Problem set 1

1\. a. distance (e, i) = 5

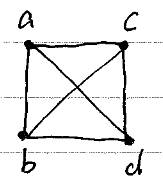
1\. b. diameter = 3

1\. c. average path length $= \frac{1}{\binom{4}{2}}$ [distance (a, b) + distance (a, c) + distance (a, d) + distance (b, c) + distance (b, d) + distance (c, d)]

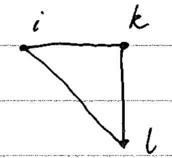
$= \frac{1}{6}[1+1+1+2+2+2]$

$= \frac{3}{2}$

2a. I find 2 cliques of network B:



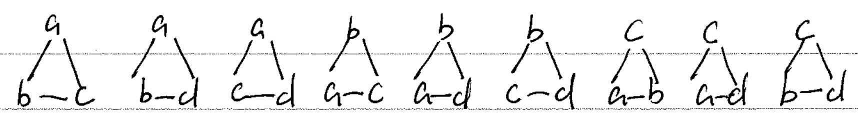
clique 1



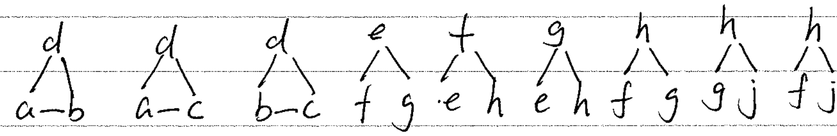
clique 2

2b. clustering coefficient of network B:

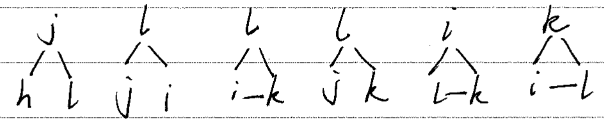
· common node:
· neighbors:



· common node:
· neighbors:



· common node:
· neighbors.



There're 24 pairs of nodes that are linked to a common node, and only 15 of these are themselves linked, therefore, the clustering coefficient is $\frac{15}{24} = \frac{5}{8}$.

3. Since $S \cup (V \setminus S) = V$, nodes in V are either in S or $V \setminus S$, edges leaving the set S must have one endpoint in S and another endpoint in $V \setminus S$. Therefore, $|e(S)|$ the number of edges leaving the set S is the number of edges crossing the cut $C(S, V \setminus S)$, that is, $|C(S, V/S)|$. For the same reason, $|e(V \setminus S)| = |C(S, V \setminus S)|$.

We can break down the definition as:

$$\min_{S \subseteq V} \frac{|C(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}} = \min\left\{ \min_{\substack{S \subseteq V \\ 1 \le |S| \le \frac{n}{2}}} \frac{|C(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}}, \min_{\substack{S \subseteq V \\ \frac{n}{2} \le |S| \le n}} \frac{|C(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}} \right\}$$

① For $1 \le |S| \le \frac{n}{2}$, $\min\{|S|, |V \setminus S|\} = |S|$

$$\Rightarrow \min_{\substack{S \subseteq V \\ 1 \le |S| \le \frac{n}{2}}} \frac{|C(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}} = \min_{\substack{S \subseteq V \\ 1 \le |S| \le \frac{n}{2}}} \frac{|e(S)|}{|S|}$$

② For $\frac{n}{2} \le |S| \le n$, $1 \le |V \setminus S| \le \frac{n}{2}$

$$\Rightarrow \min\{|S|, |V \setminus S|\} = |V \setminus S|$$

$$\Rightarrow \min_{\substack{S \subseteq V \\ \frac{n}{2} \le |S| \le n}} \frac{|C(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}}$$

$$= \min_{\substack{S \subseteq V \\ 1 \le |V \setminus S| \le \frac{n}{2}}} \frac{|e(V \setminus S)|}{|V \setminus S|}$$

Let $S' = V \setminus S$ $$= \min_{\substack{S' \subseteq V \\ 1 \le |S'| \le \frac{n}{2}}} \frac{|e(S')|}{|S'|}$$

From ① + ② $$\Rightarrow \min_{S \subseteq V} \frac{|C(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}}$$

$$= \min\left\{ \min_{\substack{S \subseteq V \\ 1 \le |S| \le \frac{n}{2}}} \frac{|e(S)|}{|S|}, \min_{\substack{S' \subseteq V \\ 1 \le |S'| \le \frac{n}{2}}} \frac{e(S')}{|S'|} \right\}$$

$$= \min_{\substack{S \subseteq V \\ 1 \le |S| \le \frac{n}{2}}} \frac{|e(S)|}{|S|}$$

4 a. There're $\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4\times 3}{2} = 6$ possible links

(i) $P(\text{no edge}) = (1-p)^6$

(ii) $P(\text{one edge}) = \binom{6}{1} p(1-p)^5 = 6p(1-p)^5$

(iii) $P(\text{more than one edge})$

$= 1 - P(\text{no edge}) - P(\text{one edge})$

$= 1-(1-p)^6 - 6p(1-p)^5$

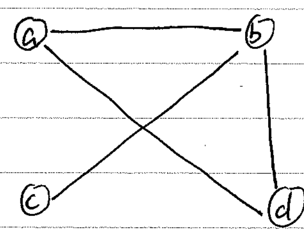
4 b. Let d_i denote the (random) degree of node i $(i=1,2,3,4)$, d_i can take any value in $\{0,1,2,3\}$. The probability that $i \in V$ has exactly d links is:

$$P(d_i = d) = \binom{3}{d} p^d (1-p)^{3-d} = \binom{3}{d} \times 0.5^3$$

Therefore, the expected degree of each node in $G(4, 0.5)$ is:

$$E(d_i) = 0\times P(d_i=0) + 1\times P(d_i=1) + 2\times P(d_i=2) + 3\times P(d_i=3)$$
$$= 1\times 3\times 0.5^3 + 2\times 3\times 0.5^3 + 3\times 1\times 0.5^3$$
$$= 0.5^3(3+6+3)$$
$$= 1.5$$

4 c.



Pset1 Question 4. d

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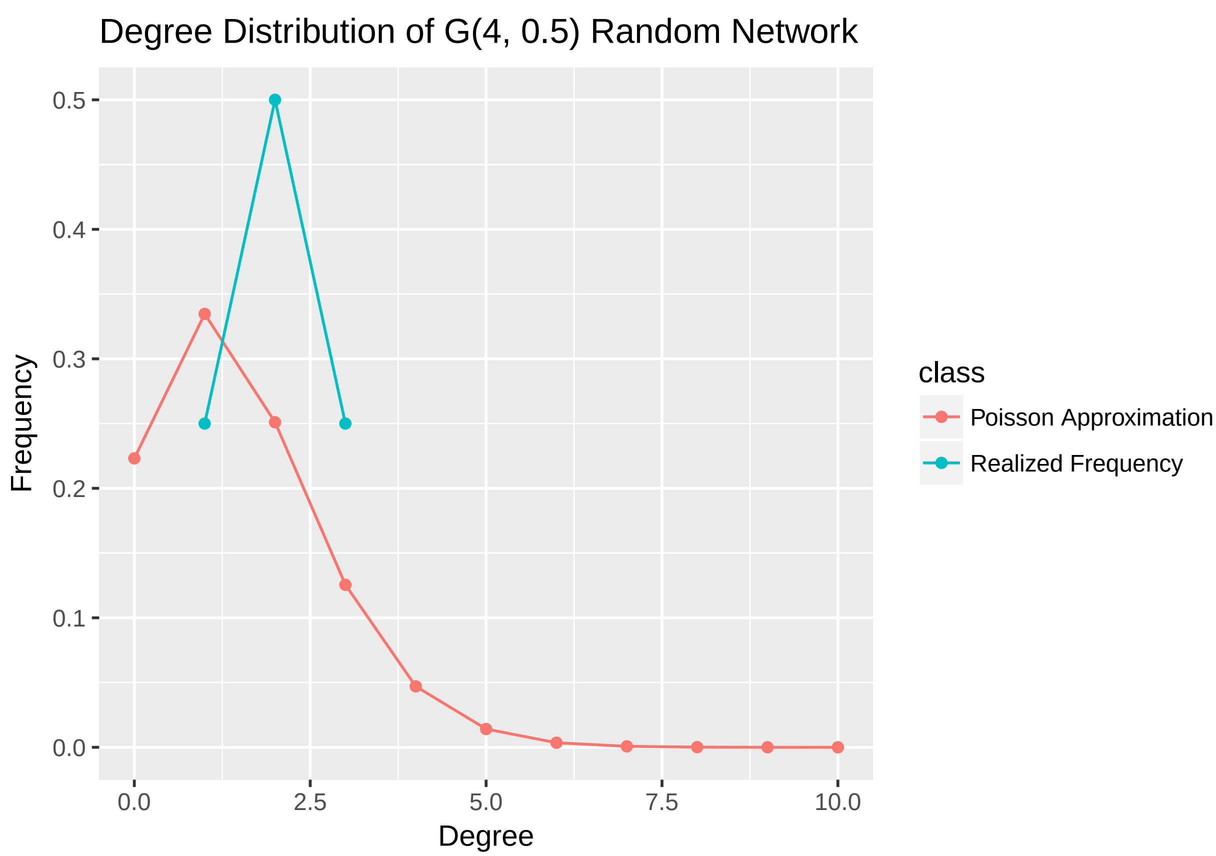
4 d

Frequency distribution of the randomly genrated network and the Poisson approximation for a probability of 0.5 on each link.

```
dat<-data.frame(degree=c(c(1,2,3),seq(0,10,1)),
                prob=c(c(0.25,0.5,0.25),dpois(seq(0,10,1),lambda = 1.5)),
                class=c(rep("Realized Frequency",3),rep("Poisson Approximation",11)))

library(ggplot2)
ggplot(dat,aes(x=degree, y=prob, color=class))+
  geom_point(aes())+geom_line()+
  xlab("Degree")+
  ylab("Frequency")+
  ggtitle("Degree Distribution of G(4, 0.5) Random Network")
```



Comment on the quality of the approximation: The Poisson distribution does not approximate the realized random network well, for such a small n=4 and large probability of link formation p=0.5.

5.a. Let $x \in [1, +\infty)$ $f(x) = (1+\frac{1}{x})^x$

$\Rightarrow \ln f(x) = x \ln(1+\frac{1}{x})$

$\Rightarrow \frac{1}{f(x)} f'(x) = \ln(1+\frac{1}{x}) + x \frac{1}{1+\frac{1}{x}} (-\frac{1}{x^2})$

$= \ln(1+\frac{1}{x}) - \frac{1}{x+1}$ ①

Let $g(x) = \ln(1+\frac{1}{x}) - \frac{1}{x+1}$

$g'(x) = \frac{1}{1+\frac{1}{x}}(-\frac{1}{x^2}) + \frac{1}{(x+1)^2} = \frac{1}{(x+1)^2} - \frac{1}{x(x+1)} = -\frac{1}{x(x+1)^2}$

for $x \geq 1$, we have $g'(x) < 0$, therefore, $g(x)$ is monotonically decreasing.

And we also know $\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} \ln(1+\frac{1}{x}) - \frac{1}{x+1} = 0$

Therefore, for $x \geq 1$, $g(x) > 0$ ②

From ①: $f'(x) = f(x) \cdot g(x)$

> 0, for $x \geq 1$

Therefore, $f(x)$ is monotonically increasing, for $x \geq 1$.

$\Rightarrow f(x) \geq f(1)$, for $x \geq 1$

$\Rightarrow (1+\frac{1}{x})^x \geq (1+\frac{1}{1})^1 = 2$, for $x \geq 1$

$\Rightarrow$ Fix an integer $k \geq 1$, $(1+\frac{1}{k})^k \geq 2$

5.b. $[(1+\frac{1}{k})^k]^n \geq 2^n$

$\geq 2^{20}$ (for $n \geq 20$)

$= 2^{10+10} = 2^{10} \times 2^{10} = 1024 \times 1024$

$> 1000 \times 1000$

$= 10^3 \times 10^3$

$= 10^6$

5.c. Return $= 1 \times (1+p)^d = \left(1+\frac{1}{\frac{1}{p}}\right)^{\frac{1}{p}\cdot pd}$

Since $0 \leq p \leq 1$ (normally I would imagine nobody is willing to give $p > 1$)

$\frac{1}{p} \geq 1$, Let $k = \frac{1}{p} \geq 1$

$\Rightarrow$ Return $= \left[\left(1+\frac{1}{k}\right)^k\right]^{pd}$

From 15b) we know that $pd \geq 20$ implies $\left[\left(1+\frac{1}{k}\right)^k\right]^{pd} \geq 1$ million

Therefore, $d \geq \frac{20}{p}$ implies return ≥ 1 million.

It would take you ceiling$\left(\frac{20}{p}\right)$ of days to become a millionaire.

For example, if daily compound interest rate is 5%, then it only takes 400 days to become a millionaire (a little longer than a year!).