1 Overview

In today's lecture we will be discussing an extension of matching networks. Whereas previously we assumed that users had binary preferences (matches were either acceptable or not), we consider situations in which users have non-binary preferences for matches. Furthermore, our previous study of matching assumes the role of an "administrator" that would compute the best matches. We are also interested in scenarios where matches occur optimally without any central administrator, the main mechanism of which are prices and payoff maximization.

2 Recap

We briefly review the results for perfect matching in bipartite graphs from last lecture.

Definition. *A graph $G = (N, E)$ is **bipartite** iff it can be partitioned into two disjoint subsets of nodes U, V such that (1) $U \cup V = N$ and (2) $\forall e \in E, \exists (u, v) \in U \times V : e = (u, v)$. In words, every node is in one of the two subsets and every edge goes between the two subsets (there are no edges within one of the subsets).*

Definition. *Let $G = (U, V, E)$ be a bipartite graph with subsets U and V. Then when $|U| = |V|$, a **perfect matching** is an assignment of nodes in U to nodes in V such that (1) $u \in U$ is assigned to $v \in V \iff (u, v) \in E$ and (2) no two nodes in U are assigned to the same $v \in V$.*

Definition. *A set of vertices $S \subseteq V$ is called a **constricted set** if $|S| > |N(S)|$ where V is one side of the bipartite graph and $N(S)$ is the neighborhood of S.*

Theorem 1 (Matching Theorem)**.** *A bipartite graph (with $|U| = |V|$) has a perfect matching iff it does not contain a constricted set of vertices.*

3 Model

3.1 Prices

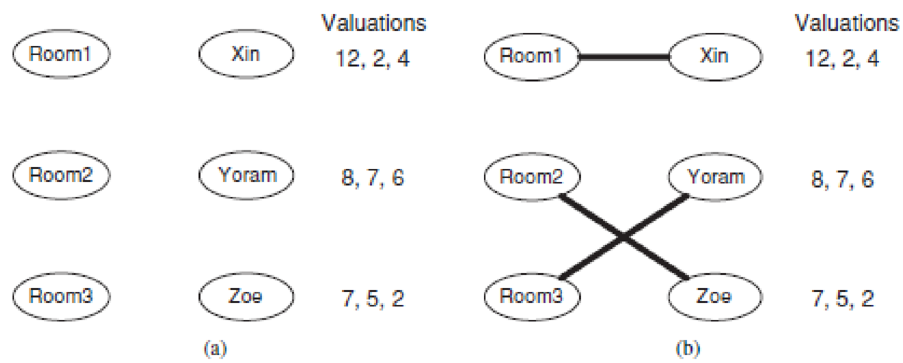

Figure 10.3. (a) A set of valuations. Each person's valuations for the objects appear as a list next to him or her. (b) An optimal assignment with respect to these valuations.

We describe our setting formally as a set of buyers trying to buy a good from a set of sellers. Sellers are parameterized by prices, and buyers are parametrized by valuations for all sellers:

Definition. *A seller i sells an items at **price** $p_i \geq 0$.*

Definition. *A buyer j's **valuation** for seller i's item is denoted v_{ij}.*

For simplicity we assume prices and valuations are nonnegative whole numbers. Seller i gets payoff p_i if they sell their item, 0 otherwise. Buyers seek to maximize their payoff:

Definition. *The **payoff** of buyer j from buying from seller i is $v_{ij} - p_i$.*

We assume that not buying anything (perhaps because all the items are too expensive) gives payoff 0.

Definition. *A buyer's **preferred seller** is the seller (possibly none) from whom buying maximizes that buyer's payoff, with ties broken arbitrarily. The set of buyers' preferred sellers induces a the **preferred-seller graph**, a bipartite graph where the nodes set is the union of the set of buyers and sellers and the edges are between each buyer and their preferred seller.*

Note that this model generalizes our previous matching problem with prices by saying that each buyer has valuation 1 for all sellers it will accept, 0 otherwise.

3.2 Market Clearing

Our ultimate goal in this model is to match each buyer with a seller, which we call "market-clearing":

> **Definition.** *A set of prices market-clearing if each buyer wants to buy from a different seller. Equivalently, a set of prices is market-clearing if there is a perfect matching in the induced preferred-seller graph.*

In Figure 10.5, only (d) has market-clearing prices; (c) does not clear because y is ambivalent between all sellers and thus clearing the market requires extra coordination between buyers.

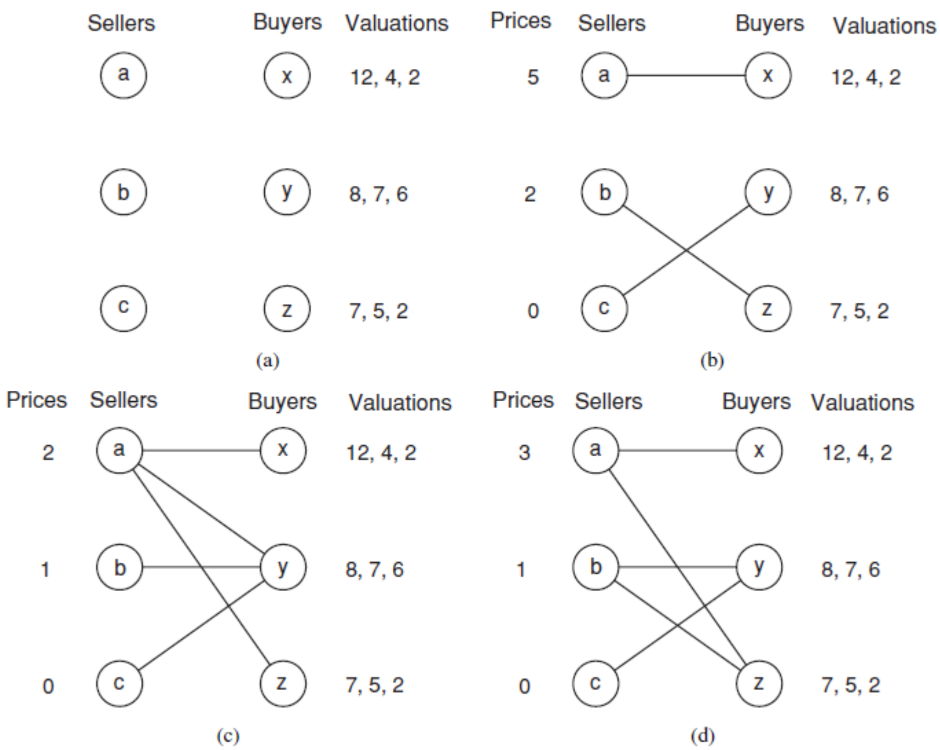

Figure 10.5. (a) Three sellers (a, b, and c) and three buyers (x, y, and z). For each buyer node, the valuations for the houses of the respective sellers appear in a list next to the node. (b) Each buyer creates a link to her preferred seller. The resulting set of edges is the preferred-seller graph for this set of market-clearing prices. (c) The preferred-seller graph for prices 2, 1, and 0 (prices that don't clear the market). (d) The preferred-seller graph for prices 3, 1, and 0 (market-clearing prices, where tie-breaking is required).

4 Main Theorem

We now arrive at the main theorem for today:

Theorem 2 (Existence and Optimality of Market-Clearing Prices)**.** *For any set of buyer valuations, there exists a set of market-clearing prices that produces a socially optimal outcome.*

Note that this theorem consists of two parts. First, for any buyer valuations, there always exists a set of market-clearing prices. Second, this set of market-clearing prices produces a socially optimal outcome in the sense that it maximizes the sum of payoffs to all sellers and buyers.

We will prove the first part by giving the following algorithm for finding these market-clearing prices given a set of buyer valuations and sellers:

```
procedure FindMarketClearingPrices
    valuations v
    prices p ← 0
    while true do
        p ← p − min_i p_i
        G ← PreferredSellerGraph(p, v)
        if G has a perfect matching then
            break
        end if
        S, N(S) ← FindConstrictedSet(G)
        for all s ∈ N(S) do
            p_s ← p_s + 1
        end for
    end while
    return p
end procedure
```

Algorithm FindMarketClearingPrices is run on a sample graph in figure 10.6. To conclude the proof of the existence of market-clearing prices, we need to show that this algorithm terminates.

We prove the second part of the theorem, the optimality of market-clearing prices, by observing that each buyer is individually optimizing their welfare given the seller prices and the utility loss from paying seller prices is exactly offset by utility gain from sellers getting paid.

5 Auctions as Markets

We can represent auctions as a specific instance of a matching problem with prices. We show a single item auction cast as a matching market with dummy nodes in Figure 10.7.

Figure 10.7. A single-item auction can be represented by the bipartite graph model: the item is represented by one seller node, and then there are additional seller nodes for which all buyers have 0 valuation. (a) The start of the bipartite graph auction. (b) The end of the bipartite graph auction, when buyer x gets the item at the valuation of buyer y.

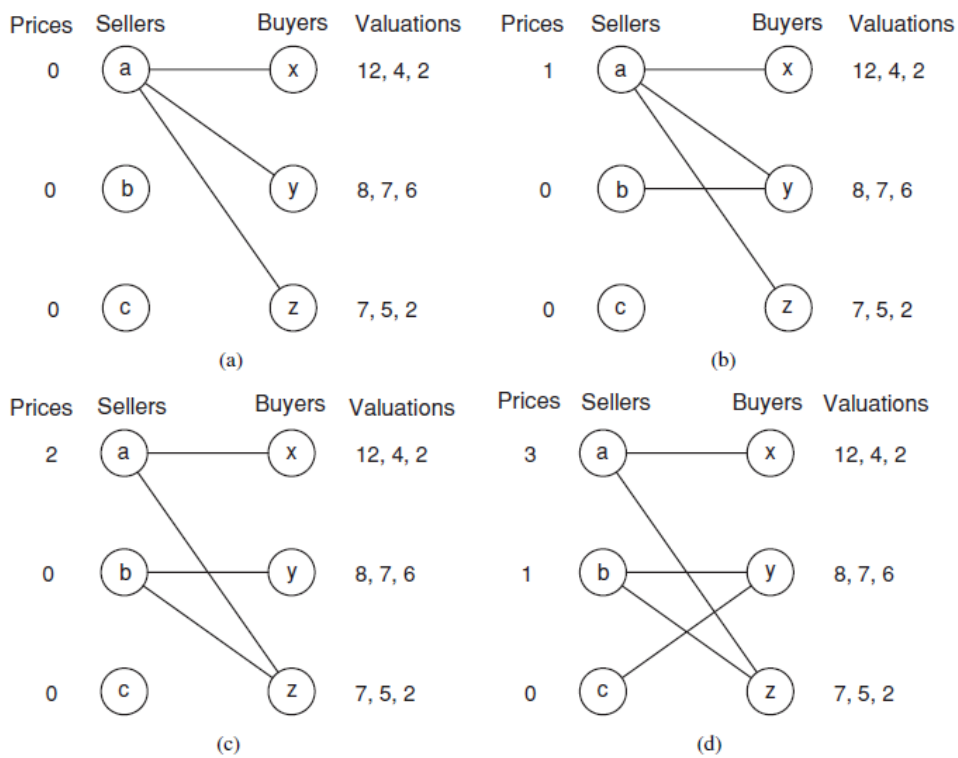

Figure 10.6. The auction procedure applied to the example from Figure 10.5. Each separate picture shows steps (i) and (ii) of successive rounds, in which the preferred-seller graph for that round is constructed. (a) In the first round, all prices start at 0. The set of all buyers forms a constricted set S, with $N(S)$ equal to the seller a. So a raises his price by one unit and the auction continues to the second round. (b) In the second round, the set of buyers consisting of x and z forms a constricted set S, with $N(S)$ again equal to seller a. Seller a again raises his price by one unit and the auction continues to the third round. (Notice that in this round, we could have alternatively identified the set of all buyers as a different constricted set S, in which case $N(S)$ would have been the set of sellers a and b. There is no problem with this – it just means that there can be multiple options for how to run the auction procedure in certain rounds, with any of these options leading to market-clearing prices when the auction comes to an end.) (c) In the third round, the set of all buyers forms a constricted set S, with $N(S)$ equal to the set of two sellers a and b. So a and b simultaneously raise their prices by one unit each, and the auction continues to the fourth round. (d) In the fourth round, when we build the preferred-seller graph, we find it contains a perfect matching. Hence, the current prices are market clearing, and the auction comes to an end.