MATH 2111 Matrix Algebra and Applications   Week 6 Tutorial

1. Let $A = \begin{bmatrix} x+y & x-y & 4 \\ 3 & 0 & x+z \\ 2x-y+z & 1 & x+y \end{bmatrix}$. Find suitable x, y, z such that: (i) A is symmetric; (ii) A is skew-symmetric.

2. Let A be an $m \times n$ matrix. Do we always have (i) A^TA (ii) AA^T being symmetric?

3. Let A be a 2×2 orthogonal matrix. Show that there exists a suitable θ such that:

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

4. Let $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$. Show that $C = \begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}$ is the inverse of B. Find a matrix D such that $DB = A$.

5. Let A be an $n \times n$ matrix. Find A^{-1} (if exists) when A satisfies:

$$\text{(i) } (A-I)(A+I) = O \quad \text{(ii) } A^5 - A^3 + A^2 = 4I \quad \text{(iii) } A^2 = A \text{ but } A \neq I.$$

6. Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Suppose that $n < m$. Can the matrix product AB (of size $m \times m$) be invertible?

7. Let A, B, C be $n \times n$ invertible matrices. Solve the following matrix equations for X.
   (i) $ABC = AXC$.
   (ii) $CXB^{-1} = C(BC)^{-1}$.
   (iii) $(ABC^{-1})^{-1}ABXBC(A^{-1}BC)^{-1} = CBA$.

8. Let $\lambda_i \neq 0$ for $i = 1, \ldots, n$. Is $\text{diag}(\lambda_1, \ldots, \lambda_n)$ invertible? If yes, what is its inverse?

9. Consider an invertible $n \times n$ matrix A, and an arbitrary $n \times k$ matrix B. Suppose that by a suitable sequence of EROs one can transform the combined matrix $[\ A \mid B\ ]$ to $[\ I_n \mid C\ ]$. What should C be?

Answers for checking:

1. (i) $(x, y, z) = (1 - s, -2 - s, s)$, where s is free (ii) $(x, y, z) = (-\frac{3}{2}, \frac{3}{2}, \frac{1}{2})$.
2. Yes for both.
3. Solve directly from definition $A^T A = I_2$.
4. $D = \begin{bmatrix} -5 & -4 \\ 5 & 3 \\ -1 & -1 \end{bmatrix}$.
5. (i) A (ii) $\frac{1}{4}(A^4 - A^2 + A)$ (iii) not exist.
6. Never.
7. (i) B (ii) C^{-1} (iii) B
8. Yes. $\text{diag}(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n})$.
9. $C = A^{-1}B$.