GOV 2001/ 1002/ E-200 Section 7 Zero-Inflated models and Intro to Multilevel Modeling[1]

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[1]These section notes are heavily indebted to past Gov 2001 TFs for slides and R code.

Logistics

Reading Assignment- Becker and Kennedy (1992), Harris and Zhao (2007) (sections 1 and 2), and Esarey and Pierce (2012)

Re-replication- Due by 6pm Wednesday, March 30th on Canvas.

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- You will receive all of the replication files from another team.
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- Aim to be helpful, not just critical!

OVERVIEW

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OUTLINE

Zero-inflated models

"Fixed" and "Random" Effects Models

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If we have an "excess" of zeroes, our modeling assumption will be wrong.

A Working Example: Fishing

You're trying to figure out how many fish people caught in a lake from a survey. People were asked:

- How many children were in the group
- How many people were in the group
- How many fish they caught

A Working Example: Fishing

The problem is, some people didn't even fish! These people have systematically zero fish.

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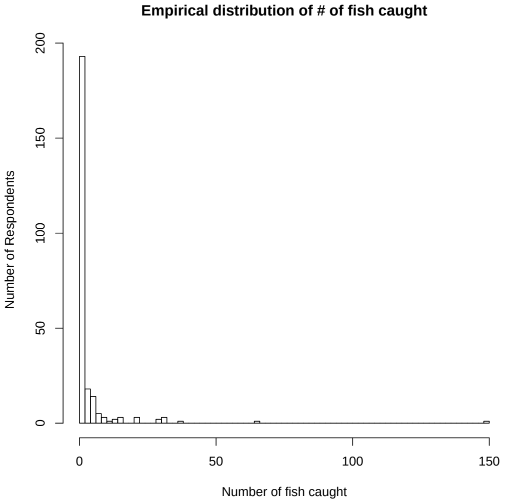

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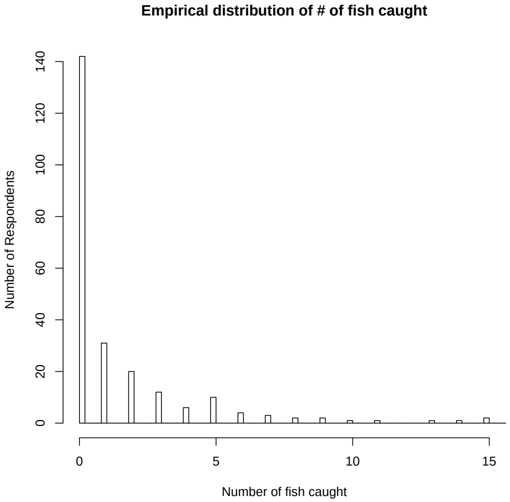

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This is a **mixture model** because our data is a mix of these two types of groups each with their own data generation process.

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and the probability that Y is 0 given $Z_i = 0$:

$$P(Y_i = 0|Q_i = 0) = 1$$

Zero-Inflated Poisson Model

So we can write Y_i as a mixture of two DGPs

$$Y_i \sim \begin{cases} 0 & \text{with probability} \quad \psi_i \\ \text{Poisson}(\lambda_i) & \text{with probability} \quad 1 - \psi_i \end{cases}$$

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And we can put covariates Z_i on ψ_i using a logit model:

$$\psi_i = \frac{e^{z_i \gamma}}{1 + e^{z_i \gamma}}$$

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$$Pr(Y_i|X_i, Z_i, \gamma, \beta) = \left[\frac{e^{Z_i\gamma}}{1+e^{Z_i\gamma}} + \frac{1}{1+e^{Z_i\gamma}} e^{-e^{X_i\beta}}\right] \mathcal{I}(Y_i = 0) \times \left[\frac{1}{1+e^{Z_i\gamma}} \frac{e^{Y_i X_i \beta} e^{-e^{X_i\beta}}}{Y_i!}\right] \mathcal{I}(Y_i > 0)$$

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$$\ell(Y|X,Z,\gamma,\beta) = \sum_{Y_i=0}\left[\log(e^{Z_i\gamma} + e^{-e^{X_i\beta}}) - \log(1+e^{Z_i\gamma})\right] + \sum_{Y_i>0}\left[Y_iX_i\beta - e^{X_i\beta} - \log(Y_i!) - \log(1+e^{Z_i\gamma})\right]$$

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How many parameters are we estimating?

LET'S PROGRAM THIS IN R

Load and get the data ready:

```
fish <- read.table("http://www.ats.ucla.edu/stat/data/fish.csv",
  sep=",", header=T)
X <- fish[c("child", "persons")]
Z <- fish[c("persons")]
X <- as.matrix(cbind(1,X))
Z <- as.matrix(cbind(1,Z))
Y <- fish$count
```

LET'S PROGRAM THIS IN R

Write out the Log-likelihood function

```
ll.zipoisson <- function(par, X, Z, Y){
  beta <- par[1:ncol(X)]
  gamma <- par[(ncol(X)+1):length(par)]

  ## Which indices are Y = 0?
  yzero <- Y==0
  ynonzero <- Y > 0

  ## First part of likelihood
  lik <- -sum(log(1 + exp(Z%*%gamma))) + # Sum over all N
    sum(log(exp(Z[yzero,]%*%gamma) + exp(-exp(X[yzero,]%*%beta)))) + #Y = 0
    sum(Y[ynonzero]*X[ynonzero,]%*%beta - exp(X[ynonzero,]%*%beta)) #Y > 0

  return(lik)

}
```

LET'S PROGRAM THIS IN R

Optimize to get the results

```
par <- rep(1,(ncol(X)+ncol(Z)))
out <- optim(par, ll.zipoisson, Z=Z, X=X,Y=Y, method="BFGS",
             control=list(fnscale=-1), hessian=TRUE)

out$par
[1] -0.3551572 -1.3403588  0.8614703  0.5308458 -0.2966534
```

Plotting to See the Relationship

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These numbers don't mean a lot to us, so we can plot the predicted probabilities of a group having not fished (i.e. predict ψ.

First, we have to simulate our gammas:

```
varcv.par <- solve(-out$hessian)
library(mvtnorm)
sim.pars <- rmvnorm(10000, out$par, varcv.par)
# Subset to only the parameters we need (gammas)
# Better to simulate all though
sim.z <- sim.pars[,(ncol(X)+1):length(par)]
```

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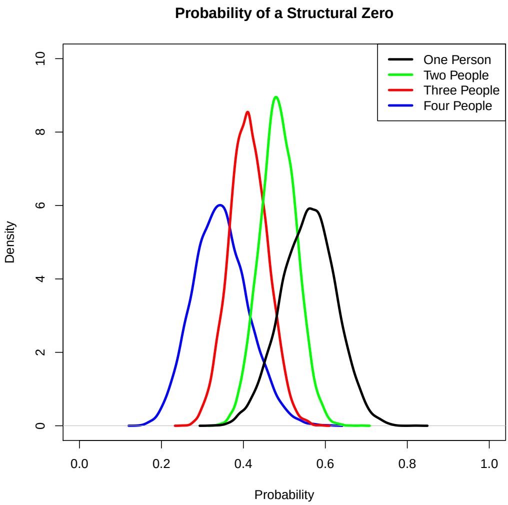
We then generate predicted probabilities of not fishing for different sized groups.

```
# Calculate the predicted probabilities for four different group sizes
person.vec <- seq(1,4)
# Each row is a different vector of covariates (our setx)
Zcovariates <- cbind(1, person.vec)
# Calculate predicted probability (inverse logit) for each set of covariates
# Remember that for each group size, this is a vector of 10000 predicted
     probabilities
exp.holder <- matrix(NA, ncol=4, nrow=10000)
for(i in 1:length(person.vec)){
  exp.holder[,i] <- exp(Zcovariates[i,]%*%t(sim.z))/(1+exp(Zcovariates[i,]%*%t(sim.z)
       ))
}
```

Plotting to See the Relationship

Using these numbers, we can plot the densities of probabilities, to get a sense of the probability and the uncertainty around those estimates



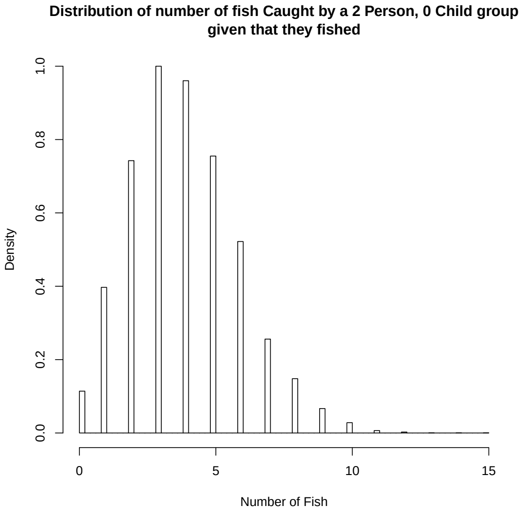
HOW MANY FISH DID SOMEONE CATCH GIVEN THAT THEY FISHED?

We can also simulate to get a distribution or expectation of $Y|Z = 1$ - let's do this for the "median" group of 2 with no children

```
# Let's plot the count distribution conditional on $Z_i = 1$
# Subset our simulation to simulated Xs
sim.x <- sim.pars[,1:ncol(X)]

# Calculate distribution of fish caught | fishing
# Set covariates to median # of children and median # of people
Xcovariates <- apply(X, 2, median)
# Remember that for each group size, this is a vector of 10000 predicted
     probabilities
set.seed(02138)
fish <- rep(NA, nrow=10000)
fish <- rpois(10000,exp(Xcovariates%*%t(sim.x)))
```

How many fish did someone catch given that they fished?



OUTLINE

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- We're going to focus on fixed vs. random effects in *hierarchical* or *multi-level* models.
- For these purposes, "fixed" effects are treated as constants while "random" effects are modeled as random variables.
- **Key point:** Ignore the labels, look at what the researcher actually wants to do.

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- Our data are now indexed by i (unit) and j (cluster).

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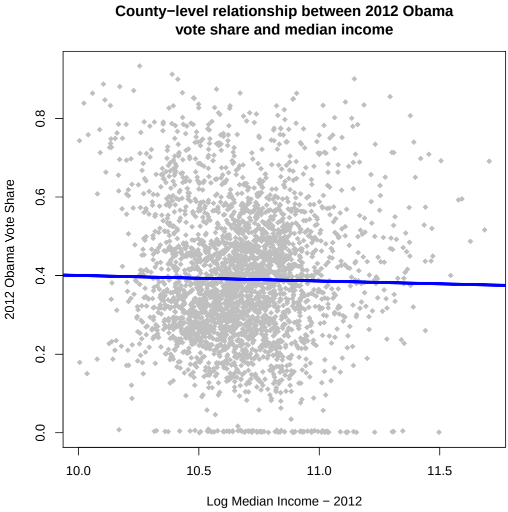
- Here i indexes counties and j indexes states.

MODEL 1: NO GROUPING



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    - Disadvantage: Can't include cluster-level predictors (perfect co-linearity).

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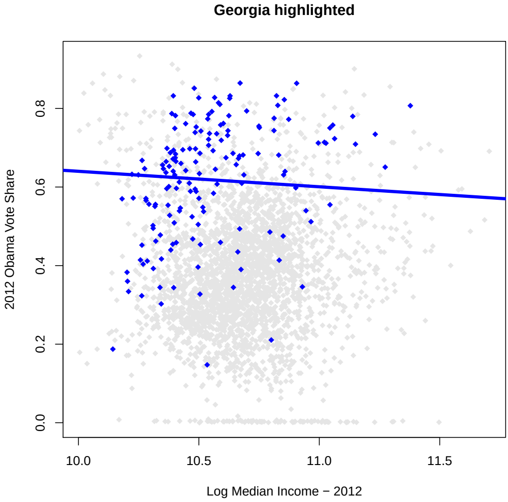

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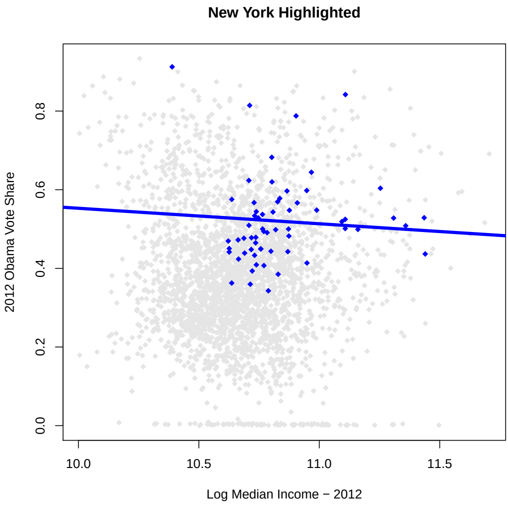

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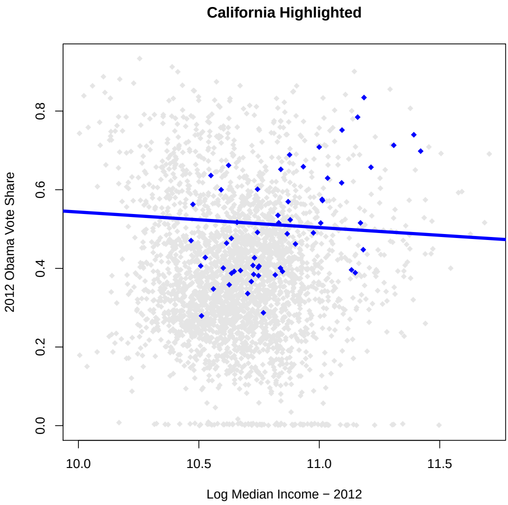

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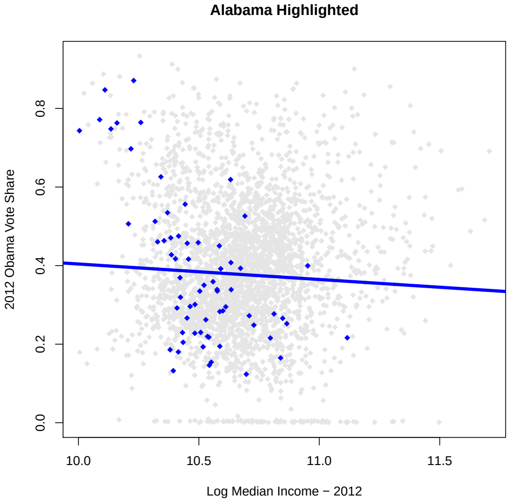

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- Alternative is to assume group-level effect η_j comes from some underlying distribution (like the error term).
- For linear random effect, typical assumption is that $\eta_j \sim \mathcal{N}(0, \sigma_\eta^2)$. η_j are i.i.d. across groupings and is independent of the individual error term ϵ_{ij}.
- **Intuition**: We're decomposing the overall error into "between-group" variance (σ_η^2) and a "within-group" variance (σ^2)

ESTIMATING RANDOM EFFECTS

- Maximum likelihood methods still work...

Estimating random effects

- Maximum likelihood methods still work...fairly well...

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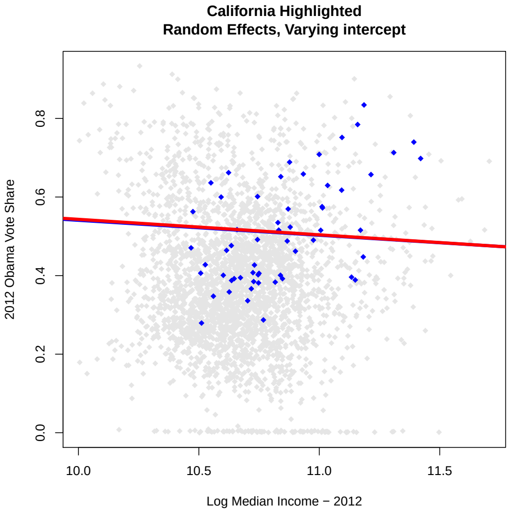
- As you get smaller sample sizes, fewer clusters or more complicated variance structures, turn to Bayesian approaches. Increasingly popular with easy-to-use software - see Gelman and Hill (2007)

Model 3: Random intercept model



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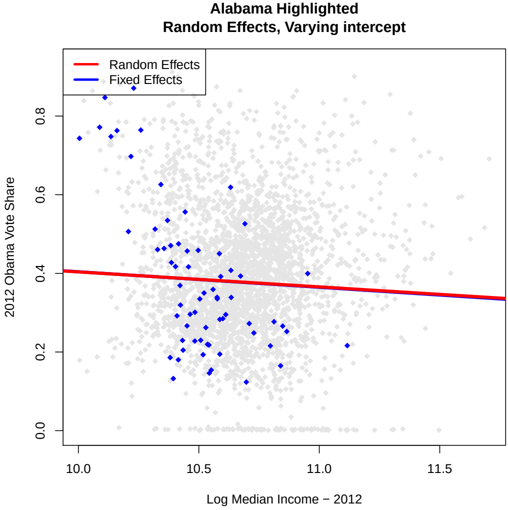

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- Point estimates identical to also subsetting the data by group and estimating j separate regressions.

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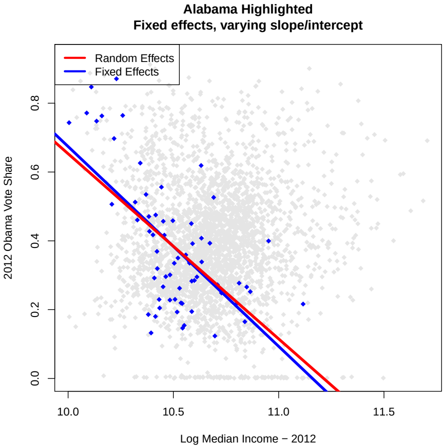
COMPARING FE AND RE



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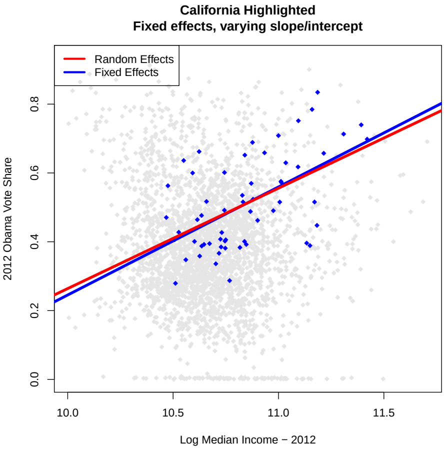

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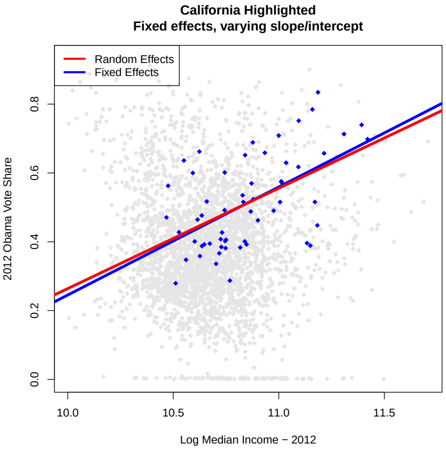

Not very different (due to lots of data), but can see RE slope slightly attenuated towards 0.

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- Multilevel modeling provides a compromise between estimating fully separate models and assuming a constant coefficient across all groups.

QUESTIONS

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