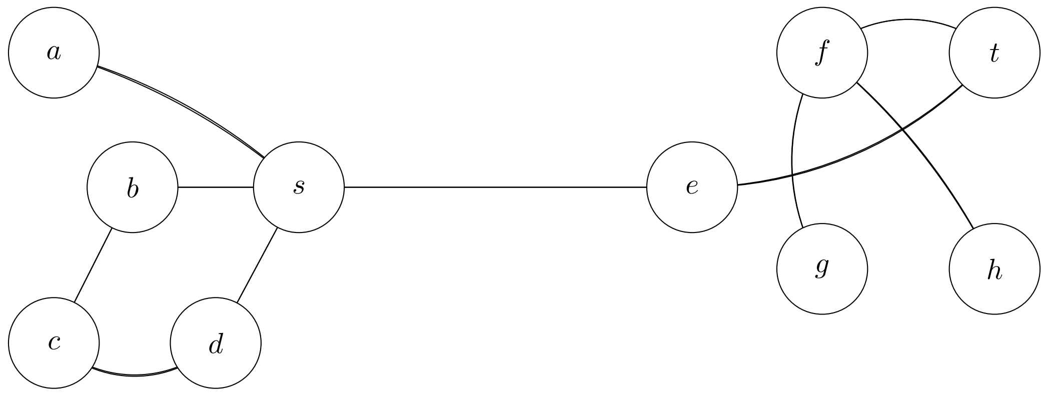
CS 134: Networks	Practice Midterm Solutions
Prof. Yaron Singer	**Friday 2/10/2017**

1. Graph Terms.



a. How many connected components are there in this graph?

Solution. There is one connected component in the graph.

b. Write down the neighborhood of $\{s, t\}$.

Solution. The neighborhood of $\{s, t\}$ is

$$\boxed{N(\{s,t\}) = \{a, b, d, e, f\}}$$

The solution $N(\{s,t\}) = \{a, b, d, e, f, s, t\}$ is admissible as well.

c. Compute the expected distance from t to a node v in the graph chosen uniformly at random.

Solution. The expected distance from t to random node in the graph $G = (V, E)$ is the average of all distances:

$$\begin{aligned}
\mathbf{E}_{v\sim V}[d(t,v)] &= \left(\frac{1}{|V|}\sum_{v\in V} d(t,v)\right) \\
&= \frac{d(t,a)+d(t,b)+d(t,c)+d(t,d)+d(t,e)+d(t,f)+d(t,g)+d(t,h)+d(t,s)+d(t,t)}{10} \\
&= 3+3+4+3+1+1+2+2+2+0 = \frac{21}{10}
\end{aligned}$$

$$\boxed{\mathbf{E}_{v\sim V}[d(t,v)] = \frac{21}{10}}$$

d. What is the size of the smallest cut in this graph?

Solution. A smallest cut in the graph is

$$C(\{a,b,c,d,s\}, \{e,f,g,h,t\}) = \{se\}$$

which has size

$$\boxed{|C(\{a,b,c,d,s\}, \{e,f,g,h,t\})| = 1.}$$

2. True or False. For each statement, indicate whether it is true or false. If it is true, nothing else is required. If it is false, correct the statement or give a counterexample.

a. Consider an Erdős-Rényi random graph $G(n, p)$, with $n \geq 3$ nodes and probability $p > 0$ of any link forming. The probability that the graph is connected is at least p^M, where $M = n(n-1)/2$ is the number of potential edges.

Solution. True. p^M is the probability that every possible edge exists; thus, this is the probability that the graph is *completely* connected. Since the set of completely connected graphs is a subset of the set of connected graphs for a given n, the probability of an Erdős-Rényi random graph being connected is at least the probability of it being completely connected, which is p^M.

b. In the BFS algorithm, the first node that enters the queue (excluding the source itself) is at distance 2 from the source.

Solution. False. the first nodes that enter the queue after the source are the nodes that are neighbors of the source, meaning that they are at distance 1 from the source (by the definition of being neighbors).

c. If n is even and G is a clique of n nodes, the expansion of G is n.

Solution. False. If n is even and G is a clique of n nodes, then a subset S of the vertices with $|S| = k$ has $k(n-k)$ edges leaving it, since each of the k vertices in S has an edge with each of the $(n-k)$ vertices not in S. Therefore, the expansion of G is

$$\alpha = \min_{1 \leq k \leq n/2} \frac{k(n-k)}{k} = \min_{1 \leq k \leq n/2} [n-k] = \frac{n}{2}$$

With the min occurring at $k = n/2$. Thus, the expansion of G is $n/2$, not n.

d. Let G be a two-dimensional square lattice of dimensions $r \times r$. Let d be a natural number smaller than r, and let v be a fixed node in G. The number of nodes at distance d or less of v in G is $\Theta(d^2)$.

Solution. True. Note that the nodes that are at a distance of exactly d from node v form a d-by-d square around v (assuming that v is not less than d distance away from the sides of the lattice). Thus, the number of nodes at exactly distance d is no greater than the perimeter of a d-by-d square, which is $4d$. Therefore, the number of nodes $n_v(d)$ within distance d or less of v is

$$\begin{aligned}
n_v(d) &= 1 + \sum_{i=1}^{d} (\text{nodes at exactly distance } i \text{ from } v) \\
&\leq 1 + \sum_{i=1}^{d} 4d = 1 + 4\sum_{i=1}^{d} d \\
&= 1 + \frac{4d(d+1)}{2} = 1 + 2d(d+1) \\
&= 2d^2 + 2d + 1 \\
\boxed{n_v(d) = \Theta(d^2)}
\end{aligned}$$

3. Power law. Let X be a discrete random variable such that

$$\mathbb{P}[X = d] = \begin{cases} c \cdot d^{-2} & \text{if } d \in \{1, 2, \ldots, \Delta\} \\ 0 & \text{otherwise.} \end{cases}$$

Here Δ is an integer and $c > 0$ is a constant that depends on Δ. This constant is set to a value that makes this a probability distribution (i.e., makes the probabilities sum up to exactly 1).

a. Indicate which of the following describes the behavior of c as a function of Δ:

(i) As $\Delta \to \infty$ we have that $c \to +\infty$.

(ii) As $\Delta \to \infty$ we have that $c \to 0$.

(iii) As $\Delta \to \infty$ we have that c converges to a finite, nonzero constant.

(iiii) None of the above.

Solution: We need that the probability distribution sums to one, i.e.

$$\sum_{d=1}^{\Delta} p(X = d) = 1$$

$$\sum_{d=1}^{\Delta} c \cdot d^{-2} = 1$$

$$c \sum_{d=1}^{\Delta} d^{-2} = 1$$

$$c = \frac{1}{\sum_{d=1}^{\Delta} d^{-2}}$$

By a standard fact from calculus, $\sum_{d=1}^{\Delta} d^{-2}$ will converge to some finite, nonzero constant as $\Delta \to \infty$, and if we take the reciprocal of that to find c,, we also get a finite, nonzero constant, so the answer is $\boxed{(iii)}$. In fact, that exact value that c will approach is $\frac{6}{\pi^2}$.

b. Let m_Δ denote the expected value of X for a given value of the constant Δ. Write a formula for m_Δ and explain briefly why $m_\Delta = O(\Delta)$ as $\Delta \to \infty$. You may freely (without explanation) use the approximations we have learned for summations.

Solution: From the definition of expected value, we have

$$\mathbb{E}[X] = \sum_{d=1}^{\Delta} d \cdot p(X - d)$$

$$= \sum_{d=1}^{\Delta} d(c \cdot d^{-2})$$

$$= c \sum_{d=1}^{\Delta} d^{-1}$$

We can treat c as a constant and we use the approximation that $\sum_{x=1}^{N} \frac{1}{x} \approx \log(N)$ to see that $\mathbb{E}[X] = O(\log(\Delta)) = O(\Delta)$.

4. Configuration model. Consider a configuration model defined on 3 nodes and degree distribution $\{2, 1, 1\}$.

a. What is the exact probability that there is a self-edge in the graph?

Solution: The probability that there is a self-edge in the graph is the probability that both half-edges out of the node with degree 2 connect to each other. The total number of possible configurations is

$$\frac{\binom{4}{2}}{2!} = 3$$

and exactly one of these has both half-edges connected to each other. Therefore, the answer is $\boxed{\frac{1}{3}}$.

b. What is the exact probability that the graph is connected?

Solution: The probability that the graph is connected is the probability that there is no self-edge. Therefore, using our answer from above, the answer is $1 - \frac{1}{3} = \boxed{\frac{2}{3}}$.

5. Diameter. Let G be an $r \times r$ square lattice.

a. Write its diameter as a function of r.

Solution: Note that in order to get from one corner (without loss of generality, the upper-left) to the diagonally opposite one (in this case, the lower-right), you have to take $r-1$ steps right and $r-1$ steps down along the lattice. Therefore, the distance between these two corners is $2(r-1)$, and because that's the greatest distance between any two vertices in the lattice, the diameter of the lattice is $\boxed{2(r-1)}$.

b. Let v be the node at a corner. How many nodes (in terms of r) are there at distance r from v? (Here, distance refers to geodesic distance in the graph G, which is the same thing as lattice distance.)

Solution: Without loss of generality, let's choose to start at the lower-left corner. The left-most node that is r steps away is $r-1$ steps up and 1 step right. The right-most node that is r steps away is 1 step up and $r-1$ steps right. Therefore, because the number of steps up and the number of steps right between the corner node and any node that is distance r away must sum to r, we know that there are $\boxed{r-1}$ nodes that are distance r away in the lattice (counting from going up $r-1$ steps and right once to going up once and right $r-1$ steps).