{-x+kt} \int_{\frac{-x+2kt}{\sqrt{4kt}}}^{+\infty} e^{-s^2} ds. \tag{20}$$

$$\tag{21}$$

The integral above is not analytic, so we use a special function to represent it. Define the **Gauss error function** as

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-s^2} ds \tag{22}$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-s^2} ds. \tag{23}$$

And the **complementary error function** as

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) \tag{24}$$

$$= \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-s^2} ds. \tag{25}$$

Then

$$\int_{\frac{-x+2kt}{\sqrt{4kt}}}^{+\infty} e^{-s^2} ds = \frac{\sqrt{\pi}}{2} \operatorname{erfc}\left(\frac{-x+2kt}{\sqrt{4kt}}\right). \tag{26}$$

Therefore,

$$u(x,t) = \frac{1}{2} e^{-x+kt} \operatorname{erfc}\left(\frac{-x+2kt}{\sqrt{4kt}}\right). \tag{27}$$

$\square$

Problem 3. *(Page 52, Q5). Prove properties (a) to (e) of the diffusion equation (1).*

Solution. Consider the diffusion equation

$$u_t = k u_{xx}, \quad (-\infty < x < +\infty, 0 < t < +\infty). \tag{28}$$

If $u(x,t)$ is a solution, then

1. For any fixed y, let $v(x,y) = u(x-y,t)$ to be a translate, then

   $$v_t - k v_{xx} = u_t - k u_{xx} = 0. \tag{29}$$

   This proves (a).

2. The derivative $w = \mathcal{L} u$, $\mathcal{L} = \partial_x, \partial_t, \partial_{xx}$, then

   $$w_t - k w_{xx} = \mathcal{L}\left(u_t - k u_{xx}\right) = 0. \tag{30}$$

   This proves (b).

3. A linear combination of solutions $p = \sum_{i=1}^N u_i$, where u_i solves the diffusion equation, satisfies

   $$p_t - k p_{xx} = \sum_{i=1}^N \left(u_t - k u_{xx}\right) = 0. \tag{31}$$

   This proves (c).

4. An integral of solutions $q(x,t) = \int_a^b u(s,t)ds$ satisfy

$$q_t - kq_{xx} = \int_a^b (u_t - ku_{xx})\, ds = 0. \tag{32}$$

So it is with a weighted integral $r(x,t) = \int_a^b u(s,t)w(s)ds$

$$r_t - kr_{xx} = \int_a^b (u_t - ku_{xx})\, w(s)ds = 0, \tag{33}$$

where $w(x)$ is an arbitrary function.

Combining with the translate property, we have the convolution property. Let $f(x,t) = \int_a^b u(x-y,t)g(y)dy$, then

$$f_t - kf_{xx} = \int_a^b (u_t - ku_{xx})\, g(y)dy = 0. \tag{34}$$

This proves (d).

5. The dilated function $h(x,t) = u(\sqrt{a}x, at)$ satisfies

$$h_t - kh_{xx} = au_t - k(\sqrt{a})^2 u_{xx} \tag{35}$$
$$= a(u_t - ku_{xx}) \tag{36}$$
$$= 0. \tag{37}$$

This proves (e).

$\square$