CS 134: Networks

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1 Overview

In today's lecture we will be discussing an extension of matching networks. Whereas previously we assumed that users had binary preferences (matches were either acceptable or not), we consider situations in which users have non-binary preferences for matches. Furthermore, our previous study of matching assumed the role of an "administrator" that would compute the best matches. We are also interested in scenarios where matches occur optimally without any central administrator, the main mechanism of which are prices and payoff maximization.

2 Recap

We briefly review the results for perfect matching in bipartite graphs from last lecture.

Definition. *A graph $G = (N, E)$ is* ***bipartite*** *iff it can be partitioned into two disjoint subsets of nodes U, V such that (1) $U \cup V = N$ and (2) $\forall e \in E, \exists (u, v) \in U \times V : e = (u, v)$. In words, every node is in one of the two subsets and every edge goes between the two subsets (there are no edges within one of the subsets).*

Definition. *Let $G = (U, V, E)$ be a bipartite graph with subsets U and V. Then when $|U| = |V|$, a* ***perfect matching*** *is an assignment of nodes in U to nodes in V such that (1) $u \in U$ is assigned to $v \in V \iff (u, v) \in E$ and (2) no two nodes in U are assigned to the same $v \in V$.*

Definition. *A set of vertices $S \subseteq V$ is called a* ***constricted set*** *if $|S| > |N(S)|$ where V is one side of the bipartite graph and $N(S)$ is the neighborhood of S.*

Theorem 1 (Matching Theorem)**.** *A bipartite graph (with $|U| = |V|$) has a perfect matching iff it does not contain a constricted set of vertices.*

3 Model

3.1 Prices

We describe our setting formally as a set of buyers trying to buy a good from a set of sellers. Sellers are parameterized by prices, and buyers are parametrized by valuations for all sellers:

Definition. *A seller i sells an items at* ***price*** *$p_i \in \mathbb{N}$.*

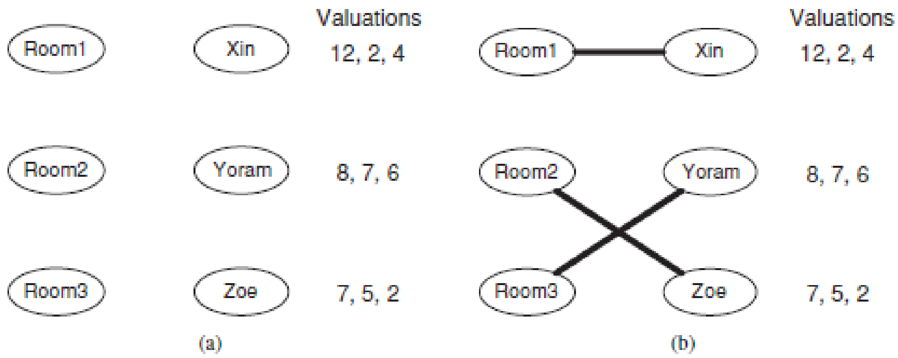

Figure 10.3. (a) A set of valuations. Each person's valuations for the objects appear as a list next to him or her. (b) An optimal assignment with respect to these valuations.

Definition. *A buyer j's* ***valuation*** *for seller i's item is denoted $v_{ij} \in \mathbb{N}$. Each buyer has a valuation for each seller.*

For simplicity we assume prices and valuations are nonnegative whole numbers. Seller i gets payoff p_i if they sell their item, 0 otherwise. Buyers seek to maximize their payoff:

Definition. *The* ***payoff*** *of buyer j from buying from seller i is $v_{ij} - p_i$. If the buyer has negative payoff from all sellers, then they prefer not to buy anything, which gives payoff 0.*

The notion of buyers' payoffs gives rise to the idea of a buyer's preferred seller:

Definition. *A buyer's* ***preferred seller*** *is the seller (possibly none) from whom buying maximizes that buyer's payoff, with ties broken arbitrarily. The set of buyers' preferred sellers induces a the* ***preferred-seller graph****, a bipartite graph where the nodes set is the union of the set of buyers and sellers and the edges are between each buyer and their preferred seller.*

Note that this model generalizes our previous matching problem with prices by saying that each buyer has valuation 1 for all sellers it will accept, 0 otherwise.

3.2 Market Clearing

Our ultimate goal in this model is to match each buyer with a seller, which we call "market-clearing":

Definition. *A set of prices market-clearing if each buyer has a different preferred seller. Equivalently, a set of prices is market-clearing if there is a perfect matching in the induced preferred-seller graph.*

In Figure 10.5, only (d) has market-clearing prices; (c) does not clear because y is ambivalent between all sellers and thus clearing the market requires extra coordination between buyers.

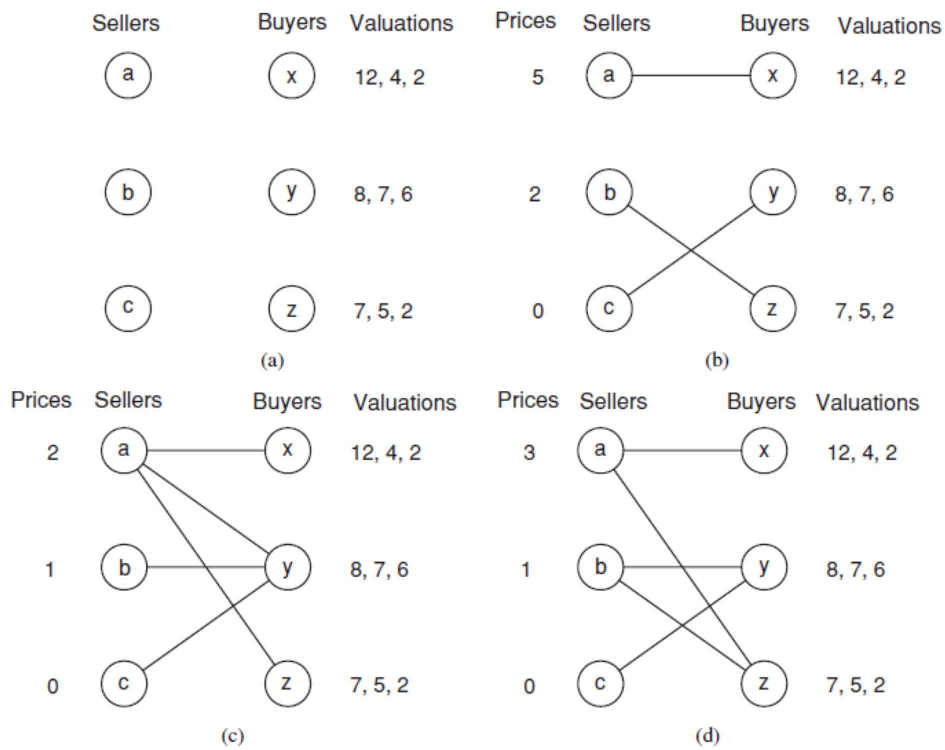

Figure 10.5. (a) Three sellers (a, b, and c) and three buyers (x, y, and z). For each buyer node, the valuations for the houses of the respective sellers appear in a list next to the node. (b) Each buyer creates a link to her preferred seller. The resulting set of edges is the preferred-seller graph for this set of market-clearing prices. (c) The preferred-seller graph for prices 2, 1, and 0 (prices that don't clear the market). (d) The preferred-seller graph for prices 3, 1, and 0 (market-clearing prices, where tie-breaking is required).

4 Main Theorem

We now arrive at the main theorems for today:

Theorem 2 (Existence of Market-Clearing Prices)**.** *For any set of buyer valuations, there exists a set of market-clearing prices that produces a socially optimal outcome, meaning that the resulting perfect matching in the resulting preferred-seller graph has the maximum possible sum of payoffs to all sellers and buyers.*

Theorem 3 (Optimality of Market-Clearing Prices)**.** *For any set of market-clearing prices, a perfect matching in the resulting preferred-seller graph has the maximum possible valuation of sellers to buyers.*

We will prove the first theorem by providing an algorithm that constructs market-clearing prices for any set of buyer valuations, but refrain from doing so now.

The second theorem says that given a set of buyer valuations and market-clearing prices for those valuations, a perfect matching in the preferred-seller graph maximizes the sum of payoffs to all sellers and buyers. We prove this property by considering the problem of maximizing total buyers' payoffs over the set $\mathbf{M}$ of all possible matches of buyers to sellers. Let $\mathbf{B}$ be the set of buyers, u_b be the payoff of buyer b, and denote by $M(b)$ the seller that buyer b buys from in matching M.

$$\max_{M \in \boldsymbol{M}} \sum_{b \in \mathbf{B}} u_b \tag{1}$$

$$= \max_{M \in \boldsymbol{M}} \sum_{b \in \mathbf{B}} v_{b,\mathbf{q}(b)} - p_{\mathbf{q}(b)} \tag{2}$$

$$= \max_{M \in \boldsymbol{M}} \sum_{b \in \mathbf{B}} v_{b,\mathbf{q}(b)} - \sum_{b \in \mathbf{B}} p_{\mathbf{q}(b)} \tag{3}$$

We observe that the prices set are independent of the matching, so maximizing buyer payoff is equivalent to maximizing the sum of buyers' valuations of their matches, $\sum_{b \in \mathbf{B}} v_{b,\mathbf{q}(b)}$. Since each buyer is individually maximizing and choose different sellers in a perfect matching, then the perfect matching maximizes buyer payoffs, and therefore buyer valuations, given the buyer valuations and seller prices.

4.1 Constructing Market Clearing Prices

We now return to prove the first theorem, which we do so by providing an algorithm, FindMarketClearingPrices, to construct market-clearing prices for any set of buyer valuations:

```
procedure FindMarketClearingPrices
    valuations v
    prices p ← 0
    while true do
        p ← p − min_i p_i
        G ← PreferredSellerGraph(p, v)
        if G has a perfect matching then
            break
        end if
        S, N(S) ← FindConstrictedSet(G)
        for all s ∈ N(S)  do
            p_s ← p_s + 1
        end for
    end while
    return p
end procedure
```

Algorithm FindMarketClearingPrices does the following. We start with all prices set to zero. Then while we do not have a perfect matching, we repeat the following: we zero the prices by subtracting off the lowest price. We then find a constricted set of buyers in the preferred-seller graph and increment the prices of all sellers in the neighborhood of the set by 1. If we cannot find a constricted set, then we have a perfect matching. We show an example of this algorithm being run in figure 10.6. In order to conclude the proof, we need to show that the algorithm terminates.

We do so by considering a "potential energy" in the market that is decreasing as the algorithm proceeds and must eventually reach zero. We define the *potential of a buyer*, given a set of prices, to be the maximum payoff they can get from any seller. The *potential of a seller* is the current price they are charging. The *potential energy of the market* is the sum of all buyers' and sellers' potentials. Note that prices can't go below zero and buyers always have the option of not buying, so the potential of buyers, sellers, and overall is lower-bounded by zero.

Since all prices begin at zero, the potential of all sellers is zero and the potential of all buyers is their maximum valuation, the sum of which we denote P_0. The potentials only change when we zero the prices or increment the prices. Note that zeroing the prices does not change the potential of the market, as the sellers' decrease in potential is offset by a commensurate increase in buyers' potential. When we increment the prices by 1, there is a similar effect, except that the size of the set of buyers affected is smaller than the size of the set of sellers affected because we have constricted. Thus, the potential in the market actually decreases by at least 1.

Therefore, since we start with some potential P_0, cannot have potential less than zero, and decrease by at least one unit of potential with every iteration of the algorithm, the must algorithm must terminate, and when it does we have a perfect matching and market-clearing prices.

5 Auctions as Markets

The decentralization of matching markets via prices is a powerful tool that allows us to model many scenarios. We consider the single-item auction, for which there are a group of n bidders that are bidding for a single item. Each bidder has a private valuation v_b for the item, and the item is given to the bidder with the highest bid, with ties broken arbitrarily. Our study of matching has thus far only considered bipartite graphs where the sizes of the partitions are equal. In order to maintain this form, we create $n-1$ dummy nodes, each of which have value 0 to each bidder.

We can then use FindMarketClearingPrices in order to run the auction. The price of the item begins at 0, at which all bidders list the item as their preferred seller, creating a constricted set. The price of the item increases until a seller drops out, listing a dummy item as their preferred seller and signifying that they have "dropped out" of the auction. The price of the real item continues to increase until all but one bidder has dropped out, who then wins the auction. The price they must pay is the highest second valuation of the item, which is the exact price at which the second to last bidder will drop out. In fact, this type of auction is known as a Dutch auction, and the results are consistent with those in game theory: the way to get each bidder to bid their own valuation truthfully is to have the winner pay the second highest bid. This entire process is depicted in Figure 10.7.

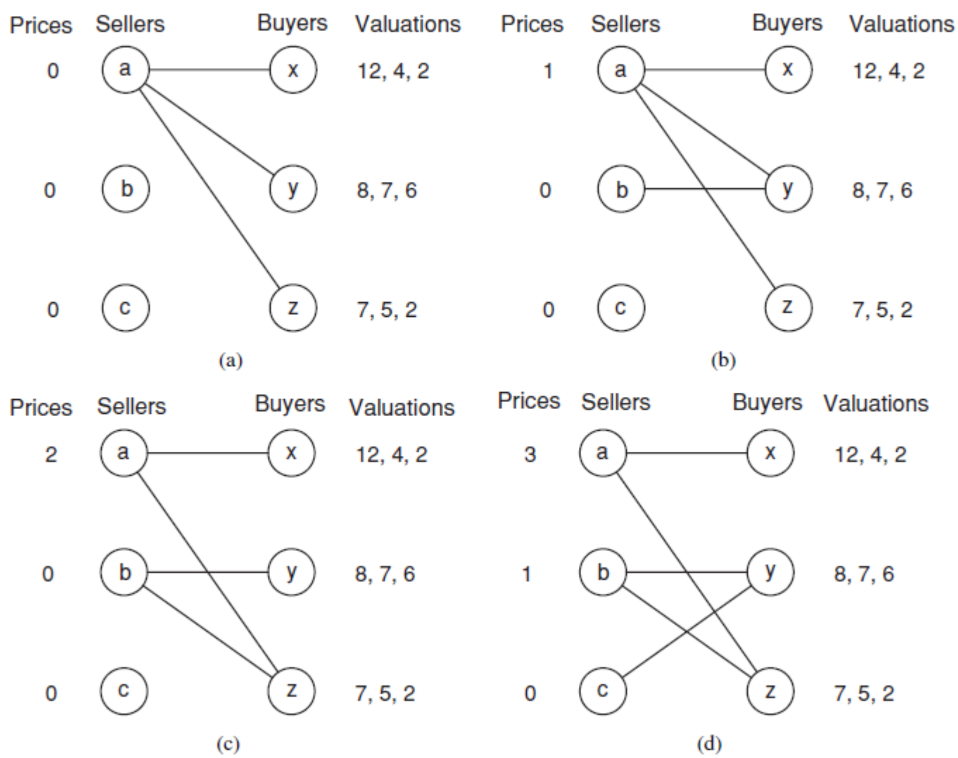

Figure 10.6. The auction procedure applied to the example from Figure 10.5. Each separate picture shows steps (i) and (ii) of successive rounds, in which the preferred-seller graph for that round is constructed. (a) In the first round, all prices start at 0. The set of all buyers forms a constricted set S, with $N(S)$ equal to the seller a. So a raises his price by one unit and the auction continues to the second round. (b) In the second round, the set of buyers consisting of x and z forms a constricted set S, with $N(S)$ again equal to seller a. Seller a again raises his price by one unit and the auction continues to the third round. (Notice that in this round, we could have alternatively identified the set of all buyers as a different constricted set S, in which case $N(S)$ would have been the set of sellers a and b. There is no problem with this – it just means that there can be multiple options for how to run the auction procedure in certain rounds, with any of these options leading to market-clearing prices when the auction comes to an end.) (c) In the third round, the set of all buyers forms a constricted set S, with $N(S)$ equal to the set of two sellers a and b. So a and b simultaneously raise their prices by one unit each, and the auction continues to the fourth round. (d) In the fourth round, when we build the preferred-seller graph, we find it contains a perfect matching. Hence, the current prices are market clearing, and the auction comes to an end.

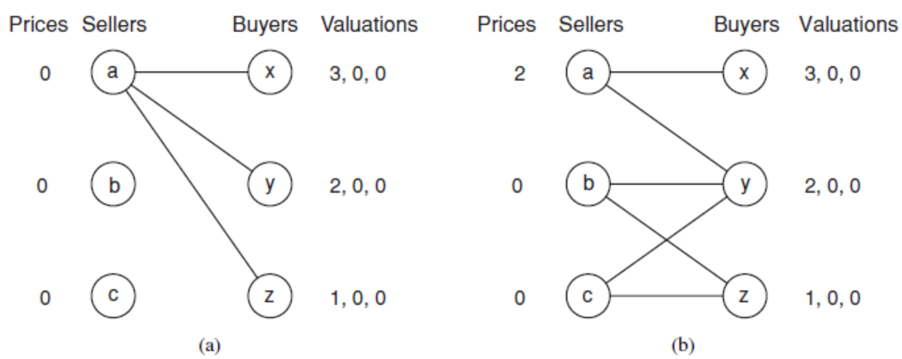

Figure 10.7. A single-item auction can be represented by the bipartite graph model: the item is represented by one seller node, and then there are additional seller nodes for which all buyers have 0 valuation. (a) The start of the bipartite graph auction. (b) The end of the bipartite graph auction, when buyer x gets the item at the valuation of buyer y.