- 2. Prove the uniqueness up to constants of the Neumann problem using the energy method.
- 3. Prove the uniqueness of the Robin problem $\partial u/\partial n + a(\mathbf{x})u(\mathbf{x}) = h(\mathbf{x})$ provided that $a(\mathbf{x}) > 0$ on the boundary.
- 4. Generalize the energy method to prove uniqueness for the diffusion equation with Dirichlet boundary conditions in three dimensions.
- 5. Prove Dirichlet's principle for the Neumann boundary condition. It asserts that among *all* real-valued functions $w(\mathbf{x})$ on D the quantity

$$E[w] = \frac{1}{2} \iiint_{D} |\nabla w|^{2} d\mathbf{x} - \iint_{\text{bdy } D} hw \, dS$$

is the smallest for w = u, where u is the solution of the Neumann problem

$$-\Delta u = 0$$
 in D , $\frac{\partial u}{\partial n} = h(\mathbf{x})$ on bdy D .

It is required to assume that the average of the given function $h(\mathbf{x})$ is zero (by Exercise 6.1.11).

Notice three features of this principle:

- (i) There is no constraint at all on the trial functions $w(\mathbf{x})$.
- (ii) The function $h(\mathbf{x})$ appears in the energy.
- (iii) The functional E[w] does not change if a constant is added to $w(\mathbf{x})$. (*Hint*: Follow the method in Section 7.1.)
- 6. Let A and B be two disjoint bounded spatial domains, and let D be their exterior. So bdy $D = \text{bdy } A \cup \text{bdy } B$. Consider a harmonic function $u(\mathbf{x})$ in D that tends to zero at infinity, which is *constant* on bdy A and *constant* on bdy B, and which satisfies

$$\iint_{\text{bdy }A} \frac{\partial u}{\partial n} \, dS = Q > 0 \qquad \text{and} \qquad \iint_{\text{bdy }B} \frac{\partial u}{\partial n} \, dS = 0.$$

[Interpretation: The harmonic function $u(\mathbf{x})$ is the electrostatic potential of two conductors, A and B; Q is the charge on A, while B is uncharged.]

- (a) Show that the solution is unique. (*Hint:* Use the Hopf maximum principle.)
- (b) Show that $u \ge 0$ in *D*. [*Hint*: If not, then $u(\mathbf{x})$ has a negative minimum. Use the Hopf principle again.]
- (c) Show that u > 0 in D.
- 7. (Rayleigh-Ritz approximation to the harmonic function u in D with u = h on bdy D.) Let w_0, w_1, \ldots, w_n be arbitrary functions such that $w_0 = h$ on bdy D and $w_1 = \cdots = w_n = 0$ on bdy D. The problem is to find