Advanced Quantitative Research Methodology, Lecture Notes: Robust Standard Errors¹

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- A better use for RSEs: as a test for misspecification

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Still allows for heteroskedasticity

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$$\Sigma = \sigma^2 \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

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$$\Sigma = \sigma^2 \prod_{n \times n}$$

Standard linear-normal regression model assumptions

What if $\Sigma \neq \sigma^2 I$ and we run a regression?

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• Can estimate V(b) by replacing σ_i^2 with e_i^2 in Σ

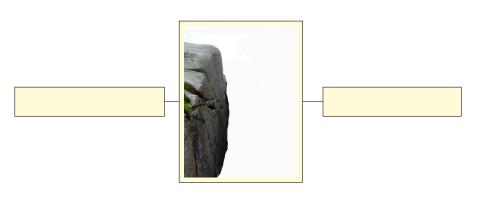
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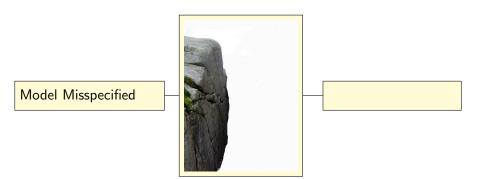
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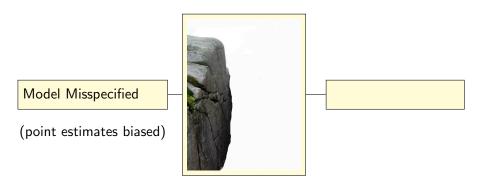
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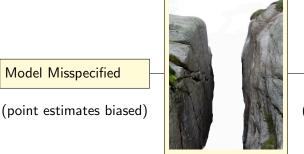
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- The idea generalizes to any MLE model





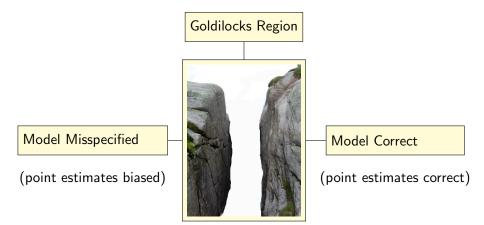


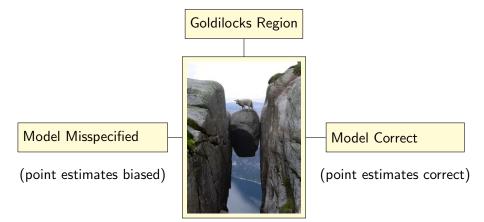




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Goldilocks Region

Model Misspecified

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Model Correct

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Biased just enough to make RSEs useful,

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but not so much as to bias everything else

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- If SE≠RSE: find missspecification, fix model, rerun until SE=RSE

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- Robust SE $(0.34) \approx$ classical SE (0.32)

Transformation of Y: Becomes Normal

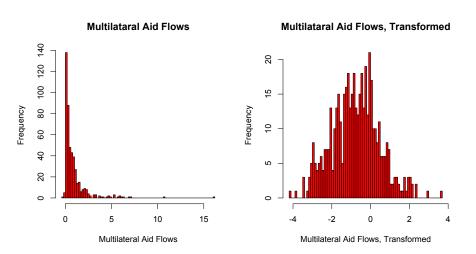
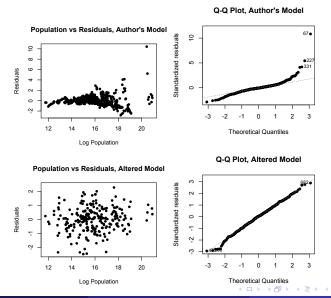


Figure: Distribution of the dependent variable before (left) and after (right) the Box-Cox transformation.

Transformation of Y: Removes Heteroskedasticty



Results: Transformed Model, Opposite Results

The misspecification \leadsto bias, not inefficiency

