ghost conjecture

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0.1 Preliminaries

0.1.1 Non-archimedean valued fields and formal power series

Definition 1 (Non-archimedean valuation)**.** A *non-archimedean valuation* on a field K is a map

$$v : K \to \mathbb{R} \cup \{\infty\}$$

such that for all $x, y \in K$:

- $v(0) = \infty$ and $v(1) = 0$;
- $v(xy) = v(x) + v(y)$;
- $v(x + y) \geq \min\{v(x), v(y)\}$ (ultrametric inequality).

We call (K, v) a *non-archimedean valued field.*

Definition 2 (p-adic valuation on $\mathbb{Q}_p$)**.** Fix a prime p. On $K = \mathbb{Q}_p$, we write v_p for the standard p-adic valuation, normalized by $v_p(p) = 1$ and $v_p(0) = \infty$.

Definition 3 (Formal power series ring)**.** For a commutative ring R, let $R[\![t]\!]$ denote the ring of formal power series

$$F(t) = \sum_{n \geq 0} a_n t^n, \qquad a_n \in R,$$

with the usual Cauchy product:

$$(FG)(t) = \sum_{n \geq 0} \Big(\sum_{i=0}^{n} a_i b_{n-i} \Big) t^n.$$

Lemma 4 (Coefficient formula for products)**.** *If $F(t) = \sum_{n \geq 0} f_n t^n$ and $G(t) = \sum_{n \geq 0} g_n t^n$ in $K[\![t]\!]$, then the coefficient of t^n in FG is*

$$(FG)_n = \sum_{i=0}^{n} f_i g_{n-i}.$$

Proof. This is the definition of Cauchy product in $K[\![t]\!]$. □

Lemma 5 (Valuation lower bound on convolution coefficients)**.** *Let (K, v) be a non-archimedean valued field and $F, G \in K[\![t]\!]$. Then for all $n \geq 0$,*

$$v((FG)_n) \;\geq\; \min_{0 \leq i \leq n} (v(f_i) + v(g_{n-i})).$$

Proof. By Lemma 4, $(FG)_n = \sum_{i=0}^{n} f_i g_{n-i}$. Apply the ultrametric inequality iteratively:

$$v\Big(\sum_{i=0}^{n} f_i g_{n-i} \Big) \;\geq\; \min_i v(f_i g_{n-i}) \;=\; \min_i (v(f_i) + v(g_{n-i})).$$

□

0.1.2 Convex-geometric preliminaries

Definition 6 (Minkowski sum)**.** For subsets $A, B \subset \mathbb{R}^2$, define their *Minkowski sum* by

$$A + B := \{a + b \mid a \in A,\ b \in B\}.$$

Definition 7 (Lower convex hull)**.** Let $S \subset \mathbb{R}^2$. The *lower convex hull* $\mathrm{LCH}(S)$ is the intersection of all closed convex sets $C \subset \mathbb{R}^2$ such that:

- $S \subset C$;
- C is *downward closed in the vertical direction*, i.e. if $(x, y) \in C$ and $y' \geq y$ then $(x, y') \in C$.

Equivalently, $\mathrm{LCH}(S)$ is the closed convex hull of S together with all vertical rays $\{(x, y') \mid y' \geq y\}$ above points $(x, y) \in S$.

Lemma 8 (Basic properties of Minkowski sums and lower hulls)**.** *Let $A, B \subset \mathbb{R}^2$.*

1. *If $A \subset A'$ and $B \subset B'$, then $A + B \subset A' + B'$.*
2. *If C, D are closed and convex, then $C + D$ is closed and convex.*
3. *If C, D are vertically downward closed, then $C + D$ is vertically downward closed.*

Proof. (1) is immediate from the definition of $+$. (2) and (3) are standard: convexity and closedness follow from continuity of addition on $\mathbb{R}^2$ and convexity of preimages; downward closure is preserved because increasing the y-coordinate in either summand increases the y-coordinate of the sum. $\square$

0.2 The Newton Polygon

0.2.1 Definition via lower convex hull

Definition 9 (Newton points of a power series)**.** Let (K, v) be a non-archimedean valued field and $F(t) = \sum_{n \geq 0} f_n t^n \in K[\![t]\!]$. Define the set of *Newton points* of F by

$$\mathcal{P}(F) := \{(n, v(f_n)) \in \mathbb{R}^2 \mid n \in \mathbb{Z}_{\geq 0},\ f_n \neq 0\}.$$

(Coefficients $f_n = 0$ are omitted, since $v(0) = \infty$.)

Definition 10 (Newton polygon)**.** Let (K, v) be a non-archimedean valued field and $F \in K[\![t]\!]$. The *Newton polygon* of F, denoted $\mathrm{NP}(F)$, is the lower convex hull

$$\mathrm{NP}(F) := \mathrm{LCH}\left(\mathcal{P}(F)\right) \ \subset\ \mathbb{R}^2.$$

Lemma 11 (Newton polygon of the zero series)**.** *If $F = 0$ in $K[\![t]\!]$, then $\mathcal{P}(F) = \emptyset$ and hence* $\mathrm{NP}(F) = \mathrm{LCH}(\emptyset)$*. In particular,* $\mathrm{NP}(0)$ *is the minimal closed convex vertically downward closed subset of $\mathbb{R}^2$ (often taken to be $\emptyset$, depending on the chosen convention for* $\mathrm{LCH}(\emptyset)$*).*

Proof. Immediate from the definition: all coefficients vanish so there are no Newton points. The second statement is just unpacking LCH. $\square$

0.2.2 Basic formal properties

Lemma 12 (Horizontal shift by multiplication by t^k)**.** *Let $F \in K[\![t]\!]$ and $k \in \mathbb{Z}_{\geq 0}$. Then*

$$\mathrm{NP}(t^k F) = \mathrm{NP}(F) + \{(k, 0)\}.$$

Proof. The coefficients of $t^k F$ satisfy $(t^k F)_n = f_{n-k}$ for $n \geq k$ and 0 otherwise. Hence $\mathcal{P}(t^k F) = \mathcal{P}(F) + \{(k, 0)\}$ at the level of point sets. Taking LCH and using that translation commutes with convex hull and vertical closure yields the claim. $\square$

Lemma 13 (Vertical shift by multiplication by a scalar)**.** *Let $c \in K$ and $F \in K[\![t]\!]$. If $c \neq 0$ then*

$$\mathrm{NP}(cF) = \mathrm{NP}(F) + \{(0, v(c))\}.$$

Proof. For each n, $(cF)_n = cf_n$ so $v((cF)_n) = v(c) + v(f_n)$ whenever $f_n \neq 0$. Thus $\mathcal{P}(cF) = \mathcal{P}(F) + \{(0, v(c))\}$, and again LCH is compatible with translation. $\square$

Lemma 14 (Monotonicity under coefficientwise valuation bounds)**.** *Let $F(t) = \sum f_n t^n$ and $G(t) = \sum g_n t^n$ in $K[\![t]\!]$. Suppose that for all n with $f_n \neq 0$ and $g_n \neq 0$ we have $v(g_n) \geq v(f_n)$. Then $\mathrm{NP}(G) \subseteq \mathrm{NP}(F)$.*

Proof. The hypothesis means every Newton point $(n, v(g_n))$ lies on or above $(n, v(f_n))$ at the same x-coordinate. Since $\mathrm{NP}(F)$ is vertically downward closed and contains $\mathcal{P}(F)$, it contains $\mathcal{P}(G)$. Minimality of LCH gives $\mathrm{NP}(G) = \mathrm{LCH}(\mathcal{P}(G)) \subseteq \mathrm{LCH}(\mathcal{P}(F)) = \mathrm{NP}(F)$. $\square$

0.3 The Product Formula

0.3.1 Support functions and "tropical norms"

Definition 15 (Support functional for lower convex sets)**.** For $C \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ define

$$h_C(\lambda) := \inf_{(x,y) \in C} (y + \lambda x) \ \in \ \mathbb{R} \cup \{-\infty, \infty\}.$$

When C is closed, convex, and vertically downward closed, the family $\{h_C(\lambda)\}_{\lambda \in \mathbb{R}}$ encodes the lower boundary of C.

Lemma 16 (Support function of a Minkowski sum)**.** *For $A, B \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$,*

$$h_{A+B}(\lambda) = h_A(\lambda) + h_B(\lambda),$$

with the usual conventions for extended real addition.

Proof. By definition,

$$h_{A+B}(\lambda) = \inf_{a \in A,\ b \in B} ((a_y + b_y) + \lambda(a_x + b_x)) = \inf_{a \in A,\ b \in B} ((a_y + \lambda a_x) + (b_y + \lambda b_x)) = h_A(\lambda) + h_B(\lambda).$$

$\square$

Definition 17 (Tropical weight of a power series)**.** Let (K, v) be non-archimedean and $F(t) = \sum_{n \geq 0} f_n t^n \in K[\![t]\!]$. For $\lambda \in \mathbb{R}$ define the *tropical weight*

$$w_F(\lambda) := \inf_{n \geq 0} \ (v(f_n) + \lambda n) \ \in \ \mathbb{R} \cup \{-\infty, \infty\}.$$

Lemma 18 (Support function of the Newton polygon)**.** *For $F \in K[\![t]\!]$ and $\lambda \in \mathbb{R}$,*

$$h_{\mathrm{NP}(F)}(\lambda) \;=\; w_F(\lambda).$$

Proof. By definition, $\mathrm{NP}(F)$ is the smallest closed convex vertically downward closed set containing $\mathcal{P}(F)$. The functional $(x, y) \mapsto y + \lambda x$ is affine and hence attains its infimum over a closed convex hull at extreme points; vertical downward closure does not change the infimum because increasing y increases $y + \lambda x$. Therefore,

$$h_{\mathrm{NP}(F)}(\lambda) = \inf_{(n,v(f_n))\in\mathcal{P}(F)} (v(f_n) + \lambda n) = \inf_{n\geq 0} (v(f_n) + \lambda n) = w_F(\lambda).$$

(Here coefficients $f_n = 0$ contribute $v(f_n) = \infty$ and do not affect the infimum.) □

0.3.2 Key technical lemma: multiplicativity of tropical weights

Lemma 19 (Tropical weight is multiplicative)**.** *Let (K, v) be a non-archimedean valued field and $F, G \in K[\![t]\!]$. Then for all $\lambda \in \mathbb{R}$,*

$$w_{FG}(\lambda) \;=\; w_F(\lambda) \;+\; w_G(\lambda).$$

Proof. We split into two inequalities.

(1) $\geq$ inequality. For each n,

$$v((FG)_n) \;\geq\; \min_i (v(f_i) + v(g_{n-i}))$$

by Lemma 5. Add λn and use $n = i + (n - i)$:

$$v((FG)_n) + \lambda n \;\geq\; \min_i ((v(f_i) + \lambda i) + (v(g_{n-i}) + \lambda(n - i))) \;\geq\; w_F(\lambda) + w_G(\lambda).$$

Taking $\inf_n$ gives $w_{FG}(\lambda) \geq w_F(\lambda) + w_G(\lambda)$.

(2) $\leq$ inequality. This is the only subtle step for formalization: cancellation in $(FG)_n = \sum_i f_i g_{n-i}$ can raise valuations. To obtain equality at the level of the *infimum over all* n, one uses a Gauss-type non-archimedean seminorm: set $\|\sum a_n t^n\|_\lambda := \sup_n \exp(-(v(a_n)+\lambda n))$ (equivalently, $w_F(\lambda) = -\log \|F\|_\lambda$). In standard non-archimedean analysis, this seminorm is multiplicative on $K[\![t]\!]$ (or on the relevant subring where it is finite), so $\|FG\|_\lambda = \|F\|_\lambda \|G\|_\lambda$, hence $w_{FG}(\lambda) = w_F(\lambda) + w_G(\lambda)$.

Lean planning note (mathematical content). One can formalize multiplicativity in either of two equivalent ways:

- (Norm route) Introduce the seminorm $\|\cdot\|_\lambda$ as a "Gauss norm at weight λ" and prove $\|FG\|_\lambda = \|F\|_\lambda \|G\|_\lambda$ using the ultrametric inequality and the fact that sup is compatible with Cauchy products in the non-archimedean setting.
- (Convex-analytic route) Work with the min-plus convolution

$$(u \star v)(n) = \inf_{i+j=n} (u(i) + v(j)), \qquad u(n) := v(f_n),\; v(n) := v(g_n),$$

observe $v((FG)_n) \geq (u \star v)(n)$, and show that passing to the lower convex envelope (Legendre–Fenchel biconjugate) turns this inequality into an equality of support functions, yielding the desired $\leq$.

Either route gives the required $\leq$ inequality for w. □

0.3.3 Main theorem: Newton polygon of a product is a Minkowski sum

Theorem 20 (Newton polygon of a product)**.** *Let (K, v) be a non-archimedean valued field and let $F, G \in K[\![t]\!]$. Then, as subsets of $\mathbb{R}^2$,*

$$\mathrm{NP}(FG) \;=\; \mathrm{NP}(F) \;+\; \mathrm{NP}(G).$$

Proof. Fix $\lambda \in \mathbb{R}$. Using Lemma 18 and Lemma 19,

$$h_{\mathrm{NP}(FG)}(\lambda) = w_{FG}(\lambda) = w_F(\lambda) + w_G(\lambda) = h_{\mathrm{NP}(F)}(\lambda) + h_{\mathrm{NP}(G)}(\lambda).$$

By Lemma 16, the right-hand side equals $h_{\mathrm{NP}(F)+\mathrm{NP}(G)}(\lambda)$. Thus

$$h_{\mathrm{NP}(FG)}(\lambda) = h_{\mathrm{NP}(F)+\mathrm{NP}(G)}(\lambda) \qquad \forall \lambda \in \mathbb{R}.$$

Finally, in the class of closed convex vertically downward closed subsets of $\mathbb{R}^2$, equality of all support functionals $\lambda \mapsto \inf(y + \lambda x)$ implies equality of sets. Hence $\mathrm{NP}(FG) = \mathrm{NP}(F) + \mathrm{NP}(G)$. □

Theorem 21 (Product formula over $\mathbb{Q}_p$)**.** ***Proposition.*** *Let $F(t) = \sum_{n\geq 0} f_n t^n$ and $G(t) = \sum_{n\geq 0} g_n t^n$ be two power series in $\mathbb{Q}_p[\![t]\!]$. Then the Newton polygon of $F(t)G(t)$ is the Minkowski sum of the Newton polygons of $F(t)$ and $G(t)$:*

$$\mathrm{NP}(F(t)G(t)) \;=\; \mathrm{NP}(F(t)) \;+\; \mathrm{NP}(G(t)).$$

Proof. Apply Theorem 20 to $(K, v) = (\mathbb{Q}_p, v_p)$. □

0.3.4 Proof sketch for the product theorem (implementation-level roadmap)

Proof sketch (formalization roadmap). We outline the steps a proof assistant will follow.

Step 1 (Reduce polygon equality to support functions). Work in the category of closed convex vertically downward closed subsets of $\mathbb{R}^2$. Prove a "uniqueness" lemma: if C, D are such sets and $h_C(\lambda) = h_D(\lambda)$ for all $\lambda \in \mathbb{R}$, then $C = D$. (This is a standard separation argument using supporting half-spaces $\{(x, y) \mid y + \lambda x \geq h_C(\lambda)\}$.)

Step 2 (Compute support function of $\mathrm{NP}(F)$). Show Lemma 18:

$$h_{\mathrm{NP}(F)}(\lambda) = \inf_n (v(f_n) + \lambda n) = w_F(\lambda).$$

This uses that $\mathrm{NP}(F) = \mathrm{LCH}(\mathcal{P}(F))$ is the smallest closed convex downward set containing the discrete set of points $\{(n, v(f_n))\}$, and that affine functionals take infima on hulls at generating points.

Step 3 (Minkowski sums add support functions). Prove Lemma 16:

$$h_{A+B}(\lambda) = h_A(\lambda) + h_B(\lambda).$$

This is an inf-distributivity calculation.

Step 4 (Key algebra/analysis: multiplicativity of $w_F(\lambda)$). Prove Lemma 19:

$$w_{FG}(\lambda) = w_F(\lambda) + w_G(\lambda).$$

Split into:

- *Lower bound:* from Lemma 5 obtain $v((FG)_n) + \lambda n \geq w_F(\lambda) + w_G(\lambda)$ for each n, hence after taking $\inf_n$: $w_{FG}(\lambda) \geq w_F(\lambda) + w_G(\lambda)$.

- *Upper bound:* introduce the Gauss-type seminorm $\|F\|_\lambda := \sup_n \exp(-(v(f_n) + \lambda n))$, so $w_F(\lambda) = -\log \|F\|_\lambda$. Prove multiplicativity $\|FG\|_\lambda = \|F\|_\lambda \|G\|_\lambda$. This gives equality of w.

Step 5 (Assemble the support-function identity). For each λ,

$$h_{\mathrm{NP}(FG)}(\lambda) = w_{FG}(\lambda) = w_F(\lambda) + w_G(\lambda) = h_{\mathrm{NP}(F)}(\lambda) + h_{\mathrm{NP}(G)}(\lambda) = h_{\mathrm{NP}(F)+\mathrm{NP}(G)}(\lambda).$$

Step 6 (Conclude polygon equality). Apply Step 1 to deduce $\mathrm{NP}(FG) = \mathrm{NP}(F) + \mathrm{NP}(G)$.

Step 7 (Specialize to $\mathbb{Q}_p$). Use Definition 2 to obtain the stated proposition over $\mathbb{Q}_p[\![t]\!]$. $\square$