CS 134: Networks

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1 Overview

Last lecture, we learned about the history of the web and some primary motivations for ranking Web pages. In the Information Age, designing good algorithms for information retrieval is particularly difficult due to the abundance of seemingly-relevant documents given a query. An understanding of the fundamental network structure of Web pages is crucial for addressing ranking questions, as we now discuss. In particular, we will analyze a ranking algorithm known as Hyperlink-Induced Topic Search (HITS).

2 Hubs and Authorities

Given a sample of "reference" pages containing links to "result" pages that are relevant to a certain query, we can determine the importance of each of the result pages by counting the number of in-links it has from the reference pages (this technique is known as **Voting By In-Links**). Furthermore, we can then determine a *reference* page's value as equal to the sum of the votes received by all pages for which it voted. We can then use these to get still more refined values for the quality of the result pages, and so on, potentially forever. This repeated refinement technique is known as the **Principle of Repeated Improvement**.

More formally, the results pages we were originally seeking - the prominent, highly endorsed answers to the queries - are called the **authorities** for the query. The reference pages are called the **hubs** for the query. We want to find the value of each page p in a network as a potential authority and as a potential hub, which we refer to as a_p and h_p respectively. To do this, we can utilize the following definitions and algorithm, based on the Principle of Repeated Improvement:

Definition. ***Authority Update Rule****: For each page p, update a_p to be the sum of the hub scores of all pages that point to it.*

Definition. ***Hub Update Rule****: For each page p, update h_p to be the sum of the authority scores of all pages that it points to.*

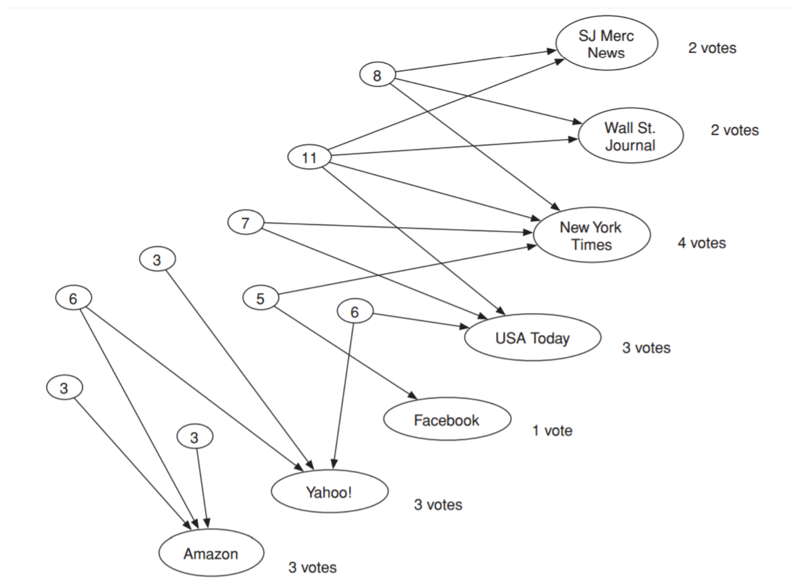


Figure 1: An example of Voting by In-Links. Note that each reference pages value is written as a number inside it (*Kleinberg 14.2*).

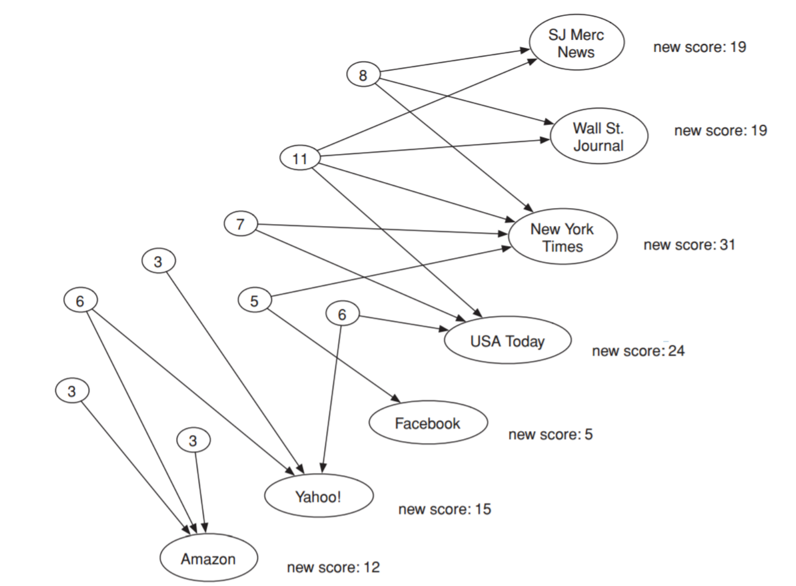


Figure 2: An example of the Principle of Repeated Improvement in action. Each result pages new score is equal to the sum of the values of all lists that point to it (*Kleinberg 14.2*).

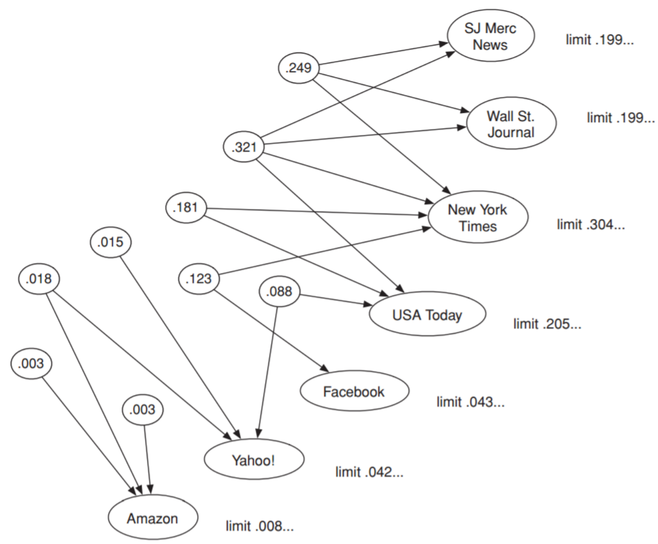


Figure 3: The limit hub and authority values for our in-links example (*Kleinberg 14.2*).

ALG 1: HITS Algorithm
input: Graph G with n nodes, number of steps t
1. Start with all hub scores and all authority scores of each page equal to 1.
2. For t iterations:
First apply the Authority Update Rule to the current set of scores;
Then apply the Hub Update Rule to the resulting set of scores;
End for
3. Normalize by dividing each authority score by the sum of all authority scores.
4. Normalize by dividing each hub score by the sum of all hub scores.
return: The hub and authority scores of each page

Just as we saw with PageRank, we would expect that the normalized values converge to limits as t goes to infinity. In fact, except for a few rare cases (characterized by a certain kind of degenerate property of the link structure), we reach the same limiting values no matter what we choose as the *initial* hub and authority values, provided only that all of them are positive. In other words, the limiting hub and authority values are purely a property of the link structure, not of the initial estimates we use to start the process of computing them. As such, they reflect the balance between hubs and authorities that provided the initial intuition for them: your authority score is proportional to the hub scores of the pages that point to you, and your hub score is proportional to the authority scores of the pages you point to.

3 Spectral Analysis of Hubs and Authorities

Thinking about hubs and authorities in a network, we define an **adjacency matrix** M as follows:

$$M_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Observe from the Hub Update Rule that the hub score of node j is

$$h_j = M_{j1}a_1 + M_{j2}a_2 + \cdots + M_{jn}a_n.$$

Along similar lines, from the Authority Update Rule the authority score of node i is

$$a_i = M_{1i}h_1 + M_{2i}h_2 + \cdots + M_{ni}h_n.$$

Then, letting $h^{(t)}$ and $a^{(t)}$ be the vectors of all hub and authority scores respectively at time step t, following the HITS algorithm we have:

$$a^{(t)} = M^T h^{(t-1)}$$

and

$$h^{(t)} = M a^{(t)}$$

where M^T is the matrix transpose of M. Observe that for the first few steps of the HITS algorithm, we get:

$$a^{(1)} = M^T h^{(0)}$$
$$h^{(1)} = M a^{(1)}$$
$$a^{(2)} = M^T h^{(1)} = M^T M M^T h^{(0)}$$
$$h^{(2)} = (MM^T)^2 h^{(0)}$$

More generally, we infer:

$$a^{(t)} = (M^T M)^{t-1} M^T h^{(0)}$$
$$h^{(t)} = (MM^T)^t h^{(0)}$$

This provides the groundwork for us to demonstrate the convergence of the scores. In other words, we first wish to show that there exists $h^{(*)}$, such that:

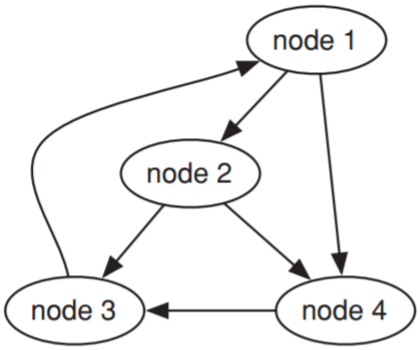
$$\lim_{t\to\infty} \frac{h^t}{c^t} = h^{(*)},$$

where c is a constant. Using our above equations we have:

$$(MM^T)h^{(*)} = c \cdot h^{(*)}$$

The constant c accounts for the fact that at each step, we normalize all the values. So the above claim is saying that the hub scores converge to some vector $h^{(*)}$ and are normalized by a factor $1/c$ at each step. Using linear algebra, we can infer that $h^{(*)}$ and c are an eigenvector and a corresponding eigenvalue of the matrix MM^T. We now utilize the following:

Definition. *(Review) A matrix A is **symmetric** if $A^T = A$.*



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Figure 4: The directed hyperlinks among Web pages can be represented using an adjacency matrix M: the entry M_{ij} is equal to 1 if there is a link from node i to node j; otherwise, $M_{ij} = 0$ (*Kleinberg 14.6*).

Theorem. *If a matrix A is symmetric then there exists n eigenvectors of A that are orthonormal. In other words, these eigenvectors form a* ***basis*** *for A.*

We observe that MM^T is symmetric since $(MM^T)^T = (M^T)^T M^T = MM^T$. Therefore, there exist eigenvectors $z_1, \cdots, z_n$ of MM^T with corresponding eigenvalues $c_1, \cdots, c_n$ that form a basis for MM^T and such that for all $x \in \mathbb{R}^n$, there exist constants $p_1, \cdots, p_n \in \mathbb{R}$ such that

$$\begin{aligned} MM^T x &= MM^T(p_1 z_1 + \cdots + p_n z_n) \\ &= p_1 c_1 z_1 + \cdots + p_n c_n z_n \end{aligned}$$

If we assume without loss of generality that $|c_1| \geq \cdots \geq |c_n|$ and $|c_1| > |c_2|$, we can write $h^{(0)}$ as a linear combination of the eigenvectors:

$$h^{(0)} = q_1 z_1 + \cdots + q_n z_n$$

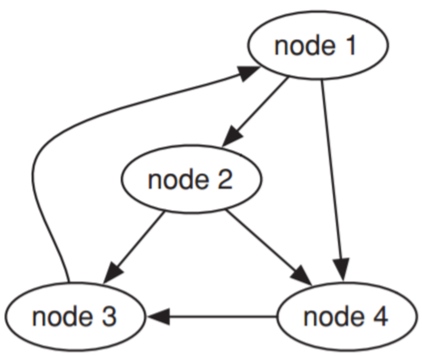
for some set of constants $q_1, \cdots, q_n \in \mathbb{R}$. Thus, we have:

$$h^{(t)} = (MM^T)^t h^{(0)}$$

$$h^{(t)} = c_1^t q_1 z_1 + \cdots + c_n^t q_n z_n$$

$$\frac{h^{(t)}}{c_1^t} = q_1 z_1 + \left(\frac{c_2}{c_1}\right)^t q_2 z_2 + \cdots + \left(\frac{c_n}{c_1}\right)^t q_n z_n.$$

From our assumption that $|c_1| > |c_i|$ for all $i \geq 2$, $\frac{h^{(t)}}{c_1^t}$ thus converges to $h^{(*)} = q_1 z_1$ as t approaches infinity. (The case where $|c_1| = |c_2|$ will be discussed in section.)



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 2 \\ 4 \end{bmatrix}$$

Figure 5: By representing the link structure using an adjacency matrix, the Hub and Authority Update Rules become matrix-vector multiplication. In this example, we show how multiplication by a vector of authority scores produces a new vector of hub scores (*Kleinberg 14.6*).