Econ 1034 / CS 134: Networks	**Problem set 5**
Prof. Yaron Singer	Due: **Wednesday 3/8/2017**

1. (20 points) Use the Iterated Deletion of Strictly Dominated Strategies to find all Nash equilibria of the following game:

	L	M	R
U	2, 4	2, 1	3, 2
D	1, 2	3, 3	2, 4

Give your intuition for why this algorithm never eliminates a strategy that is part of a Nash equilibrium.

Solution by Luca Schroeder:

Solution. First, we notice that for the column player, strategy M is strictly dominated by strategy R. (If the row player plays U, the column player has payoff 2 for playing R and payoff 1 for playing M. If the row player plays D, the column player has payoff 4 for playing R and payoff 3 for playing M.)

	L	R
U	2, 4	3, 2
D	1, 2	2, 4

We now consider the reduced game above with strategy M removed. We notice that for the row player, strategy D is strictly dominated by strategy U. (If the column player plays L, the row player has payoff 2 for playing U and payoff 1 for playing D. If the column player plays R, the row player has payoff 3 for playing U and payoff 2 for playing D.)

	L	R
U	2, 4	3, 2

We now consider the further reduced game with strategy D removed. We notice that for the column player, strategy R is strictly dominated by strategy L. (If the row player plays U, the column player has payoff 4 from playing strategy L and payoff 2 for playing strategy R.)

	L
U	2, 4

From lecture and section we know that when the process of iterated deletion of strictly dominated strategies leads to one outcome, this outcome is the unique Nash equilibrium of the game. So the unique Nash equilibrium of the game above is (U, L).

Recall that in a Nash equilibrium each agent is playing their best response to the strategies of all other players. In the Iterated Deletion of Strictly Dominated Strategies, we only remove strictly dominated strategies; a strictly dominated strategy obviously cannot ever be a best response (since by definition there is always another strategy or set of strategies that produces a strictly better payoff, regardless of what other players do), so this algorithm only eliminates strategies that would never be part of a Nash equilibrium (or, to phrase it more in accordance with the prompt, this algorithm never eliminates a strategy that is part of a Nash equilibrium).

2. (25 points) Consider five strangers travelling in the last row of a bus from Boston to New York City. Each of them has the option to read a book or to sleep during the trip. Each of them controls a bulb of 60 watts, and a bulb of an immediate neighbor provides 30 watts. Each of them prefers as much light as possible if more than 100 watts of light reaches them (they can't read with less than 100 watts of light, but as long as they can read, they want to read with as much light as possible) but as little as possible otherwise (if they can't read, they want to sleep with as little light as possible).

- **a. (7 points)** Transform this situation into a game (list the set of players, the set of strategies and an appropriate payoff function)

- **b. (6 points)** List any dominated strategies you can find in this game.

- **c. (6 points)** List any dominant strategies you can find in this game.

- **d. (6 points)** List all equilibria you can find in this game.

Solution by Matthew Beatty:

a.

The set of players: $\{P_1, P_2, P_3, P_4, P_5\}$. Each player represents one of the five strangers travelling. They are in numerical order in the row, i.e. P_1 and P_5 are at the ends.

The set of strategies: each player has two strategies $\{S_{on}, S_{off}\}$. S_{on} represents turning their light bulb on, and S_{off} represents turning their light bulb off.

An appropriate payoff function U for player i given a chosen strategy S_i and and the amount of lights in watts reaching player i is called W:

$$U_i(S_i) = \begin{cases} 100 - W & \text{W} < 100 \\ W & \text{W} > 100 \end{cases}$$

b.

The strategy S_{on} is strictly dominated for the players sitting on the ends of the row (w.l.o.g. P_1 and P_5). This is because they can only achieve $W = 90$ using their light and the light of their single neighbor. Considering the payoff function above, there is no way for $W > 100$ for P_1 and P_5, so they will always choose S_{off}.

There are no dominated strategies for the players in the middle of the row (P_2, P_3 and P_4). For any of the middle players, if their two neighbors turn their lights on, they will also choose S_{on} to maximize their payoff. In this case, $U(S_{on}) = 120$. However, if none or only one of their neighbors turn their lights on, $W > 100$ is not possible for the player, so they will choose S_{off}.

c.

The strategy S_{off} is dominant for the players sitting on the ends of the row. As shown in part b), the end players will never be able to reach $W > 100$, so they will try to minimize W with S_{off}.

There are no dominant strategies for the middle players. Also shown in part b), since there are no dominated strategies for the players in the middle of the row, there cannot be any dominating strategies for these players. Their strategy will always depend on their neighbors.

d.

The only equilibrium for the game will be all players choosing S_{off}. First, we know that S_{off} strictly dominates S_{on} for the players at the end of the row. This means that P_1 and P_5 will turn their lights off. As explained above, if any of a middle player's neighbors turns their light off, the middle player will as well. This means that P_2 and P_4 will choose S_{off} because it is now the payoff maximizing strategy. Finally, if both neighbors choose S_{off}, then by the same logic P_3 will also choose S_{off}. This means that all of the five players will choose S_{off}, and this is a Nash equilibrium.

3. (25 points) Redo Exercise 2 assuming that the five strangers are seated in a round table (so that each has a right and left neighbor) instead of in a row.

Solution by Amanda Glazer:

(a) Still have that the set of players is the 5 strangers on the bus. Each player has 2 strategies: turn their light on or keep their light off.

The payoff function is a bit different because at the round table everyone will have the same payoff. The possibilities (when adding up the wattage) are 0, 30, 60, 90 and 120 watts which gives a payoff of 4, 3, 2, 1 and 5 respectively.
(b) & (c) No strategies are strictly dominated or are dominant. Because it is better for a person to keep their light off if one or none of their neighbors has their light on. But it is better for a person to keep their light on if both neighbors have their light on. Therefore, the best strategy is dependent upon what the neighbors do.
(d) N.E. are when all lights are on or all lights are off because no one would change their strategy in those cases without lessening their utility.

4. Borrow-a-Book Network Game (30 points) A *network game* (N, g) is a special class of game in which

- the set of players N is connected by a network g,
- the set of available strategies for each player is $\{0, 1\}$, and
- the payoff function $u_i(x_i, x_{N_i(g)})$ of each player i depends only on the action x_i of player i and the actions[1] $x_{N_i(g)}$ of her neighbors $N_i(g)$ in the network (N, g).

Each player in a *borrow-a-book network game* chooses whether to buy a book. If a player does not buy the book, then he can freely borrow it from any of his neighbors who bought it. Indirect borrowing is not permitted, so the player cannot borrow a book from a neighbor of a neighbor. If none of a player's neighbors has bought the book, then the player would prefer to pay the cost of buying the book himself rather than not having access to the book. This is problem is a classic *free rider* problem, but defined on a network. Formally:

$$u_i(1, x_{N_{i(g)}}) = 1 - c$$
$$u_i(0, x_{N_{i(g)}}) = \begin{cases} 1 & \text{if } x_j = 1 \text{ for some } j \in N_i(g) \\ 0 & \text{if } x_j = 0 \text{ for all } j \in N_i(g) \end{cases}$$

where $1 > c > 0$ denotes the cost of buying the book.

a. (5 points) Characterize the set of Nash Equilibria of a borrow-a-book game in which the players are linked with a clique.

b. (5 points) Characterize the set of Nash Equilibria of a borrow-a-book game in which the players are linked with a ring.

[1] $x_{N_i(g)}$ is the vector that contains all elements of x that correspond to the neighbors of i

c. **(10 points)** For any network G, let $m(G)$ be the minimum number of books bought in any equilibrium of the borrow-a-book game in which players are linked by G. Suppose we obtain G' by adding some links to G. Is it possible that $m(G') > m(G)$?

d. **(10 points)** Consider a borrow-a-book game whose players are linked by graph $G = (N, g)$. Give a condition on $N' \subset N$ that guarantees that the set of players N' are the only ones buying the book in some equilibrium.

Solution by Nihal Gowravaram:

- We proceed by primarily taking this problem as a thought experiment (won't be nearly as formal in considering mixed equilbria as in the previous problem).

a. If the graph is a clique, then every vertex is connected to every other vertex. Therefore, if there are multiple book purchasers, it is advantageous for a book purchaser to instead borrow the book from another purchaser. In contrast, if there are no book purchasers, it is advantageous for an individual to buy the book (since $c < 1$). Therefore, from these two observations, the set of Nash Equilibria are $\boxed{(x_1, \cdots, x_N) \text{ where exactly one } x_i = 1 \text{ (otherwise 0)}}$. (Note I ignore mixed equilibria here, as they would be far too messy to enumerate).

b. If the graph is a ring, then it is only connected to its two adjacent neighbors. Therefore, if a neighbor purchases the book, it is advantageous to borrow instead of purchasing. In constrast, if neither neighbor has purchased the book, it is advantageous to purchase the book. Hence, the equilibria will be situations in which one purchases the book, or a neighbor has purchased the book. Therefore, from these two observations, the set of Nash Equilbiria are of form $(x_1, \cdots, x_N) = \{(1, 0, 1, 0, \cdots)\}, \{(0, 1, 0, 1, \cdots)\}, \{(1, 0, 0, 1, 0, \cdots)\}, \{(0, 1, 0, 0, 1, \cdots)\}$, etc - essentially any strategy in which their are either 1 or 2 non-purchasers between every 2 purchasers of the book. (Note I ignore mixed equilibria here, as they would be far too messy to enumerate).

c. We claim it is indeed possible. Consider the following example below (Figure 1).

Note that in the first graph (without the extra edge) the nodes in red represent the equilbria book purchasers in the equilbria with the fewest number of book purchasers (2). However, note that after adding an edge, we have connected the two book purchasers. Hence, since they are neighbors, both will not purchase the book in any equilibria (otherwise advantageous to borrow). Hence, at most one will purchase the book, forcing the other's 3 neighbors to each purchase the book as well. Hence, in this new graph, the minimum number of books bought in equilibrium is 4. Hence it is indeed possible that $m(G') > m(G)$.

d. The set of players N' are the only ones buying the book in some equilibrium if

- For every player $i \in N \backslash N'$, there exists $j \in N_i(g) \cap N'$ such that $x_j = 1$ (i.e. every player not in set is neighbors with someone in the set).
- For every player $i \in N'$, $N_i(g) \cap N' = \emptyset$ (i.e. no two neighbors in the set are neighbors).

The conditions above guarantee that the set of players N' are the only ones buying the book in some equilibrium.

Figure 1: Counterexample for 4c.