Gibbs sampling for LDA

1 Derivation of The Full Conditional Posterior Distribution

We will use collapsed Gibbs sampling to estimate unknown parameters. The full conditional posterior distribution is $P(z_i^d|\mathbf{z_{-i}}, \mathcal{C}, \alpha, \eta)$ where z_i^d is the i-th word in document d. $\mathbf{z_{-z_i^d}}$ is the assignment of all topics except z_i^d, $\mathcal{C}$ is a list of observed words.

According to Bayes' rule, $P(z_i^d|\mathbf{z_{-i}}, \mathcal{C}, \alpha, \eta) = \dfrac{P(\mathbf{z}, \mathcal{C}|\alpha, \eta)}{P(\mathbf{z_{-i}}, \mathcal{C}|\alpha, \eta)}$, when sampling z_i^d, all other topics are considered to be known, so the denominator can be ignored. Let $(\theta^0, \ldots, \theta^D)$ be topic proportions for D documents and θ_k^d is the weight of k-th topic in document d There are K topics in total. Topics are independent given θ and $(\theta^0, \ldots, \theta^D)$ are independent given α. Similarly let $(\psi^0, \ldots, \psi^K)$ be word probabilities for K topics and ψ_v^k is the weight of v-th word in topic k. There are V unique words in total. Words are independent given ψ and $\mathbf{z}$, and $(\psi^0, \ldots, \psi^K)$ are independent given η.

$$
\begin{aligned}
P(z_i^d|\mathbf{z_{-i}}, \mathcal{C}, \alpha, \eta) \propto P(\mathbf{z}, \mathcal{C}|\alpha, \eta) &= \iint P(\mathbf{z}, \mathcal{C}, \theta, \psi|\alpha, \eta)\, d\theta, d\psi \\
&= \iint P(\mathcal{C}|\psi, \mathbf{z}) P(\psi|\eta) P(\mathbf{z}|\theta) P(\theta|\alpha)\, d\theta, d\psi \\
&= \int P(\mathcal{C}|\psi, \mathbf{z}) P(\psi|\eta)\, d\psi \int P(\mathbf{z}|\theta) P(\theta|\alpha)\, d\theta
\end{aligned} \tag{1}
$$

Then we derive the formula for two integrals. Let $m^k = (m_0^k, \ldots, m_V^k)$ be the number of times a word v assigned to topic k in all documents. Let $n^d = (n_0^d, \ldots, n_K^d)$ be the number of words assigned to each topic in document d.

$$
\begin{aligned}
\int P(\mathbf{z}|\theta)P(\theta|\alpha)\,d\theta &= \int\cdots\int P(\mathbf{z}|\theta^0,\ldots,\theta^D)\cdot P(\theta^0,\ldots,\theta^D|\alpha)\,d\theta^0\ldots d\theta^D \\
&= \int\cdots\int \prod_{d=1}^{D}\prod_{k=0}^{K}(\theta_k^d)^{n_k^d}\cdot\prod_{d=0}^{D}\frac{\prod_{k=0}^{K}(\theta_k^d)^{\alpha_k-1}}{B(\alpha)}\,d\theta^0\ldots d\theta^D \\
&= \prod_{d=0}^{D}\int\frac{\prod_{k=0}^{K}(\theta_k^d)^{n_k^d+\alpha_k-1}}{B(\alpha)}\,d\theta^d \\
&= \prod_{d=0}^{D}\frac{B(\alpha+n^d)}{B(\alpha)}
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\int P(\mathcal{C}|\psi,\mathbf{z})P(\psi|\eta)\,d\psi &= \int\cdots\int P(\mathcal{C}|\psi^0,\ldots,\psi^K,\mathbf{z})\cdot P(\psi^0,\ldots,\psi^K|\eta)\,d\psi^0\ldots d\psi^K \\
&= \int\cdots\int \prod_{k=0}^{K}\prod_{v=0}^{V}(\psi_v^k)^{m_v^k}\cdot\prod_{k=0}^{K}\frac{\prod_{v=0}^{V}(\psi_v^k)^{\eta_v-1}}{B(\eta)}\,d\psi^0\ldots d\psi^K \\
&= \prod_{k=0}^{K}\int\frac{\prod_{v=0}^{V}(\psi_v^k)^{m_v^k+\eta_v-1}}{B(\eta)}\,d\psi^k \\
&= \prod_{k=0}^{K}\frac{B(m^k+\eta)}{B(\eta)}
\end{aligned} \tag{3}
$$

Let M^k be number of words with topic k in all documents. To sample a topic z_p^q for w_p^q (p-th word in document q), full conditional posterior distribution (1) is reduced to:

$$
\begin{aligned}
P(z_p^q|\mathbf{z}_{-\mathbf{i}},\mathcal{C},\alpha,\eta) &\propto \prod_{d=0}^{D}\frac{B(\alpha+n^d)}{B(\alpha)}\cdot\prod_{k=0}^{K}\frac{B(m^k+\eta)}{B(\eta)} \\
&\propto \prod_{d=0}^{D}B(\alpha+n^d)\cdot\prod_{k=0}^{K}B(m^k+\eta) \\
&= \prod_{d=0}^{D}\frac{\prod_{k=0}^{K}\Gamma(\alpha_k+n_k^d)}{\Gamma\sum_{k=0}^{K}(\alpha_k+n_k^d)}\cdot\prod_{k=0}^{K}\frac{\prod_{v=0}^{V}\Gamma(\eta_v^k+m_v^k)}{\Gamma\sum_{v=0}^{V}(\eta_v^k+m_v^k)} \\
&\propto \frac{\prod_{k=0}^{K}\Gamma(\alpha_k+n_k^q)}{\Gamma\sum_{k=0}^{K}(\alpha_k+n_k^q)}\cdot\prod_{k=0}^{K}\frac{\Gamma(\eta_v^k+m_{w_p^q}^k)}{\Gamma\sum_{v=0}^{V}(\eta_v^k+m_v^k)}
\end{aligned} \tag{4}
$$

$$[Continue] \quad \propto \prod_{k=0}^{K} \Gamma(\alpha_k + n_k^q) \cdot \prod_{k=0}^{K} \frac{\Gamma(\eta_v^k + m_{w_p^q}^k)}{\Gamma(\sum_v \eta_v^k + M^k)}$$

$$= \left(\prod_{k \neq z_p^q} \Gamma(\alpha_k + n_k^q) \frac{\Gamma(\eta_v^k + m_{w_p^q}^k)}{\Gamma(\sum_v \eta_v + M^k)} \right) \cdot \Gamma(\alpha_{z_p^q} + n_{z_p^q}^q) \frac{\Gamma(\eta^{z_p^q} + m_{w_p^q}^{z_p^q})}{\Gamma(\sum_v \eta_v + M^{z_p^q})} \tag{5}$$

Let $(n_k^d)^-$ be defined same way as n_k^d, only without the count for (z_p^q, w_p^q). Let $(m_v^k)^-$ be defined same way as m_v^k without the count for (z_p^q, w_p^q).

$$[Continue] \quad = \left(\prod_{k \neq z_p^q} \Gamma(\alpha_k + n_k^{q-}) \frac{\Gamma(\eta_v^k + m_{w_p^q}^{k\ -})}{\Gamma(\sum_v \eta_v + M^k)} \right) \cdot \Gamma(\alpha_{z_p^q} + (n_{z_p^q}^q)^- + 1) \frac{\Gamma(\eta^{z_p^q} + (m_{w_p^q}^{z_p^q})^- + 1)}{\Gamma(\sum_v \eta_v + M^{z_p^q} + 1)}$$

$$= \left(\prod_{k \neq z_p^q} \Gamma(\alpha_k + n_k^{q-}) \frac{\Gamma(\eta_v^k + m_{w_p^q}^{k\ -})}{\Gamma(\sum_v \eta_v + M^k)} \right)$$

$$\times \left(\alpha_{z_p^q} + (n_{z_p^q}^q)^-\right) \Gamma\left(\alpha_{z_p^q} + (n_{z_p^q}^q)^-\right) \frac{\left(\eta^{z_p^q} + (m_{w_p^q}^{z_p^q})^-\right) \Gamma\left(\eta^{z_p^q} + (m_{w_p^q}^{z_p^q})^-\right)}{(\sum_v \eta_v + M^{z_p^q}) \Gamma(\sum_v \eta_v + M^{z_p^q})}$$

$$= \left(\prod_{k=0}^{K} \Gamma(\alpha_k + n_k^{q-}) \frac{\Gamma(\eta_v^k + m_{w_p^q}^{k\ -})}{\Gamma(\sum_v \eta_v + M^k)} \right) \times \frac{\left(\alpha_{z_p^q} + (n_{z_p^q}^q)^-\right) \left(\eta^{z_p^q} + (m_{w_p^q}^{z_p^q})^-\right)}{(\sum_v \eta_v + M^{z_p^q -})}$$

$$\propto \frac{\left(\alpha_{z_p^q} + (n_{z_p^q}^q)^-\right) \left(\eta^{z_p^q} + (m_{w_p^q}^{z_p^q})^-\right)}{\sum_v \eta_v + M^{z_p^q -}} \tag{6}$$

In conclusion, if we want to sample a topic for i-th word in d-th document, the probability distribution is

$$P(z_i^d = j | w_i^d = v, \mathbf{z_{-i}}, \mathcal{C}, \alpha, \eta) = \frac{\left(\alpha_j + (n_j^d)^-\right) \left(\eta_v^j + (m_v^j)^-\right)}{\sum_v \eta_v^j + M^{j-}} \tag{7}$$