CS 134: Networks
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1 Game Theory

Definition. *A **strategic game** consists of:*

- *A finite set of agents (players)* $N = \{a_1, \dots, a_n\}$*;*
- *For each agent* $\forall a_i \in N$ *there is a nonempty set of strategies (actions)* S_i*;*
- *For each agent* $\forall a_i \in N$ *there is a utility function* $u_i : S_1 \times S_2 \times \dots S_n \to \mathbb{R}$*;*

Assumption 1. *Throughout the course we will make the following assumptions:*

1. *Agents act simultaneously;*
2. *Agents are rational (meaning utility maximizers);*
3. *Agents have full information about the actions of other agents and their payoffs;*
4. *Agents do not coordinate in advance on their strategies.*

1.1 Nash Equilibrium

Notation. *For a given vector* $\boldsymbol{x} = (x_1, \dots, x_n)$ *we will use* x_{-i} *to denote* $\boldsymbol{x}$ *without the component* x_i*:* $x_{-i} = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$*.*

Definition. *A **Nash equilibrium (NE)** of a strategic game (N,A,U) is a profile of actions* $a^\star$ *such that for every agent* a_i *we have that:*

$$u_i(s^\star) \geq u_i(s_i, s^\star_{-i}), \ \forall s_i \in S_i.$$

1.2 Classic Games

Prisoner's Dilemma. Two suspects, Alice and Bob, of a crime are arrested and put into separate cells. Each prisoner is in solitary confinement with no means of speaking to or exchanging messages with the other. The police admit they don't have enough evidence to convict the pair on the principal charge. They plan to sentence both to a year in prison on a lesser charge. Simultaneously, the police offer each prisoner a bargain. Here's how it goes:

- If Alice and Bob both confess to the crime, each of them serves 3 years in prison;
- If Alice confesses but Bob denies the crime, Alice will be set free whereas Bob will serve 4 years in prison (and vice versa);

- If Alice and Bob both deny the crime, both of them will only serve 1 year in prison.

	Don't Confess	**Confess**
Don't Confess	$-1, -1$	$-4,\ 0$
Confess	$0, -4$	$-3, -3$

Battle of the Sexes. This is a two-player coordination game. Imagine two friends who agreed to meet for the evening. One prefers going to a movie, the other prefers going to a play, but both would prefer to go to the same place rather than different ones.

	Movie	**Play**
Movie	$2, 1$	$0, 0$
Play	$0, 0$	$1, 2$

Matching Pennies. The game is played between two players, Alice and Bob. Each player has a penny and must secretly turn the penny to heads or tails. The players then reveal their choices simultaneously. If the pennies match (both heads or both tails) Alice wins. If the pennies do not match Bob wins.

	Tails	**Heads**
Tails	$1, 0$	$0, 1$
Heads	$0, 1$	$1, 0$

1.3 Strictly competitive games

Definition. *A strategic game* $(N = (a_1, a_2), S = (S_1, S_2), U = (u_1, u_2)$ *is* ***strictly competitive*** *if for any* $s \in S$ *and* $s' \in S$ *we have that* $u_1(s) \geq u_1(s')$ *if and only if* $u_2(s) \leq u_2(s')$.

	Tails	**Heads**
Tails	$10, 3$	$0, 1$
Heads	$5, 7$	$0, 0$

Definition. *A strategic game* $(N = (a_1, a_2), S = (S_1, S_2), U = (u_1, u_2)$ *is* ***zero-sum*** *if for any* $s \in S$ *we have that* $u_1(s) + u_2(s) = 0$.

Definition. *Let* $(N = (a_1, a_2), S = (S_1, S_2), U = (u_1, u_2)$ *be a strictly competitive game. The action* $x^\star$ *is a* ***maxminimizer*** *for* a_1 *if:*

$$\min_{y \in S_2} u_1(x^\star, y) \geq \min_{y \in S_2} u_1(x, y) \forall x \in S_1.$$

Similarly, $y^\star$ *is a* ***maxminimizer*** *for* a_2 *if:*

$$\min_{x \in S_1} u_2(x, y^\star) \geq \min_{x \in S_1} u_1(x, y) \forall y \in S_2.$$

Notation. *Given some function* $f : X \to \mathbb{R}$ *we define* $\arg\max$ *to be:*

$$\arg\max_{x \in X} f(x) = \{y \ : \ f(y) \geq f(x) \ \forall x \in X\}.$$

Lemma. *Let* $(N = (a_1, a_2), S = (S_1, S_2), U = (u_1, u_2))$ *be a zero sum game. Then:*

$$\max_{y \in S_2} \min_{x \in A_1} u_2(x, y) = -\min_{y \in S_2} \max_{x \in S_1} u_1(x, y).$$

Theorem. *Let* $G = (N = (a_1, a_2), S = (S_1, S_2), U = (u_1, u_2))$ *be a zero sum game.*

- *If* $(x^\star, y^\star)$ *is a NE of* G *then* $x^\star$ *is a maxminimizer for* a_1 *and* $y^\star$ *is a maxminimizer for* a_2.
- *If* $x^\star$ *is a maximinimizer for* a_1 *and* $y^\star$ *is a maxminimizer for* a_2 *then* $(x^\star, y^\star)$ *is a NE.*

1.4 Mixed Strategies and Equilibria

Theorem (Nash's theorem)**.** *Every finite game has a mixed Nash equilibrium.*

Definition. *A strategy of a player* $a_i^\star$ *is a* ***best response*** *to a profile of strategies* a_{-i} *if:*

$$a_i^\star \in \arg\max_{a_i \in S_1} u_i(a_i, a_{-i}).$$

Definition. *A* ***Nash equilibrium*** *is a strategy profile* $a^\star$ *in which for every player* i *plays best response:*

$$a_i^\star \in B_i(a_{-i}^\star).$$