CS 134: Networks	Problem set 6
Prof. Yaron Singer	Due: **Wednesday 3/22/2017**

1. (30 points) Consider a variant of the branching process discussed in lecture. The base graph is a rooted, infinite k-ary tree, with $k \geq 2$ an integer (each node has k children and one parent, except for the root which has no parents). We define level n to be the set of nodes in this tree at distance n from the origin.

Let $A_v \in \{0, 1\}$ indicate whether node v is active. If $A_v = 1$, v is active, and if $A_v = 0$, v is inactive. The root node r (the sole node at level $n = 0$) is assumed to be active. For any active node v at level n, the number of its active children at level $n + 1$ is a random variable Z_v (and all sets of active children that have the same size are equally likely).

All the Z_v are mutually independent and distributed identically, according to a probability distribution function P, so that $\mathbb{P}(Z_v = \ell) = P(\ell)$. Assume that $0 < P(0) < 1$ and $0 < P(1) < 1$.

Given these data, the whole (random) tree of active nodes can be constructed inductively. Let X_n be the number of active nodes at level n.

- **a. (5 points)** What is $\mathbb{E}[X_n]$ in terms of P? (The expectation is taken across all realizations of the random tree.)
- **b. (2 points)** Let q_n be the probability that $X_n \neq 0$. What is q_0?
- **c. (3 points)** Is q_n increasing in n, decreasing in n, or neither? Argue from the definition of X_n and the basic properties of the process, without doing any calculations.
- **d. (5 points)** For $n > 0$, we have $q_n = f(q_{n-1})$. Write out the formula for f and explain your answer. (The function f will involve the values of P.)
- **e. (5 points)** Recall the function $f : [0, 1] \to [0, 1]$ you described above. Give the sign of its first derivative and the sign of the second derivative. (Here P is again arbitrary, subject to the assumptions given in the statement.)
- **f. (5 points)** Define $q_\infty = \lim_{n\to\infty} q_n$ when this limit exists. Describe how you can find q_∞ by looking at a plot of some possible q_n vs. n. Give a verbal statement of the meaning of q_∞ (this is required to get full points on this part).
- **g. (5 points)** Give a necessary and sufficient condition for $q_\infty > 0$ in terms of the values of P only. Explain this condition using a plot of some possible q_n vs. n and your answer in (e).

Solution by Michael Ge:

a.

Consider X_0 which always has exactly one active node:

$$\mathbb{E}[X_0] = 1$$

The expectation of X_1 considers all active nodes in X_0 (of which there is one) and adds $\mathbb{E}[Z_v]$ on average:

$$\mathbb{E}[X_1] = E[Z_v]$$

We inductively note that X_n adds $\mathbb{E}[Z_v]$ nodes for each of the active nodes in X_n, of where there is $\mathbb{E}[X_{n-1}]$ in expectation. Thus, we arrive at the following conclusion:

$$\mathbb{E}[X_n] = \mathbb{E}[Z_v]E[X_{n-1}] = \mathbb{E}[Z_v]^2 E[X_{n-2}] = \ldots = \mathbb{E}[Z_v]^n = \left(\sum_{\ell=0}^{k} \ell P(\ell)\right)^n$$

Where $\mathbb{E}[Z_v]$ is expanded by the definition of expected value.

b.

Since the root is always active, the probability that $X_0 \neq 0$ is 1, so $q_0 = 1$.

c.

It must be that q_n is decreasing in n. At every step of the process, branches have a nonzero chance of dying off. Once a branch dies off, it can never generate new active nodes for any of the children on the same branch. As n increases, more and more branches die off, causing X_n, the number of active children at level n, to trend toward slower growth. As the growth rate slows, the probability that X_n has more than one active node decreases since more branches will die off with fewer chances to regenerate new branches. Logically, once $X_n = 0$, $X_{n+i} = 0$ for all $i > 0$, so an increase is not possible.

d.

To find $\mathbb{P}(X_n \neq 0)$, we can find the complement: the probability that X_n is 0 for every possible configuration of X_n. We first note that if the parent fails (which happens with probability $P(0)$, then the child fails with probability 1. For any child of the root node, the probability that this child fails is:

$$1 - q_{n-1}$$

We then consider all possible combinations of this branch failing, weighting by their respective probabilities. For i children, the probability that each child fails is $P(i)(1 - q_{n-1})^i$ since all i must undependently fail. The probability that activation fails to persist to n levels is:

$$\sum_{i=0}^{k} (1 - q_{n-1})^i P(i)$$

Thus, the probability that activation succeeds at persisting to n levels is:

$$q_n = 1 - \sum_{i=0}^{k} (1 - q_{n-1})^i P(i)$$

e.

$$f'(q) = \sum_{i=0}^{k} i(1 - q_{n-1})^{i-1} P(i)$$

The first derivative is positive.

$$f''(q) = -\sum_{i=0}^{k} i(i-1)(1 - q_{n-1})^{i-2} P(i)$$

The second derivative is negative.

The function must be increasing and concave.

f.

We can find q_∞ by comparing the function $f(x)$ with $y = x$ to search for the intersection by repeated functional compositions (as done in class). This process of first finding $(1, f(1))$, then $(f(1), f(f(1)))$, etc. asymptotically approaches this intersection.

$q_\infty = 0$ or 1 depending on the starting value of q. q_∞ is the asymptotic probability of a contagion model dying off. If $q_\infty = 0$, then the process dies off, otherwise the process has a probability of continuing indefinitely.

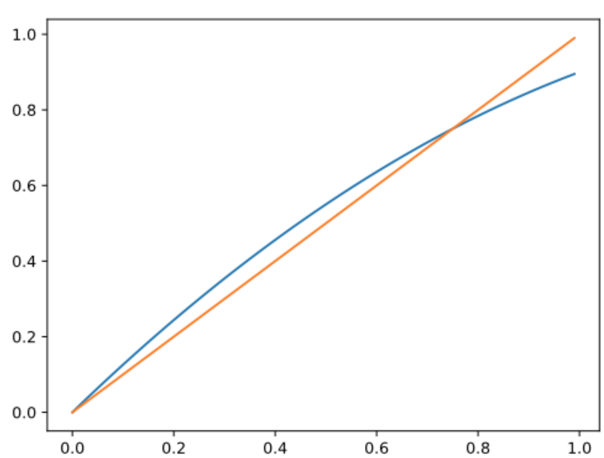
g.

$E(Z_v)$ is the analogous term of R_0. If $E(Z_v)$ is less than 1, then every generation will have less than the same number of active children on average and die out, but if $E(Z_v)$ is greater than 1, then the process will proliferate. This is equal to:

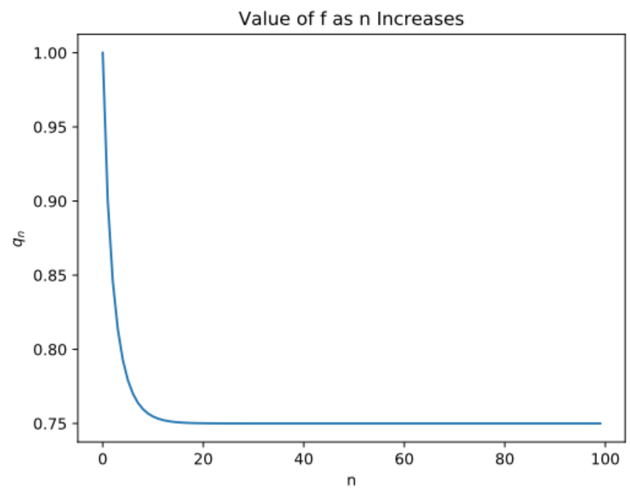
$$\sum_{\ell=1}^{k} \ell P(\ell) > 1$$

For an example $k = 2$ and $P(0) = .1, P(1) = .5, P(2) = .4$ (yieliding an $f > 1$), we can plot the function f beside $y = x$ and observe the intersection. By compositing the f function, we find q_n approaches 0.75 which is greater than 0.

The following is the asymptotic visualization.



And the following is the value of q_n as n increases to 100.



2. (20 points) Suppose we have a probability distribution P on the numbers $\{0, 1, \ldots, k\}$ as given in Problem 1. Suppose we have another distribution $\widetilde{P}$ on the same set. We say that $\widetilde{P}$ *first-order stochastically dominates* P if the following condition holds. For every strictly increasing function $r : \{0, \ldots, k\} \to \mathbb{R}$, we have $\sum_{\ell=0}^{k} \widetilde{P}(\ell)r(\ell) > \sum_{\ell=0}^{k} P(\ell)r(\ell)$. In other words, the expectation of $r(Z)$ when Z is drawn from $\widetilde{P}$ always exceeds the expectation of $r(Z)$ when Z is drawn from P. In this sense, $\widetilde{P}$ tends to take higher values.[1]

- **a. (5 points)** Show that if $\widetilde{P}$ first-order stochastically dominates P and $r : \{0, \ldots, k\} \to \mathbb{R}$ is a strictly *decreasing* function, then $\sum_{\ell=0}^{k} \widetilde{P}(\ell)r(\ell) < \sum_{\ell=0}^{k} P(\ell)r(\ell)$.
- **b. (7 points)** Let f be constructed as in Problem 1(d) based on P, and let $\widetilde{f}$ be constructed in the same way from $\widetilde{P}$. For $q \in (0, 1)$, can you give an inequality relating $\widetilde{f}(q)$ and $f(q)$? State it and justify it. You don't need to give a full proof, but give a clear reason.
- **c. (8 points)** Define $\widetilde{q}_\infty$ as in Problem 1, but now with Z_v having distribution $\widetilde{P}$. When $\widetilde{q}_\infty$ and q_∞ are both positive, give an inequality relating the two and explain it.

Solution (courtesy of Jacob Gollub):

- If $r : \{0, \ldots, k\} \to \mathbb{R}$ is a strictly decreasing function, then $-r : \{0, \ldots, k\} \to \mathbb{R}$ is a strictly increasing function.
  So $\sum_{\ell=0}^{k} -\widetilde{P}(\ell)r(\ell) > \sum_{\ell=0}^{k} -P(\ell)r(\ell)$ and $\sum_{\ell=0}^{k} \widetilde{P}(\ell)r(\ell) < \sum_{\ell=0}^{k} P(\ell)r(\ell)$

- For any given q, since $\widetilde{P}$ stochastically dominates P and $(1-q)^j$ is strictly decreasing with j (for $0 < q < 1$), $\sum_{j=0}^{k} P(j)(1-q)^j > \sum_{j=0}^{k} \widetilde{P}(j)(1-q)^j$ and we know $\widetilde{f}(q) > f(q)$. This holds for all q because if $\widetilde{q}_n \geq q_n$ then $\sum_{j=0}^{k} P(j)(1-q_n)^j > \sum_{j=0}^{k} \widetilde{P}(j)(1-q_n)^j$ and we are subtracting a smaller summation from 1 so $\widetilde{f}(q_{n+1}) > f(q_{n+1})$. By induction, we know this inequality holds for all q_n.

[1] If you're curious, Wikipedia will tell you many different characterizations of first-order stochastic dominance.

$\widetilde{q}_\infty$ is the probability that all active nodes never die out when Z_v follows distribution $\widetilde{P}$. When $\widetilde{q}_\infty$ and q_∞ are both positive, we know that at every step of this branching process, you can expect to pass on more active nodes when Z_v has distribution $\widetilde{P}$. Then the probability of all nodes dying out must be less when following distribution $\widetilde{P}$ since we always expect to see more active nodes at every level. Therefore, the probability of never running out of nodes must be greater and $\widetilde{q}_\infty \geq q_\infty$.

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3. (15 points) Let P be an $n \times n$ matrix with every entry a nonnegative real number, and the (i,j) entry denoted by p_{ij}. Assume that for every $i \in \{1,\ldots,n\}$ we have $\sum_{j=1}^{n} p_{ij} = 1$. A matrix having these properties is called *row-stochastic*. Assume Q is another row-stochastic matrix.

a. (3 points) Write down a formula for the (i,j) entry of the matrix product $R = QP$.

b. (4 points) For every $i \in \{1,\ldots,n\}$, compute $\sum_{j=1}^{n} r_{ij}$ (give a number) and justify your answer. Note that this is the sum of all entries in row i of R.

c. (4 points) Compute the sum of all the entries in row i of P^2 and P^3, justifying your answer. Compute the row sum of all the entries in row i of P^k, where $k \geq 1$, and justify your answer.

d. (4 points) For each of the three different 5×5 matrices P below, compute $L = P^{100}$ to three decimal digits of precision using your favorite technology. Also compute LP for each case. What do you notice?

$$P_1 = \begin{bmatrix} 0.1 & 0.4 & 0 & 0.3 & 0.2 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0.6 & 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.1 & 0.1 & 0.2 \end{bmatrix}, P_2 = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.3 & 0.4 & 0 & 0.2 \\ 0.3 & 0.1 & 0.1 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.3 & 0.2 & 0.2 \end{bmatrix}, P_3 = \begin{bmatrix} 0.1 & 0.3 & 0.4 & 0.2 & 0 \\ 0 & 0 & 0.5 & 0.4 & 0.1 \\ 0.2 & 0.2 & 0.3 & 0.2 & 0.1 \\ 0 & 0.3 & 0.5 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.1 & 0 & 0.1 \end{bmatrix}$$

e. (No credit, just "for fun") A transition matrix is one in which each entry represents the probability of traveling from state i to j in the Markov process—and we'll be covering transition matrices more thoroughly as we move into influence models. View P as the transition matrix of a Markov process on the state space $\{1,2,\ldots,n\}$, with p_{ij} being the probability of going to state j conditional on being in state i. Give a probabilistic interpretation of P^k, and also probabilistic interpretations of your answer in (d). If you've never encountered Markov chains before in this formalism, try Googling around a bit, reading Wikipedia, watching some instructional videos on the subject...

Solution:

a. To find the (i, j) entry of R (which we'll denote r_{ij}), we take the dot product of row i of Q and column j of P:

$$r_{ij} = \sum_{k=1}^{n} q_{ik} p_{kj}$$

b. We find the sum of all entries in the ith row of R by using our result from (a):

$$\begin{aligned}
\sum_{j=1}^{n} r_{ij} &= \sum_{j=1}^{n} \sum_{k=1}^{n} q_{ik} p_{kj} \\
&= \sum_{k=1}^{n} q_{ik} \left(\sum_{j=1}^{n} p_{kj} \right) \\
&= \sum_{k=1}^{n} q_{ik} (1) \\
&= 1
\end{aligned}$$

Thus, because all the rows in R sum to 1 for any $R = QP$, we have shown that the dot product of two row stochastic matrices is also row stochastic.

c. We can rewrite P^2 as $P \cdot P$, and since we showed in part (b) that the dot product of two row stochastic matrices is also row stochastic, then P^2 must also be row stochastic. Thus, all of the rows in P^2 sum to 1.

By similar logic, we can rewrite P^3 as $P \cdot P^2$. Since P and P^2 are row stochastic, P^3 must also be row stochastic, so its rows also sum to 1.

By induction, we will show that P^k for any $k \geq 1$ is row stochastic:

Base Case: For $k = 1$, we have $P^1 = P$ which we are given is row stochastic.

Inductive Step: For any $k > 1$, if P^{k-1} is row stochastic, then P^k must also be row stochastic. This is because we can rewrite P^k as $P \cdot P^{k-1}$, and since P^k is the dot product of two row stochastic matrices, P^k must also be row stochastic, and thus all of its rows sum to 1.

d. Below are my results:

$$L_1 = \begin{bmatrix} 0.232055063913 & 0.184857423795 & 0.194690265487 & 0.244837758112 & 0.143559488692 \\ 0.232055063913 & 0.184857423795 & 0.194690265487 & 0.244837758112 & 0.143559488692 \\ 0.232055063913 & 0.184857423795 & 0.194690265487 & 0.244837758112 & 0.143559488692 \\ 0.232055063913 & 0.184857423795 & 0.194690265487 & 0.244837758112 & 0.143559488692 \\ 0.232055063913 & 0.184857423795 & 0.194690265487 & 0.244837758112 & 0.143559488692 \end{bmatrix}$$

$$L_1P_1 = \begin{bmatrix} 0.232055063913 & 0.184857423795 & 0.194690265487 & 0.244837758112 & 0.143559488692 \\ 0.232055063913 & 0.184857423795 & 0.194690265487 & 0.244837758112 & 0.143559488692 \\ 0.232055063913 & 0.184857423795 & 0.194690265487 & 0.244837758112 & 0.143559488692 \\ 0.232055063913 & 0.184857423795 & 0.194690265487 & 0.244837758112 & 0.143559488692 \\ 0.232055063913 & 0.184857423795 & 0.194690265487 & 0.244837758112 & 0.143559488692 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 0.159402241594 & 0.107845579078 & 0.295143212951 & 0.296637608966 & 0.14097135741 \\ 0.159402241594 & 0.107845579078 & 0.295143212951 & 0.296637608966 & 0.14097135741 \\ 0.159402241594 & 0.107845579078 & 0.295143212951 & 0.296637608966 & 0.14097135741 \\ 0.159402241594 & 0.107845579078 & 0.295143212951 & 0.296637608966 & 0.14097135741 \\ 0.159402241594 & 0.107845579078 & 0.295143212951 & 0.296637608966 & 0.14097135741 \end{bmatrix}$$

$$L_2P_2 = \begin{bmatrix} 0.159402241594 & 0.107845579078 & 0.295143212951 & 0.296637608966 & 0.14097135741 \\ 0.159402241594 & 0.107845579078 & 0.295143212951 & 0.296637608966 & 0.14097135741 \\ 0.159402241594 & 0.107845579078 & 0.295143212951 & 0.296637608966 & 0.14097135741 \\ 0.159402241594 & 0.107845579078 & 0.295143212951 & 0.296637608966 & 0.14097135741 \\ 0.159402241594 & 0.107845579078 & 0.295143212951 & 0.296637608966 & 0.14097135741 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 0.14089347079 & 0.195215437483 & 0.376288659794 & 0.201691779011 & 0.085910652921 \\ 0.14089347079 & 0.195215437483 & 0.376288659794 & 0.201691779011 & 0.085910652921 \\ 0.14089347079 & 0.195215437483 & 0.376288659794 & 0.201691779011 & 0.085910652921 \\ 0.14089347079 & 0.195215437483 & 0.376288659794 & 0.201691779011 & 0.085910652921 \\ 0.14089347079 & 0.195215437483 & 0.376288659794 & 0.201691779011 & 0.085910652921 \end{bmatrix}$$

$$L_3P_3 = \begin{bmatrix} 0.14089347079 & 0.195215437483 & 0.376288659794 & 0.201691779011 & 0.085910652921 \\ 0.14089347079 & 0.195215437483 & 0.376288659794 & 0.201691779011 & 0.085910652921 \\ 0.14089347079 & 0.195215437483 & 0.376288659794 & 0.201691779011 & 0.085910652921 \\ 0.14089347079 & 0.195215437483 & 0.376288659794 & 0.201691779011 & 0.085910652921 \\ 0.14089347079 & 0.195215437483 & 0.376288659794 & 0.201691779011 & 0.085910652921 \end{bmatrix}$$

These results show that for large k, all the rows of P^k converge to the same values. This is because all of the L matrices above have approximately the same entries for each row, and all of the LP matrices are virtually the same as L, which shows that P^k converges to some value as $k \to \infty$. All of the rows in all of the above matrices sum to 1, showing our result from (c).

4. Braess' Paradox (Easley and Kleinberg, 8.4 Q1) (15 points) There are 1,000 cars that must travel from town A to town B. There are two possible routes that each car can take: the upper route through town C or the lower route through town D. Let x be the number of cars traveling on the edge A-C and let y be the number of cars traveling on the edge D-B. The directed graph in Figure 1 indicates that travel time per car on edge A-C is $x/100$ if x cars use edge A-C, and similarly the travel time per car on edge D-B is $y/100$ if y cars use edge D-B. The travel time per car on each of edges C-B and A-D is 12 regardless of the number of cars on these edges. Each driver wants to select a route to minimize his travel time. The drivers make simultaneous choices.

- **a.** **(3 points)** Find Nash equilibrium values of x and y.
- **b.** **(6 points)** Now the government builds a new (one-way) road from town C to town D. The new road adds the path A-C-D-B to the network. This new road from C to D has a travel time of 0 per car regardless of the number of cars that use it. Find a Nash equilibrium for the game played on the new network. What are the equilibrium values of x and y? What happens to total cost of travel (the sum of total travel times for the 1,000 cars) as a result of the availability of the new road?
- **c.** **(6 points)** Suppose now that conditions on edges C-B and A-D are improved so that the travel times on each edge are reduced to 5. The road from C to D that was constructed in part (b) is still available. Find a Nash equilibrium for the game played on the network with the smaller travel times for C-B and A-D. What are the equilibrium values of x and y? What is the total cost of travel? What would happen to the total cost of travel if the government closed the road from C to D?

Solution by Bennett Parsons:

- **a.** Nash equilibrium occurs at even flow – that is, when $x = 500$ and $y = 500$.
- **b.** We now have that the Nash equilibrium occurs only when all cars travel this new route A-C-D-B. The equilibrium values for x and y are now both 1000. The total cost of travel is now $20 * 1000 = 20000$, which is greater than the previous total cost of travel which was only $17 * 1000 = 17000$.
- **c.** Now, the Nash equilibrium is the same as part a, where half the drivers will travel A-C-B and the other half will travel A-D-B; the total cost is $10 * 1000 = 10000$. Since no drivers will use road C-D in this model (assuming Nash equilibrium), closing the C-D road will have no affect on the total cost of travel.

5. Learning Influence Locally (20 points) Let's say we have a graph $G = (V, E)$ and some opinion data for each node $n \in V$. Assume each node has an opinion that changes over time based on the influence of its neighbors for a particular observed cascade. For a particular observed cascade i, let $opinion_i$ be a vector of timestamped opinions, where $opinion_i[n]$ is a vector such that $opinion_i[n][t]$ is equal to the opinion of node n at time t during observed cascade i, where t ranges from 0 to $\tau - 1$ so that $opinion_i[n]$ has length τ, where opinions are either 0 (node is inactive) or 1 (active).

Now consider one way of modeling how opinions spread across a graph, the Independent Cascade (IC) model. In the IC model, each directed edge (n, m), has some corresponding edge weight $e_{(n,m)} \in (0, 1]$.

If n is activated at time step t, then at $t+1$, n will try activate each of its neighbors $m \in N(n)$ and will succeed with probability $e_{(n,m)}$. Node n will only try to activate its neighbors once, in the time step after n was activated. Moreover, once a node is active, it cannot become inactive.

If we want to fit the independent cascade (IC) model locally to the opinion data, we can use the maximum likelihood estimate (MLE) of the edge weight from n to m, defined as

$$\frac{\sum_i (opinion_i[n][t_{i,n}] * opinion_i[m][t_{i,n}+1] * (1 - opinion_i[m][t_{i,n}]))}{\sum_i opinion_i[n][t_{i,n}]}$$

where $t_{i,n}$ is the first time n becomes active in cascade i, and we ignore the cascade if n never becomes active. Basically: for a node n, we average across all cascades the likelihood of it (potentially) causing its neighbor m's activation.

The file *network1.txt* is a directed graph, where $a \quad b$ corresponds to an edge from a to b. The file *node_opinions.pk* is an array of opinions, where each entry *node_opinions*$[i]$ is a dictionary equivalent to the vectors $opinion_i$ described above, such that $opinion_i[n]$ gives the vector of timestamped opinions described.

(Whew! That was a lot. But don't worry, we'll break it down. It's simpler than you'd think.)

a. **(2 points)** Load the file *opinions.pk*. If you're using python, you can use python's 'pickle' library and the 'load' function to easily turn the file into an array; if you're using another language, we've provided *node_opinions.json*, a JSON-encoded file that you can easily load in any language using the standard JSON encoding and the respective JSON library for your language (though you might have to do some additional string to integer conversion). What is τ and how many cascades are there? At what timestep (using 0-indexing) does node 13 first have its opinion activated in cascade 6? We will use the convention that if a node is not activated during a cascade then its activation time is τ. (You'll be tempted to do this last part visually by printing the opinion vector, but perhaps do it algorithmically now...)

b. **(2 points)** Now load the file *network1.txt*. How many nodes are there? What is the average (out-)degree?

c. **(4 points)** Write a function $doesInfluence(n, m, t, o)$ that takes nodes n and m, time t, and cascade dictionary o, and returns 1 if n is activated at time t and m is activated at time $t+1$ and not active at time t, or 0 otherwise (or if $t+1 >= \tau$). For argument $(10, 4, 1, opinion_0)$, what does your function output?

d. **(10 points)** Using the MLE described above, approximate the edge weights of each $e \in E$. What's the estimated weight of edge $(1,2)$? Of $(26,21)$? Additionally, submit your estimates as a separate file formatted as a .csv, where entries in the first column correspond to the first node in the edge, entries in the second column correspond to the second node in the edge, and entries in the third column correspond to the edge weight.

e. **(2 points)** What node has the highest average edge weight (for outgoing edges)? What node has the lowest average edge weight for outgoing edges? What node, on average, is activated first (in cascades where it activates)? What node, on average, is activated last?

Solution: Thanks, Matthew Aguirre!

Solution:

a. There are 100 cascades in opinions.pk, each with $\tau = 30$. Node 13 first has its opinion activated at time step 2 in cascade 6.

b. There are 30 nodes in the graph in network1.txt, with average out-degree 5.23.

c. See code below. doesInfluence(10, 4, 1, 0) = 1.

d. Full predictions attached on Canvas. The estimated weight of edge (1,2) is 0.67 and the estimated weight of edge (26,21) is 0.303.

e. The desired information is below, per the provided script:

```
11 has highest average outgoing edge weight while 28 has the lowest.
11 is activated last, on average, while 14 is activated earliest.
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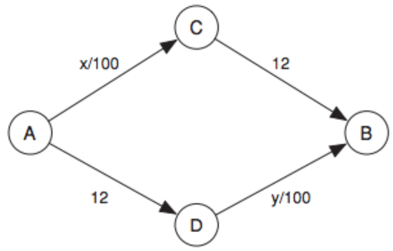

Figure 1: A traffic network.