CS 134: Networks	Section 4
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1 Classic Games

Recall that a game consists of

- a set of players $N = \{a_1, ..., a_n\}$
- a set of strategies for each player S_i
- a payoff (or utility) function for each player $u_i : S_1 \times S_2 \times ...S_n \to \Re$

Two-player games with finite number of strategies are often expressed with a *payoff matrix*. For example, the game Prisoner's Dilemma can be represented in the payoff matrix below.

	Don't Confess	**Confess**
Don't Confess	$-1, -1$	$-4,\ 0$
Confess	$0, -4$	$-3, -3$

2 Nash Equilibrium

In words, a Nash equilibrium is the set of strategies where no player can independently change their strategy and achieve a higher payoff. If the assumptions in lecture are true (agents act rationally, simultaneously, independently, and with full knowledge of the game), then they should choose strategies in a Nash equilibrium, if possible.

2.1 Pure Nash Equilibrium

A *pure-strategy Nash equilibrium* is a profile of strategies $x = (x_1, \dots, x_n)$ such that

$$\begin{aligned} u_i(1, x_{N_i(g)}) \geq u_i(0, x_{N_i(g)}) \quad \text{if } x_i = 1 \\ u_i(0, x_{N_i(g)}) \geq u_i(1, x_{N_i(g)}) \quad \text{if } x_i = 0 \end{aligned}$$

where $x_{N_i(g)}$ is the vector that contains all elements of x that correspond to the neighbors of i.

So the equilibrium condition requires that each player choose the action that offers the highest payoff in response to the actions of her neighbors: no player should regret the choice that she has made given the actions taken by the other players.

2.2 Finding Nash Equilibria

A game can have

- a suboptimal Nash equilibrium (Prisoner's Dilemma)
- multiple Nash equilibria (Battle of the Sexes)
- no Nash equilibria (Matching Pennies)

One useful way to find Nash equilibria in games is through *Iterated Deletion of Strictly Dominated Strategies*.

- A (strictly) *dominant* strategy of a player is a strategy that is a (strict) best response to any set of strategies played by her opponents. That is, s_i is a (strictly) dominant strategy iff it is a (strict) best response to every possible s_{-i}.

- A strategy of a player is *dominated* if there is another strategy that does at least as good against any set of strategies of her opponents, and better for at least one set of opponent strategies. That is, s_i is dominated if there is another strategy s_i' such that $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$ for all possible s_{-i} and $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for some s_{-i}.[1] For a strategy to be *strictly dominated*, there must be another strategy that does strictly better against any possible set of opponent strategies; that is, s_i is strictly dominated if there is another strategy s_i' such that $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for all possible s_{-i}.

2.2.1 Iterated Deletion of Strictly Dominated Strategies

The process of iterated deletion of strictly dominated strategies proceeds as follows:

- We start with any n-player game, find all the strictly dominated strategies, and delete them.

- We then consider the reduced game in which these strategies have been removed. In this reduced game there may be strategies that are now strictly dominated, despite not having been strictly dominated in the full game. We find these strategies and delete them.

- We continue this process, repeatedly finding and removing strictly dominated strategies until none can be found.

Note that we only delete *strictly* dominated strategies; this is in order to preserve the following property: the set of Nash equilibria of the original game coincides with the set of Nash equilibria for the final reduced game, consisting only of strategies that survive iterated deletion of strictly dominated strategies – i.e. the process does not delete any strategy that is part of a Nash equilibrium. Moreover, **when the process of iterated deletion of strictly dominated strategies leads to one outcome, this outcome is the unique Nash equilibrium of the game**.[2]

2.3 Mixed Nash Equilibrium

A *mixed strategy* is a strategy in which a player assigns a probability to each of their actions, and then randomly selects an action based on that probability distribution.

Nash's theorem states that every finite game has a mixed Nash equilibrium.

Exercise. What are the mixed Nash equilibria of Battle of the Sexes and Matching Pennies?

2.4 Network Games

A *network game* (N, g) is a special class of game in which

- the set of players N is connected by a network g,

- the set of available strategies for each player is $\{0, 1\}$, and

[1] Some more notation: u_i is the utility function for player i. So $u_i(s_i, s_{-i})$ gives the payoff for player i if it players strategy s_i against s_{-i}.

[2] Were we to iteratively eliminate weakly dominated strategies in addition to strictly dominated strategies, we would still be guaranteed that at least one Nash equilibrium would remain, but it is possible that some Nash equilibria from the original game are eliminated during the process.

- the payoff function of each player depends only on the actions of her neighbors in the network (N, g).

2.5 Examples

2.5.1 Couples game

Imagine learning a skill that is most easily enjoyed when there is at least one friend to practice it with, such as playing tennis. In the context of network games, this situation can be formalized by assuming that a player prefers to take action 1 if at least one neighbor takes action 1, but prefers to take action 0 otherwise. We can think of this as having a cost c of investing in the skill, along with a benefit $b = 1$ if there is a partner to play with. Formally:

$$u_{d_i}(1, m) = \begin{cases} 1 - c & \text{if } m \geq 1 \\ -c & \text{if } m = 0 \end{cases}$$
$$u_{d_i}(0, m) = 0$$

2.5.2 Triplets game

Imagine learning a skill that is most easily enjoyed when there is at least *two* friends to practice it with, such as playing poker. This situation can be formalized using the semi-anonymous network game in which each player i has the following payoff function:

$$u_{d_i}(1, m) = \begin{cases} 1 - c & \text{if } m \geq 2 \\ -c & \text{if } m < 2 \end{cases}$$
$$u_{d_i}(0, m) = 0$$

2.6 Complements and Substitutes

It will prove very useful to study the following two cases separately:

1. *strategic complementarity*: a player has an *increasing* incentive to take a given action as *more* neighbors take that action. The couples and triplets games are of this type.

2. *strategic substitutability*: a player has a *decreasing* incentive to take a given action as *more* neighbors take that action. The borrow-a-book game is of this type.

3 Questions/ Discussion

1. Characterize the set of Nash Equilibria of an arbitrary couples game on a connected network.

2. Characterize the set of all Nash Equilibria of a triplets game with the network depicted in Figure 1. Can you find a "maximum" and a "minimum" equilibrium?

3. As we will see in lecture, games of strategic complements always have a maximum Nash equilibrium. Can you think of an algorithm that will find the maximum equilibrium of a triplets game for any network?

Figure 1