Appendix B

Big O and Little o Notation

It is often useful to talk about the *rate* at which some function changes as its argument grows (or shrinks), without worrying to much about the detailed form. This is what the $O(\cdot)$ and $o(\cdot)$ notation lets us do.

A function $f(n)$ is "of constant order", or "of order 1" when there exists some non-zero constant c such that

$$\frac{f(n)}{c} \rightarrow 1 \tag{B.1}$$

as $n \rightarrow \infty$; equivalently, since c is a constant, $f(n) \rightarrow c$ as $n \rightarrow \infty$. It doesn't matter how big or how small c is, just so long as there is some such constant. We then write

$$f(n) = O(1) \tag{B.2}$$

and say that "the proportionality constant c gets absorbed into the big O". For example, if $f(n) = 37$, then $f(n) = O(1)$. But if $g(n) = 37(1 - \frac{2}{n})$, then $g(n) = O(1)$ also.

The other orders are defined recursively. Saying

$$g(n) = O(f(n)) \tag{B.3}$$

means

$$\frac{g(n)}{f(n)} = O(1) \tag{B.4}$$

or

$$\frac{g(n)}{f(n)} \rightarrow c \tag{B.5}$$

as $n \rightarrow \infty$ — that is to say, $g(n)$ is "of the same order" as $f(n)$, and they "grow at the same rate", or "shrink at the same rate". For example, a quadratic function $a_1 n^2 + a_2 n + a_3 = O(n^2)$, no matter what the coefficients are. On the other hand, $b_1 n^{-2} + b_2 n^{-1}$ is $O(n^{-1})$.

Big-O means "is of the same order as". The corresponding little-o means "is ultimately smaller than": $f(n) = o(1)$ means that $f(n)/c \to 0$ for any constant c. Recursively, $g(n) = o(f(n))$ means $g(n)/f(n) = o(1)$, or $g(n)/f(n) \to 0$. We also read $g(n) = o(f(n))$ as "$g(n)$ is ultimately negligible compared to $f(n)$".

There are some rules for arithmetic with big-O symbols:

- If $g(n) = O(f(n))$, then $c\,g(n) = O(f(n))$ for any constant c.
- If $g_1(n)$ and $g_2(n)$ are both $O(f(n))$, then so is $g_1(n) + g_2(n)$.
- If $g_1(n) = O(f(n))$ but $g_2(n) = o(f(n))$, then $g_1(n) + g_2(n) = O(f(n))$.
- If $g(n) = O(f(n))$, and $f(n) = o(h(n))$, then $g(n) = o(h(n))$.

These are not *all* of the rules, but they're enough for most purposes.