CS 134: Networks

Prof. Yaron Singer | Spring 2017

1 Overview

In this lecture we will cover basic concepts from game theory.

1.1 Reasoning about strategic behavior

Here the rows are one player's actions, columns are the other's; entries are payoffs for (row,column) when those actions are played.

2 Game Theory

Let us begin by defining a game formally.

Definition. *A **strategic game** consists of:*

- *A finite set of agents (players) $N = \{a_1, \dots, a_n\}$;*
- *For each agent $\forall a_i \in N$ there is a nonempty set of strategies (actions) S_i;*
- *For each agent $\forall a_i \in N$ there is a utility function $u_i : S_1 \times S_2 \times \dots S_n \to \mathbb{R}$;*

Assumption 1. *Throughout the course we will make the following assumptions:*

1. *Agents act simultaneously;*
2. *Agents are rational (meaning utility maximizers);*
3. *Agents have full information about the actions of other agents and their payoffs;*
4. *Agents do not coordinate in advance on their strategies.*

There are many interesting models that challenge each one of these assumptions. Throughout this course, we are interested in analyzing basic scenarios, and unless stated otherwise, we will make the above assumptions about the model.

2.1 Nash Equilibrium

Notation. *For a given vector $\boldsymbol{x} = (x_1, \dots, x_n)$ we will use x_{-i} to denote $\boldsymbol{x}$ without the component x_i: $x_{-i} = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.*

Definition. *A **Nash equilibrium (NE)** of a strategic game (N,A,U) is a profile of actions $a^\star$ such that for every agent a_i we have that:*

$$u_i(s^\star) \geq u_i(s_i, s^\star_{-i}), \ \forall s_i \in S_i.$$

That is, the Nash equilibrium occurs when, for a given set of strategies, no player can improve her payoff (when all other strategies held constant) by changing her strategy.

2.2 Classic Games

Prisoner's Dilemma. Two suspects, Alice and Bob, of a crime are arrested and put into separate cells. Each prisoner is in solitary confinement with no means of speaking to or exchanging messages with the other. The police admit they don't have enough evidence to convict the pair on the principal charge. They plan to sentence both to a year in prison on a lesser charge. Simultaneously, the police offer each prisoner a bargain. Here's how it goes:

- If Alice and Bob both confess to the crime, each of them serves 3 years in prison;
- If Alice confesses but Bob denies the crime, Alice will be set free whereas Bob will serve 4 years in prison (and vice versa);
- If Alice and Bob both deny the crime, both of them will only serve 1 year in prison.

	Don't Confess	**Confess**
Don't Confess	$-1, -1$	$-4,\ 0$
Confess	$0, -4$	$-3, -3$

Even though they're both better off if neither confesses, the Nash Equilibrium is when both confess, since they don't trust each other which is a suboptimal outcome.

Battle of the Sexes. This is a two-player coordination game. Imagine two friends who agreed to meet for the evening. One prefers going to a movie, the other prefers going to a play, but both would prefer to go to the same place rather than different ones.

	Movie	**Play**
Movie	$2, 1$	$0, 0$
Play	$0, 0$	$1, 2$

In this example there are two Nash equilibria that occur at (Movie,Movie) and (Play,Play).

Matching Pennies. The game is played between two players, Alice and Bob. Each player has a penny and must secretly turn the penny to heads or tails. The players then reveal their choices simultaneously. If the pennies match (both heads or both tails) Alice wins. If the pennies do not match Bob wins.

	Tails	**Heads**
Tails	$1, 0$	$0, 1$
Heads	$0, 1$	$1, 0$

In this game there is no Nash equilibrium.

2.3 Brief summary

From the above games we analyzed we can infer the following:

- The Nash equilibrium may be suboptimal (prisoner's dilemma);
- The Nash equilibrium may not be unique (battle of sexes);
- The Nash equilibrium amy not exist.

2.4 Existence of Nash Equilibrium

As we learned from the example above, not every game has a NE. This naturally leads us to the following question:

Which games always have at least a single Nash equilibrium?

Before we address this question, what is good about knowing when we have equilibrium? Here are some thoughts:

- If we know that a game has an equilibrium, we can hope to find it.
- If a game has an equilibrium it is consistent with some steady state solution.

2.5 Strictly competitive games

Definition. *A strategic game $(N = (a_1, a_2), S = (S_1, S_2), U = (u_1, u_2)$ is **strictly competitive** if for any $s \in S$ and $s' \in S$ we have that $u_1(s) \geq u_1(s')$ if and only if $u_2(s) \leq u_2(s')$.*

	Tails	**Heads**
Tails	$10, 3$	$0, 1$
Heads	$5, 7$	$0, 0$

A special case of strictly competitive games is zero-sum games:

Definition. *A strategic game $(N = (a_1, a_2), S = (S_1, S_2), U = (u_1, u_2)$ is **zero-sum** if for any $s \in S$ we have that $u_1(s) + u_2(s) = 0$.*

Exercise. *Prove that any zero-sum game is strictly competitive.*

Definition. *Let $(N = (a_1, a_2), S = (S_1, S_2), U = (u_1, u_2)$ be a strictly competitive game. The action $x^\star$ is a **maxminimizer** for a_1 if:*

$$\min_{y \in S_2} u_1(x^\star, y) \geq \min_{y \in S_2} u_1(x, y) \forall x \in S_1.$$

*Similarly, $y^\star$ is a **maxminimizer** for a_2 if:*

$$\min_{x \in S_1} u_2(x, y^\star) \geq \min_{x \in S_1} u_1(x, y) \forall y \in S_2.$$

Notation. *Given some function $f : X \to \mathbb{R}$ we define* $\arg\max$ *to be:*

$$\arg\max_{x \in X} f(x) = \{y \ : \ f(y) \geq f(x) \ \forall x \in X\}.$$

Remark. *Note that the maxminimizer of a_1 strategy is a solution to the following optimization problem:*

$$x^\star = \arg\max_{x\in A_1}\min_{y\in A_2} u_1(x,y)$$

And similarly, if $y^\star$ is the maxminimizer of a_2 then:

$$y^\star = \arg\max_{y\in A_2}\min_{x\in A_1} u_2(x,y).$$

Lemma. *Let $(N=(a_1,a_2), S=(S_1,S_2), U=(u_1,u_2))$ be a zero sum game. Then:*

$$\max_{y\in S_2}\min_{x\in A_1} u_2(x,y) = -\min_{y\in S_2}\max_{x\in S_1} u_1(x,y).$$

Proof. For any function we have that $\min_z(-f(z)) = -\max_z f(z)$. Therefore:

$$\min_{x\in S_1} u_2(x,y) = \min_{x\in S_1}(-u_1(x,y)) = -\max_{x\in S_1} u_1(x,y)$$

Hence:

$$\max_{y\in S_2}[\min_{x\in S_1} u_2(x,y)] = -\min_{y\in S_2}[-\min_{x\in S_1} u_2(x,y)] = -\min_{y\in S_2}\max_{x\in S_1} u_1(x,y).$$

□

Theorem. *Let $G=(N=(a_1,a_2), S=(S_1,S_2), U=(u_1,u_2))$ be a zero sum game.*

- *If $(x^\star,y^\star)$ is a Nash equilibrium of G then $x^\star$ is a maxminimizer for a_1 and $y^\star$ is a maxminimizer for a_2.*
- *If $x^\star$ is a maximinimizer for a_1 and $y^\star$ is a maxminimizer for a_2 then $(x^\star,y^\star)$ is a Nash equilibrium.*

Proof. From the fact that $(x^\star,y^\star)$ is a NE we get that:

$$u_2(x^\star,y^\star) \geq u_2(x^\star,y)\ \forall y\in S_2$$

Since $u_1=-u_2$ this implies that:

$$u_1(x^\star,y^\star) \leq u_1(x^\star,y)\ \forall y\in S_2$$

Hence:

$$u_1(x^\star,y^\star) = \min_{y\in S_2} u_1(x^*,y) \leq \max_{x\in S_1}\min_{y\in S_2} u_1(x,y)$$

Similarly, since $(x^\star,y^\star)$ is a NE we get that:

$$u_1(x^\star,y^\star) \geq u_1(x,y^\star)\ \forall x\in S_1$$

Hence:

$$u_1(x^\star,y^\star) \geq \min_{y\in S_2} u_1(x,y)\ \forall x\in S_1$$

Thus:

$$u_1(x^\star,y^\star) \geq \max_{x\in S_1}\min_{y\in S_2} u_1(x,y)$$

Since $u_1(x^\star,y^\star) \geq \max_{x\in S_1}\min_{y\in S_2} u_1(x,y)$ and $u_1(x^\star,y^\star) \leq \max_{x\in S_1}\min_{y\in S_2} u_1(x,y)$ it must be that: $u_1(x^\star,y^\star) = \max_{x\in S_1}\min_{y\in S_2} u_1(x,y)$. The argument for a_2 is analogous. □

2.6 Mixed Strategies and Equilibria

As we saw before, there are some games for which there is no Nash equilibrium. We will now formalize the notion of *mixed* strategy and *mixed* Nash equilibrium.

Theorem (Nash's theorem)**.** *Every finite game has a mixed Nash equilibrium.*

Before we prove the theorem, let's introduce a useful definition.

Definition. *A strategy of a player $a_i^\star$ is a* ***best response*** *to a profile of strategies a_{-i} if:*

$$a_i^\star \in \arg\max_{a_i \in S_1} u_i(a_i, a_{-i}).$$

It will be useful to think about a best response set-valued function; we will use $B_i(a_{-i})$ to denote the set of strategies that are best response for a_{-i}. The following alternative definition of Nash equilibrium will be useful.

Definition. *A* ***Nash equilibrium*** *is a strategy profile $a^\star$ in which for every player i plays best response:*

$$a_i^\star \in B_i(a_{-i}^\star).$$

A pure strategy provides a complete definition of how a player will play a game. In particular, it determines the move a player will make for any situation he or she could face. A player's strategy set is the set of pure strategies available to that player.

A mixed strategy is an assignment of a probability to each pure strategy. This allows for a player to randomly select a pure strategy. Since probabilities are continuous, there are infinitely many mixed strategies available to a player, even if their strategy set is finite.

Of course, one can regard a pure strategy as a degenerate case of a mixed strategy, in which that particular pure strategy is selected with probability 1 and every other strategy with probability 0.