Advanced Quantitative Research Methodology, Lecture Notes: Theories of Inference[1]

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3. A more reasonable, limited goal. Let $M = \{M^*, \theta\}$, where M^* is assumed & θ is to be estimated:

$$\mathrm{P}(\theta|y, M^*) \equiv \mathrm{P}(\theta|y)$$

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8. The two differ on the rest

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6. $L(\theta|y)$ is a function: for y fixed at the observed values, it gives the "likelihood" of any value of θ.

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10. The likelihood principle: the data only affect inferences through the likelihood function

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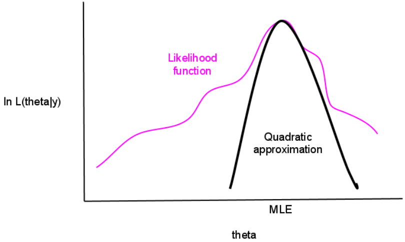

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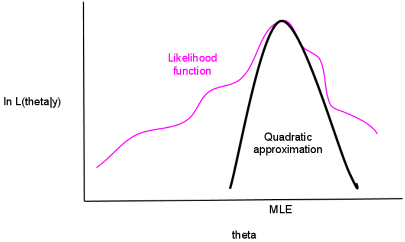

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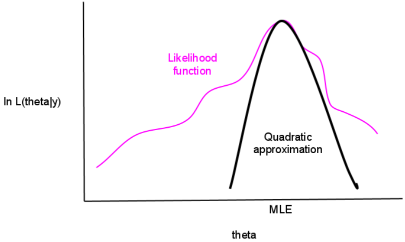

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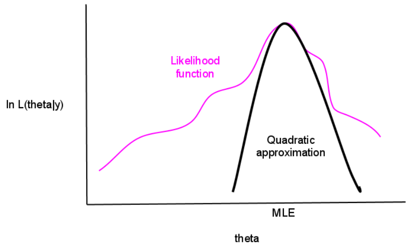

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- Uncertainty of point estimate: curvature at the maximum

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- $P(\theta)$, the prior density — the way Bayes differs from likelihood

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4. A philosophical assumption that nonsample information should matter (as it always does) *and* be formalized and included in all inferences.

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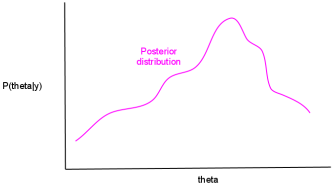
4. If variables A and B are both unknown, then the distribution of A alone is $P(A) = \int \mathrm{P}(A,B)dB = \int P(A|B)P(B)dB$.

The posterior density, P($\theta|y$)



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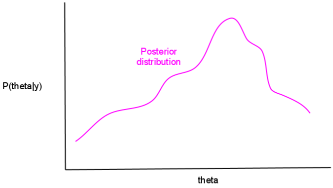

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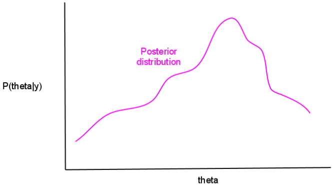

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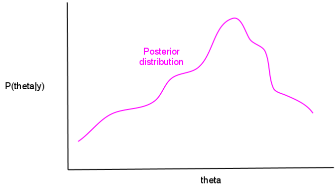

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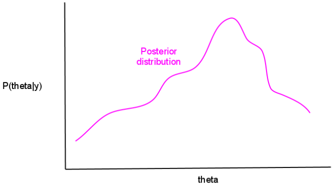

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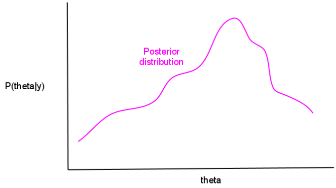

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- If $P(\theta) = 1$, i.e., is uniform in the relevant region, then $L(\theta|y) = P(\theta|y)$.

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- Few fights now between Bayesians and likelihoodists

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- Choose Type I error, probability of deciding H_1 is right when H_0 is really true: say $\alpha = 0.05$

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And then first collect your data. You may not revise your hypothesis or your theory.

Neyman-Pearson Hypothesis Testing

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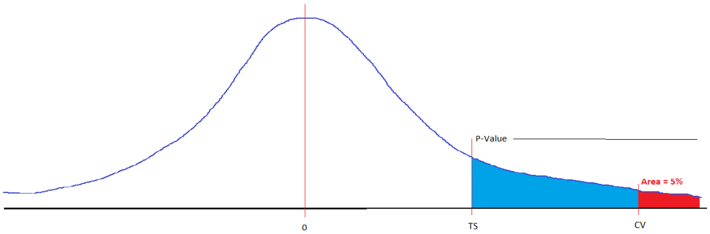

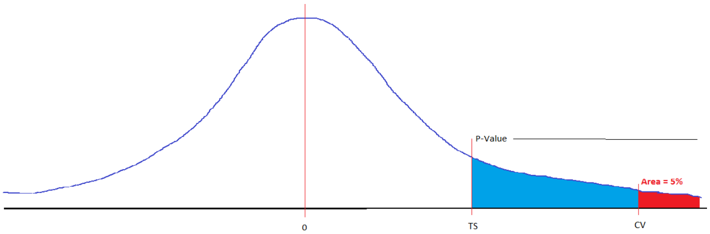

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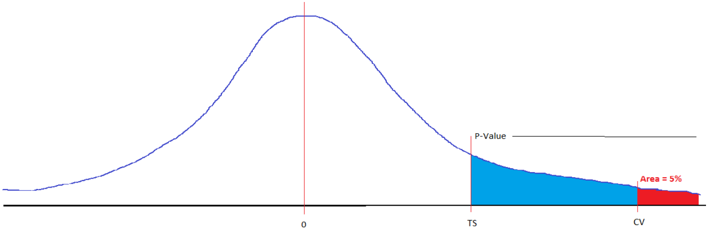

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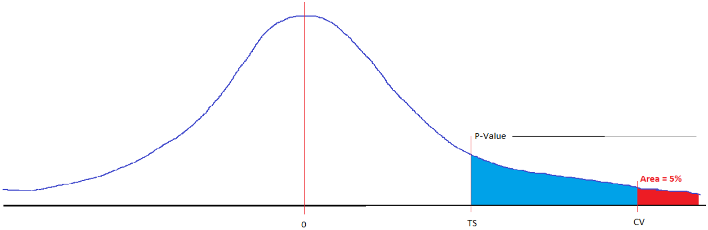

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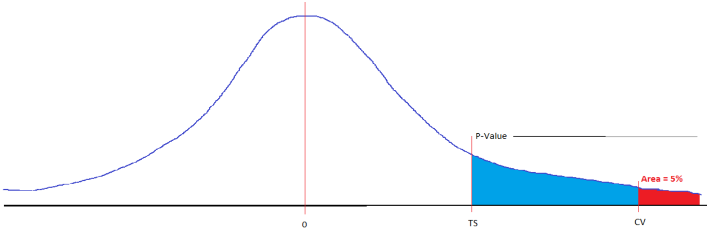

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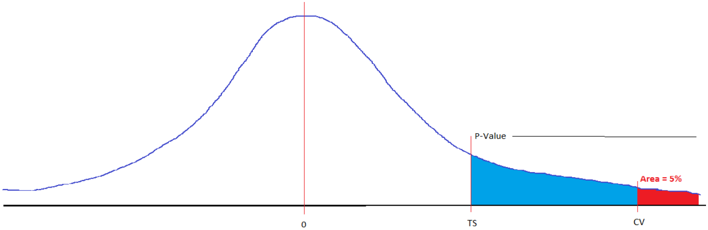

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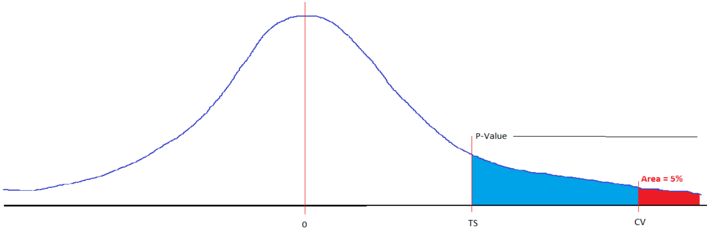

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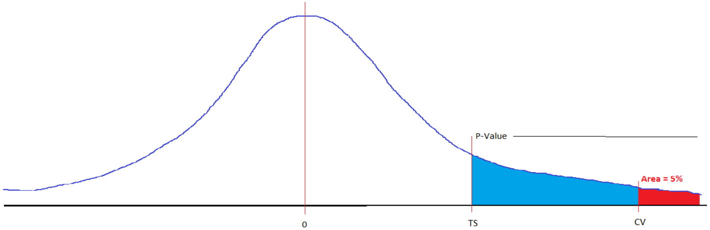

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- We can use likelihood to compute hypothesis tests and p-values.

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4. Methods for applied researchers: either useful or irrelevant

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- The key: No assumptions can be trusted; all theories of inference condition on assumptions and so data analysts always struggle trying to understand and get around them

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- What can you do with this probability density?

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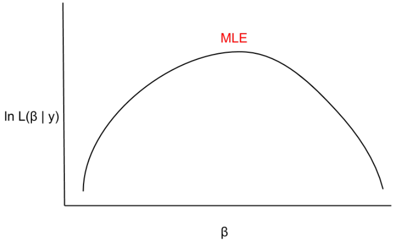
$$\doteq \sum_{i=1}^{n} -\frac{1}{2}(y_i-\beta)^2 = -\frac{1}{2}\sum_{i=1}^{n}(y_i-\beta)^2$$

Log-likelihood interpretation



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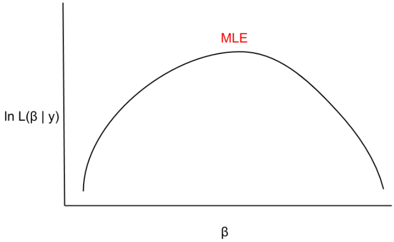

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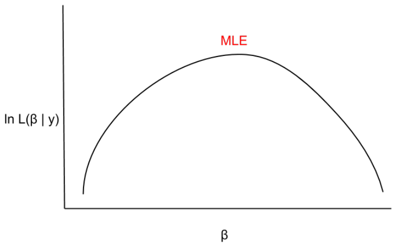

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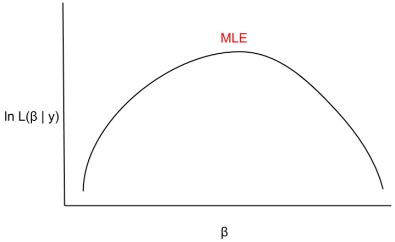

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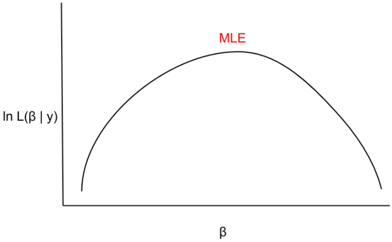

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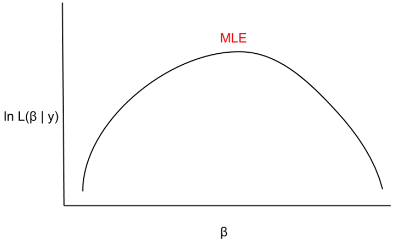

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5. No reason to summarize this curve with only the MLE

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   - In fact, it's a great idea!

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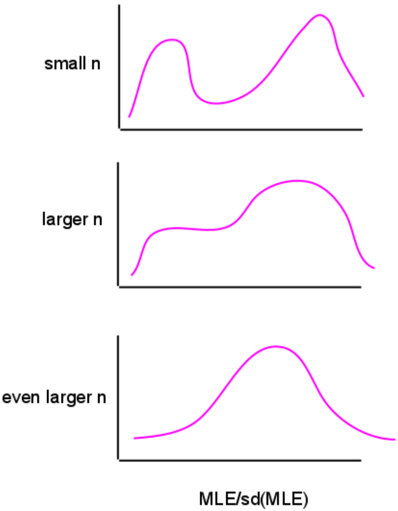
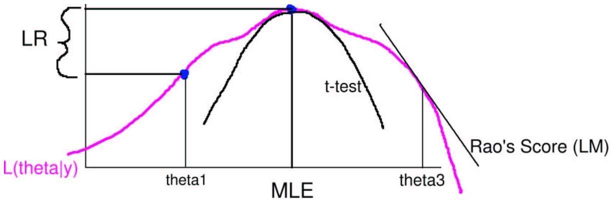
Asymptotic Properties of the MLE

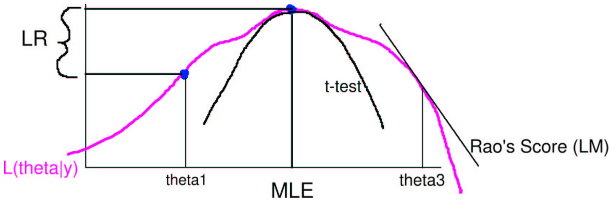
1. Consistency (from the Law of Large Numbers). As $n \to \infty$, the sampling distribution of the MLE collapses to a spike over the parameter value
2. Asymptotic normality (from the central limit theorem):
   - As $n \to \infty$, the distribution of MLE/se(MLE) converges to a Normal.
   - Why do we care? If N is large enough, the asymptotic distribution is a good approximation in finite samples
   - Do the LLN and CLT (the 2 most important theorems in statistics) contradict each other?
3. Asymptotic efficiency. The MLE contains as much information as can be packed into a point estimator.

Sampling distributions of the MLE: CLT vs LLN




LR
t-test
Rao's Score (LM)
L(theta|y)
theta1
MLE
theta3

Uncertainty: Likelihood Ratios for nested models



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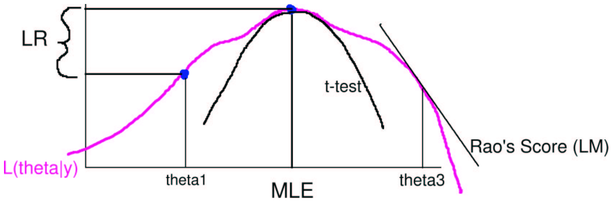

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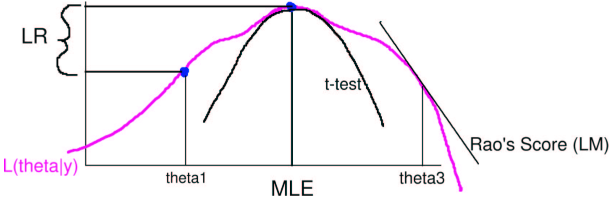

- L^* is the likelihood value for the unrestricted model
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- $\implies L^* \geq L^*_R \implies \frac{L^*_R}{L^*} \leq 1$

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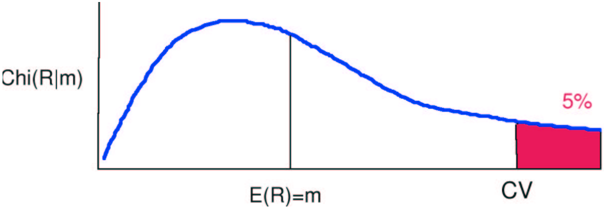

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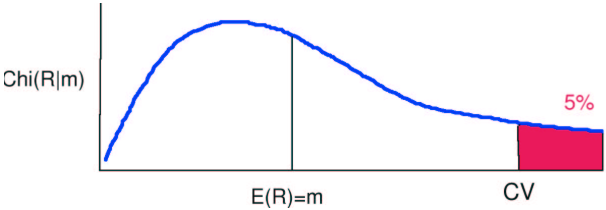

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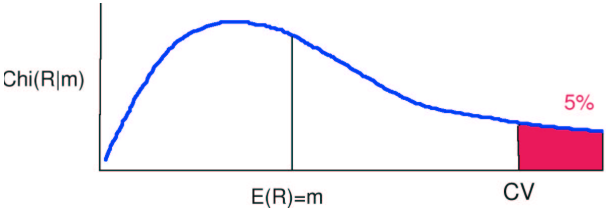

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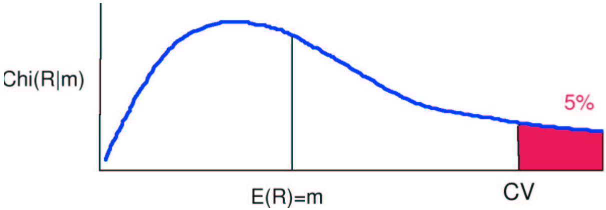

- If restrictions have no effect, $E(R) = m$.
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- Disadvantage: Too many likelihood ratio tests may be required to test all points of interest

Standard Errors

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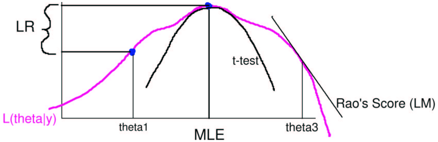

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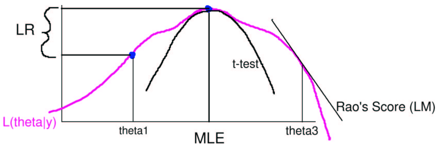

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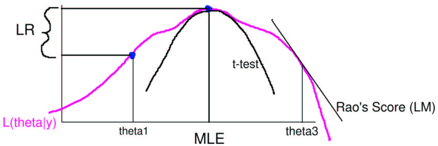

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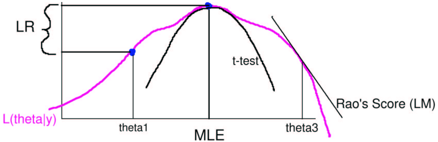

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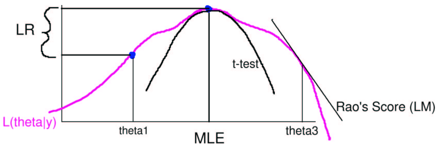

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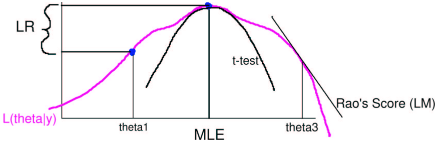

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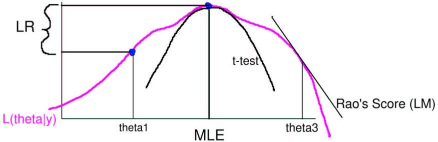

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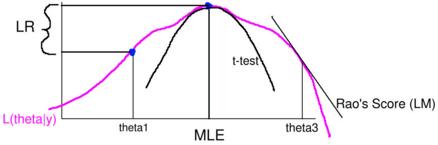

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$$\ln L(\beta|y) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta)^2$$

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    - the larger the likelihood ratio would be in comparing the MLE with *any* other parameter value.

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E_i	The number of electoral college votes for each state in 2016

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R Code for the Log-Likelihood

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- Mathematical Form:

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- An R function:

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ll.normal <- function(par, X, Y) {
X <- as.matrix(cbind(1, X))
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- Calling it:

```
ll.normal(c(2,1,2,1,33,4,3.2),x,y)
ll.normal(c(2,1,2,1,33,4,3.7),x,y)
ll.normal(c(2,1,2,1,33,4,3.5),x,y)
```

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- Repeat Steps 1–3 $M = 1,000$ times, and plot a histogram of the results.

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Actual Results (calculated before the election) for 1992

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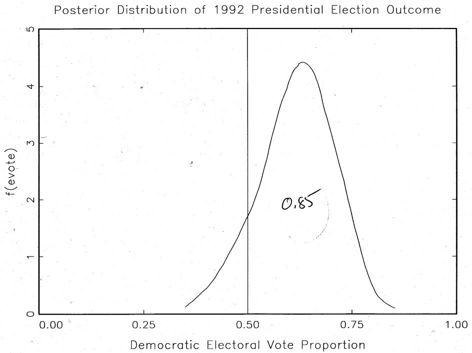

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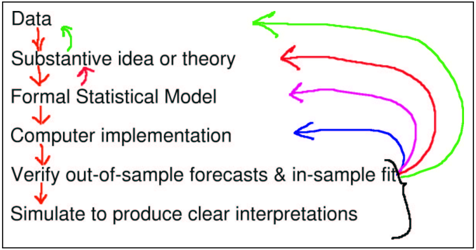
- For what applications would this model be informative?

An Outline of the Research Process



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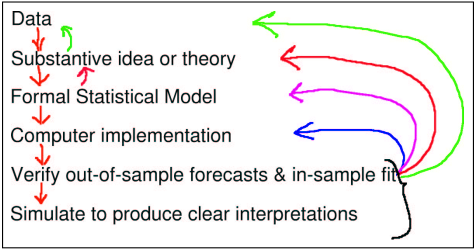

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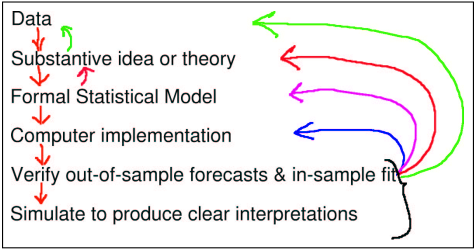
An Outline of the Research Process



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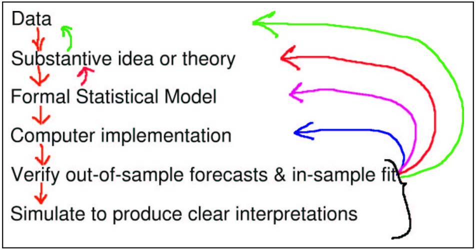

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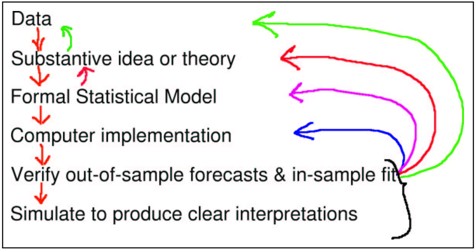

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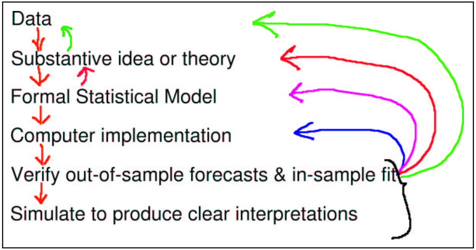

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