CS 134: Networks Lecture 9, Wednesday, 3/1
Prof. Yaron Singer Spring 2017

1 Overview

In this lecture, we review the concepts of game theory and Nash equilibria, and discuss methods for determining the Nash equilibria of different games. We also expand into considering games with more than two players.

2 Game Theory Review

2.1 Definition of a Game

Remember that a game is formally defined as:

Definition. *A* ***strategic game*** *consists of:*

- *A finite set of agents (players)* $N = \{a_1, \ldots, a_n\}$*;*
- *For each agent* $\forall a_i \in N$ *there is a nonempty set of strategies (actions)* S_i*;*
- *For each agent* $\forall a_i \in N$ *there is a utility function* $u_i : S_1 \times S_2 \times \ldots S_n \to \mathbb{R}$*;*

2.2 Nash Equilibrium

Games can also have a *Nash equilibrium* in which no player can improve their payoff by changing strategies, when all other players' strategies are held constant. A Nash equilibrium is formally defined as:

Definition. *A* ***Nash equilibrium (NE)*** *of a strategic game (N,A,U) is a profile of actions* $a^\star$ *such that for every agent* a_i *we have that:*

$$u_i(s^\star) \geq u_i(s_i, s^\star_{-i}), \ \forall s_i \in S_i.$$

2.3 Mixed Strategies and Equilibria

A pure strategy provides a complete definition of how a player will play a game. In particular, it determines the move a player will make for any situation he or she could face. A player's strategy set is the set of pure strategies available to that player.

A *mixed strategy* is an assignment of a probability to each pure strategy. This allows for a player to randomly select a pure strategy. Since probabilities are continuous, there are infinitely many mixed strategies available to a player, even if their strategy set is finite.

A *mixed Nash equilibrium* is a Nash equilibrium in which players play the most optimal mixed strategies, and cannot increase their payoff to switch to a different mixed strategy.

3 Iterated Deletion of Strictly Dominated Strategies

One useful way to find Nash equilibria in games is through *Iterated Deletion of Strictly Dominated Strategies*.

3.1 Strictly Dominated Stratgies

- A (strictly) *dominant* strategy of a player is a strategy that is a (strict) best response to any set of strategies played by her opponents. That is, s_i is a (strictly) dominant strategy iff it is a (strict) best response to every possible s_{-i}.

- A strategy of a player is *dominated* if there is another strategy that does at least as good against any set of strategies of her opponents, and better for at least one set of opponent strategies. That is, s_i is dominated if there is another strategy s'_i such that $u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all possible s_{-i} and $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$ for some s_{-i}.[1] For a strategy to be *strictly dominated*, there must be another strategy that does strictly better against any possible set of opponent strategies; that is, s_i is strictly dominated if there is another strategy s'_i such that $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$ for all possible s_{-i}.

3.2 IDSDS Process

The process of iterated deletion of strictly dominated strategies proceeds as follows:

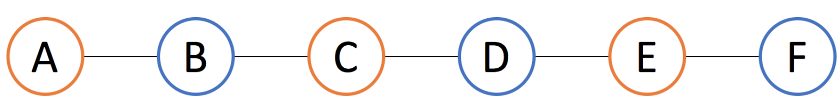
- We start with any n-player game, find all the strictly dominated strategies, and delete them.

- We then consider the reduced game in which these strategies have been removed. In this reduced game there may be strategies that are now strictly dominated, despite not having been strictly dominated in the full game. We find these strategies and delete them.

- We continue this process, repeatedly finding and removing strictly dominated strategies until none can be found.

Note that we only delete *strictly* dominated strategies; this is in order to preserve the following property: the set of Nash equilibria of the original game coincides with the set of Nash equilibria for the final reduced game, consisting only of strategies that survive iterated deletion of strictly dominated strategies – i.e. the process does not delete any strategy that is part of a Nash equilibrium. Moreover, **when the process of iterated deletion of strictly dominated strategies leads to one outcome, this outcome is the unique Nash equilibrium of the game.**[2]

3.3 Example: Shopping Mall Placement



[1] Some more notation: u_i is the utility function for player i. So $u_i(s_i, s_{-i})$ gives the payoff for player i if it players strategy s_i against s_{-i}.

[2] Were we to iteratively eliminate weakly dominated strategies in addition to strictly dominated strategies, we would still be guaranteed that at least one Nash equilibrium would remain, but it is possible that some Nash equilibria from the original game are eliminated during the process.

In this game, player 1 has the option to build a mall in towns A, C, and E (red), and player 2 has the option to build a mall in towns B, D, and F (blue). The residents from each town will shop at whichever mall is closer to them and can only travel in a straight line between A and F. The payoff for each player is based how many towns shop at their mall. The payoff matrix for the shopping mall game for both players is below.

	B	**D**	**F**
A	$1, 5$	$2, 4$	$3, 3$
C	$4, 2$	$3, 3$	$4, 2$
E	$3, 3$	$2, 4$	$5, 1$

Question: What is the Nash equilibrium of this game?

Intuitively, the optimal solution for each player to maximize their payoff is to build as close to the center as possible: player 1 builds at C and player 2 builds at D. (Note that this also isn't the "ideal" placement of malls to minimize the distance traveled for everyone.) To formally determine the Nash equilibrium, we can use IDSDS.

A is a strictly dominated strategy for player 1 because, no matter what player 2 does, player 1 can obtain a higher payoff by switching to strategy C. This allows us to delete A as a strategy for player 1, resulting in a reduced game, with payoff matrix below.

	B	**D**	**F**
C	$4, 2$	$3, 3$	$4, 2$
E	$3, 3$	$2, 4$	$5, 1$

Strategy F is now a strictly dominated strategy for player 2, as player 2 would always benefit from switching to strategy D instead. Deleting strategy F:

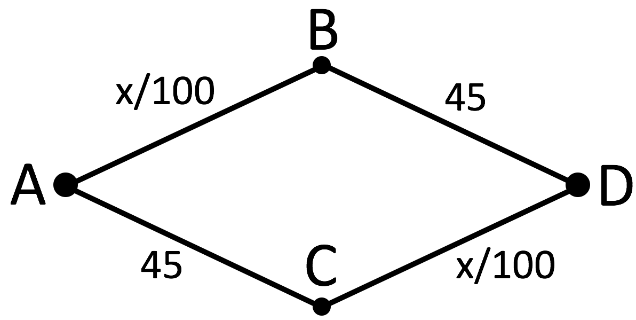
	B	**D**
C	$4, 2$	$3, 3$
E	$3, 3$	$2, 4$

In this 2×2 game matrix, strategy E for player 1 is strictly dominated and can be deleted. Crucially, strategy E is only a strictly dominated strategy in this reduced game. It required strategy F to be deleted to be a strictly dominated strategy. Finally, strategy B is also now strictly dominated and can be deleted. In this final version, players 1 and 2 both only have one strategy remaining, C and D respectively. As the conclusion of IDSDS, (C,D) is the only Nash equilibrium of the game.

4 Traffic Congestion

There are two roads that connect points A and D: one road goes through B, and the other road goes through C. There are 4000 players $\{x_1, ..., x_4 000\}$ who are traveling from A to D. Each player i has two possible strategies $S_i = \{ABD, ACD\}$. Above each road is the amount of time (in minutes) it takes for a player to cross that road. Due to congestion, the time it takes to cross roads AB and CD depends on the number of players taking that road x. Everyone wants to get from A to D as fast as possible, so the payoff for each player is $-1*$(time taken to get from A to D).

4.1 Congestion Game: Original



The Nash equilibrium for this game occurs when 2000 people take route ABD and 2000 people take route ACD. We can prove that this is a Nash equilibrium as follows.

Proof. Assume 2000 players take ABD and 2000 players take ACD. The payoff for each of these players is

$$-(\frac{x_ABD}{100} + 45) = -(45 + \frac{x_ACD}{100}) = -65$$

Without loss of generality, we can look at player x_1 with strategy ABD. If x_1 switches strategies to ACD, assuming all other strategies stay the same, they would have a new payoff of

$$-(45 + \frac{x_ACD}{100}) = -(45 + \frac{2001}{100}) = -65.01 < -65$$

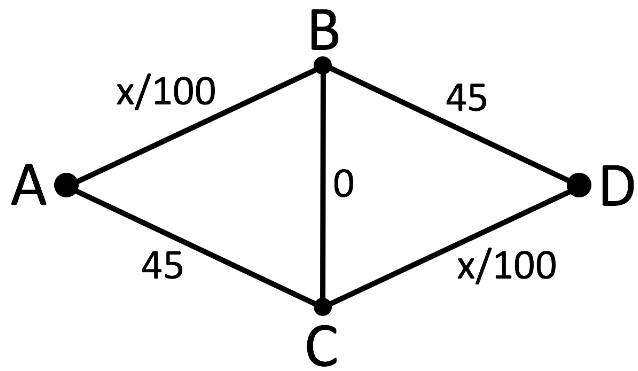
so x_1 would not want to switch strategies. Since we assumed generality, no player would want to switch strategies, and thus this is a Nash equilibrium. □

An even split is also the only Nash equilibrium. If there were an uneven split, players could improve their payoff by changing their strategy to travel along the less congested route.

In this scenario, the Nash equilibrium is also the socially optimal set of strategies. In other words, a city designer or administrator would also choose the situation at Nash equilibrium to minimize the total travel time for everyone.

4.2 Congestion Game: Modified

In the modified scenario, a one-directional bridge is built from B to C that takes zero minutes to cross and is never congested. The times for the other four roads remain the same, and the set of players remain the same. However, each player now has three options $S_i = \{ABD, ACD, ABCD\}$.



In this game, the Nash equilibrium occurs when all 4000 players take route ABCD.

Proof. Assume all 4000 players take ABCD. Without loss of generality, we can look at the payoffs for player x_1 for all possible strategies, and show that the optimal strategy is to also take ABCD.

$$u_1(ABD, s_-1) = -(\frac{x_A BD}{100} + 45) = -(\frac{4000}{100} + 45) = -85$$

$$u_1(ACD, s_-1) = -(\frac{x_A CD}{100} + 45) = -(\frac{4000}{100} + 45) = -85$$

$$u_1(ABCD, s_-1) = -(\frac{x_A BD}{100} + \frac{x_A BD}{100}) = -(\frac{4000}{100} + \frac{4000}{100}) = -80$$

Player x_1's best strategy is to travel along route ABCD. Since with generality, no player would want to switch strategies from ABCD, this is a Nash equilibrium. □

We can also show that an even split is no longer a Nash equilibrium.

Proof. Assume 2000 players take ABD and 2000 players take ACD. Without loss of generality, we can look at the payoffs for player x_1 for all possible strategies, and show that the optimal strategy is to also take ABCD.

$$u_1(ABD, s_-1) = -(\frac{x_A BD}{100} + 45) = -(\frac{2000}{100} + 45) = -65$$

$$u_1(ACD, s_-1) = -(\frac{x_A CD}{100} + 45) = -(\frac{2000}{100} + 45) = -65$$

$$u_1(ABCD, s_-1) = -(\frac{x_A BD}{100} + \frac{x_A BD}{100}) = -(\frac{2000}{100} + \frac{2000}{100}) = -40$$

Player x_1's best strategy is to travel along route ABCD. Regardless what their strategy originally was, x_i would improve their payoff by switching strategies to ABCD. □

Note that unlike the previous scenario, this is not the socially optimal set of strategies. Even though each player is optimizing their own payoffs, they don't get the highest possible payoff in the Nash equilibrium. Furthermore, this suboptimal equilibrium is a result of adding a congestion-free road, which intuitively should have improved the situation.