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Midterm Exam of MATH2352, Spring 2015

(Please return the problem sheet with your exam booklet)

Problems (The numbers in brackets are credits, totalled to 60.):

(8) 1. f and f_y exist and are continuous everywhere except on

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$$ty = 0 \quad \text{and} \quad 1 + y^2 = t^2.$$

The first equation corresponds to the two axes, and the second equation represent two curves in the (t, y) plane. At all regions bounded by any two of these curves there exists a unique solution.

(8) 2. Let $Q(t)$ denote the amount of toxic waste in grams. Then,

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$$\begin{aligned}\frac{dQ(t)}{dt} &= \text{rate in} - \text{rate out} \\ &= 25 \times 10^6 \times c(t) - 25 \times 10^6 \times \frac{Q(t)}{100},\end{aligned}$$

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with initial conditions $Q(0) = 10 \times 10^6$.

(8) 3. Multiply the equation by $\mu = y$, we obtain

$$y^2 + (2xy - y^2 e^y)y' = 0.$$

Let $M = y^2, N = 2xy - y^2 e^y$. Apparently $M_y = 2y = N_x$, hence the new equation is **exact**. Let $\Psi(x, y)$ be the function such that $\Psi_x = M, \Psi_y = N$, then

$$\Psi(x, y) = xy^2 + C(y).$$

Differentiate Ψ w.r.t y we obtain

$$\Psi_y = 2xy + C_y(y) = N = 2xy - y^2 e^y,$$

yielding

$$C_y(y) = -y^2 e^y.$$

It follows that

$$\begin{aligned}C(y) &= -\int y^2 e^y dy + D \\ &= -y^2 e^y + 2\int y e^y dy + D \\ &= -y^2 e^y + 2ye^y - 2\int e^y dy + D \\ &= -y^2 e^y + 2ye^y - 2e^y + D\end{aligned}$$

The solution to the ODE is thus

$$xy^2 - (y^2 - 2y + 2)e^y + D = 0.$$

(20) 4. (a) Take $u = \ln x$, then, by the chain rule,

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$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{x}\frac{dy}{du},$$
$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\frac{dy}{du} + \frac{1}{x^2}\frac{d^2y}{du^2}.$$

Plug in into the equation we obtain

$$\alpha\frac{d^2y}{du^2} + (\beta - \alpha)\frac{dy}{du} + \delta y = 0. \tag{1}$$

Let $a = \alpha, b = \beta - \alpha$ and $c = \delta$.

(b) The characteristic equation is $ar^2 + br + c = 0$. Its solutions are

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$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expressions for fundamental solutions vary according to three cases.

i. (4) For $b^2 - 4ac > 0$,

$$y_1(u) = e^{r_1 u}, \quad y_2(u) = e^{r_2 u}.$$

ii. (4) For $b^2 - 4ac = 0$,

$$y_1(u) = e^{r_1 u}, \quad y_2(u) = ue^{r_1 u}.$$

iii. (4) For $b^2 - 4ac < 0$, let $\lambda = -b/(2a)$ and $\mu = \sqrt{|b^2 - 4ac|/(2a)}$. Then

$$y_1(u) = e^{\lambda u}\cos(\mu u), \quad y_2(u) = e^{\lambda u}\sin(\mu u).$$

(c) In term of $x = e^u$ the fundamental solutions become, according to the cases

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$$y_1(x) = x^{r_1}, \quad y_2(x) = x^{r_2}.$$

$$y_1(x) = x^{r_1}, \quad y_2(x) = x^{r_1}\ln x.$$

and

$$y_1(x) = x^{\lambda}\cos(\mu\ln x), \quad y_2(x) = x^{\lambda}\sin(\mu\ln x).$$

(8) 5. The solutions can be derived directly, or can be obtained by quoting the result of

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

with

$$u_1(t) = -\int\frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt$$
$$u_2(t) = +\int\frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt$$

and

$$W(y_1, y_2)(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = \det \begin{pmatrix} e^t & (1+t) \\ e^t & 1 \end{pmatrix} = -te^t$$

It follows that

$$\begin{aligned} u_1(t) &= \int \frac{(1+t)e^{2t}}{te^t} dt \\ &= \int \frac{(1+t)e^t}{t} dt, \\ u_2(t) &= -\int \frac{e^{2t}}{t} dt. \end{aligned}$$

5 for $g(t) = t \cdot e^{2t}$

The general solutions to the ODE are

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t).$$

(8) 6. The corresponding characteristic equation is

3 for e^{rt}

$$\begin{aligned} &a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0 \\ =&a_n (r - r_1)(r - r_2) \ldots (r - r_n), \end{aligned}$$

5.

Let r_k be the k-th root with multiplicity m_k, then the corresponding fundamental solution are

$$e^{r_k t}, te^{r_k t}, \ldots, t^{m_k - 1} e^{r_k t}.$$

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If $(r_k, \bar{r}_k) = \lambda_k \pm i\mu_k$ are a pair of complex roots with multiplicity m_k, then the corresponding fundamental solutions are

$$e^{\lambda_k t} \cos(\mu_k t), e^{\lambda_k t} \sin(\mu_k t), te^{\lambda_k t} \cos(\mu_k t), te^{\lambda_k t} \sin(\mu_k t), \ldots,$$
$$t^{m_k - 1} e^{\lambda_k t} \cos(\mu_k t), t^{m_k - 1} e^{\lambda_k t} \sin(\mu_k t).$$

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(12) 7. (a) The Wrongskian is defined by 1 if $W(t_0)$

(4)

$$W(y_1, y_2)(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$

(b) Plug in $y_1(t)$ and $y_2(t)$ into the ODE we obtain

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(8)

$$\begin{aligned} y_1'' + p(t) y_1' + q(t) y_1 &= 0, \\ y_2'' + p(t) y_2' + q(t) y_2 &= 0, \end{aligned}$$

Multiply the 1st eqtn by y_2 and the 2nd eqtn by y_1, and subtract eqtn 1 from 2, we obtain

$$(y_1 y_2'' - y_2 y_1'') + p(t)(y_1 y_2' - y_2 y_1') = 0,$$

or, one can verify

$$W' + p(t)W = 0,$$

with initial condition

$$W(t_0) = 1.$$

Solve the equation we obtain

$$W(t) = \exp\left[-\int_{t_0}^{t} p(s)ds\right].$$

====================== The End! ======================