**CS 134: Networks**

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# 1 Overview

In today's lecture we will be discussing matching markets, or markets defined on networks that involve matching in some way. Examples include:

- Kidney exchanges - matching donors to those who need kidney donations
- Medical residencies - matching applicants to residency programs
- School matching - matching students to public schools; Al Roth won the Nobel Prize in 2012 for proposing a matching algorithm for the NYC public school system
- Dating websites - matching people to each other
- Uber - matching drivers to those requesting rides
- Sponsored search - matching firms to search queries based on auctions

# 2 Matching Basics

We define matchings on general graphs before defining matchings on bipartite graphs.

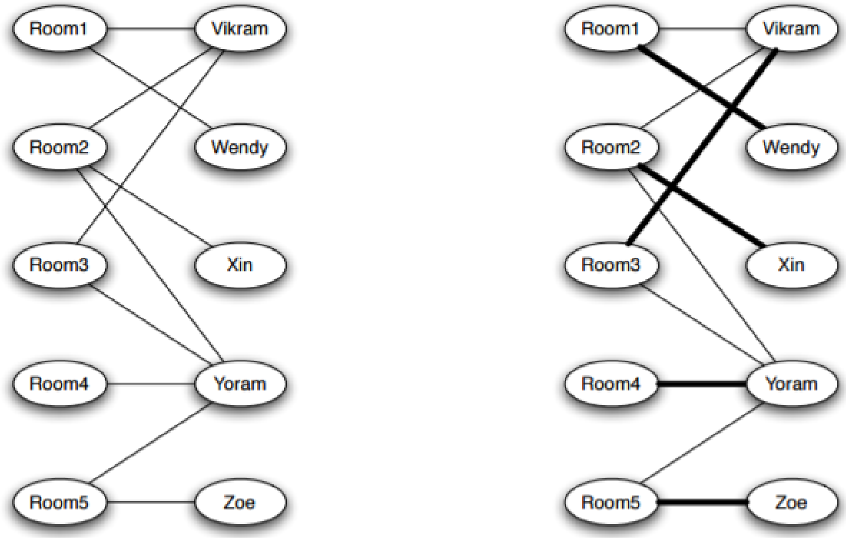
**Definition.** *For a given graph $G = (V, E)$, a* ***matching*** *is a set of edges $M \subseteq E$ s.t. the endpoints of the edges in $M$ are all unique. That is, if for some $(u, v) \in E$ we have that $(u, v) \in M$ then $(u, w) \notin M$ for any $(u, w) \in E$.*

We will now define two concepts: maximal and maximum matching. A matching is *maximal* if adding an edge violates the matching, and *maximum* if there are no other matchings in the graph with more edges.

**Definition.** *A matching $M$ is called* ***maximal*** *if for any $e \in E \setminus M$ we have that $M \cup \{e\}$ is not a matching. A matching $M$ is called* ***maximum*** *if for any other matching $M' \subseteq E$, we have that $|M'| \leq |M|$.*

## 2.1 Matching on bipartite graphs

To begin, we will analyze matching markets defined on bipartite graphs; but first, we define the concept of a bipartite graph:




(a) *Bipartite Graph* (b) *A Perfect Matching*

Figure 10.1: (a) An example of a bipartite graph. (b) A perfect matching in this graph, indicated via the dark edges.

**Definition.** *A graph $G = (N, E)$ is* ***bipartite*** *iff it can be partitioned into two disjoint subsets of nodes $U, V$ such that (1) $U \cup V = N$ and (2) $\forall e \in E, \exists (u, v) \in U \times V : e = (u, v)$. In words, every node is in one of the two subsets and every edge goes between the two subsets (there are no edges within one of the subsets).*

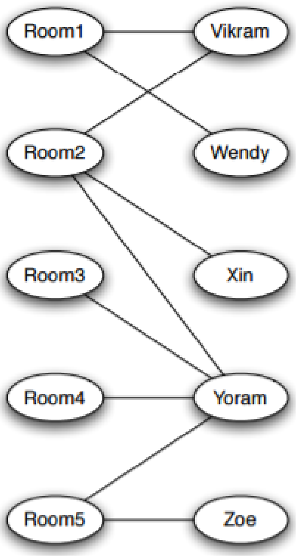
Moreover, we can think about assignments from nodes in $U$ to nodes in $V$ (that is, assignments have the type $U \to V$). We can also think about a matching assignment as a selection of a subset of the edges in the graph. Now, we can define what it means for a matching assignment to be perfect:

**Definition.** *Let $G = (U, V, E)$ be a bipartite graph with subsets $U$ and $V$. Then when $|U| = |V|$, a* ***perfect matching*** *is an assignment of nodes in $U$ to nodes in $V$ such that (1) $u \in U$ is assigned to $v \in V \iff (u, v) \in E$ and (2) no two nodes in $U$ are assigned to the same $v \in V$.*

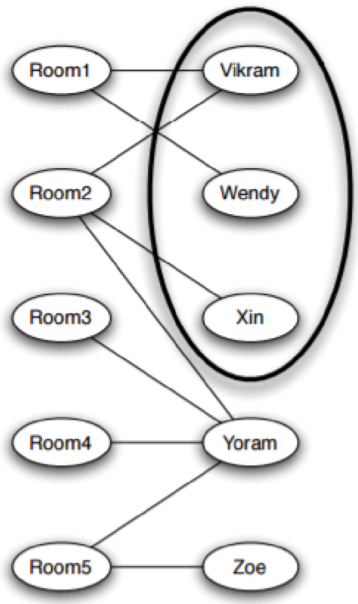
Now, how can we easily show that a graph has no perfect matching? To do this, we can utilize the concept of a *constricted set*:

**Definition.** *A set of vertices $S \subseteq V$ is called a* ***constricted set*** *if $|S| > |N(S)|$ where $V$ is one side of the bipartite graph and $N(S)$ is the neighborhood of $S$.*

Note that intuitively, a constricted set is a set of nodes on one side of the bipartite graph with fewer edges leaving the set than there are nodes in the set; the existence of a constricted set in a bipartite graph would mean that it is impossible to give a perfect matching for those nodes in the constricted set. This fact is the crux of the matching theorem.




(a) *Bipartite graph with no perfect matching*




(b) *A constricted set demonstrating there is no perfect matching*

Figure 10.2: (a) A bipartite graph with no perfect matching. (b) A constricted set demonstrating there is no perfect matching.

# 3 The Matching Theorem

**Theorem 1** (Matching Theorem)**.** *If a bipartite graph (with $|U| = |V|$) does not have a perfect matching, then it must contain a constricted set of vertices.*

Therefore, we can easily show that a graph has no perfect matching by identifying a constricted set of vertices in the bipartite graph. Remarkably, this is the only condition for there not to exist a perfect matching.

## 3.1 Alternating and Augmenting paths

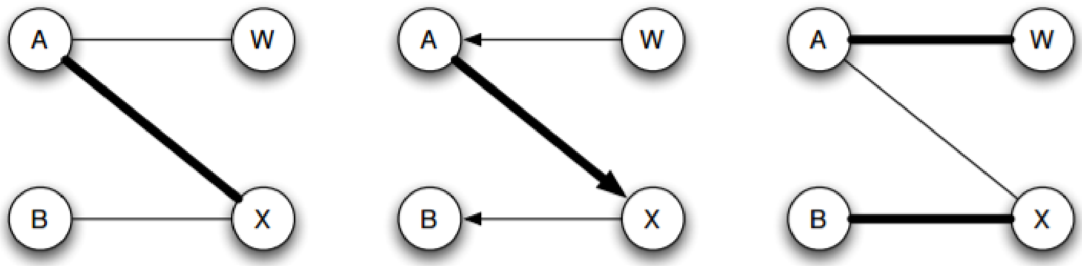
To prove the matching theorem, we will use the following concepts.

| |
|---|
| **Definition.** *A path is **alternating** if it is simple (does not repeat any nodes) and alternates between edges in a particular match and not in the match. A path is **augmenting** if it is an alternating path with unmatched endpoints. Intuitively, the existence of an augmenting path means that we can augment, or increase the size of, a matching.* |

See Figures 10.8 and 10.9 for depictions of alternating and augmenting paths respectively.

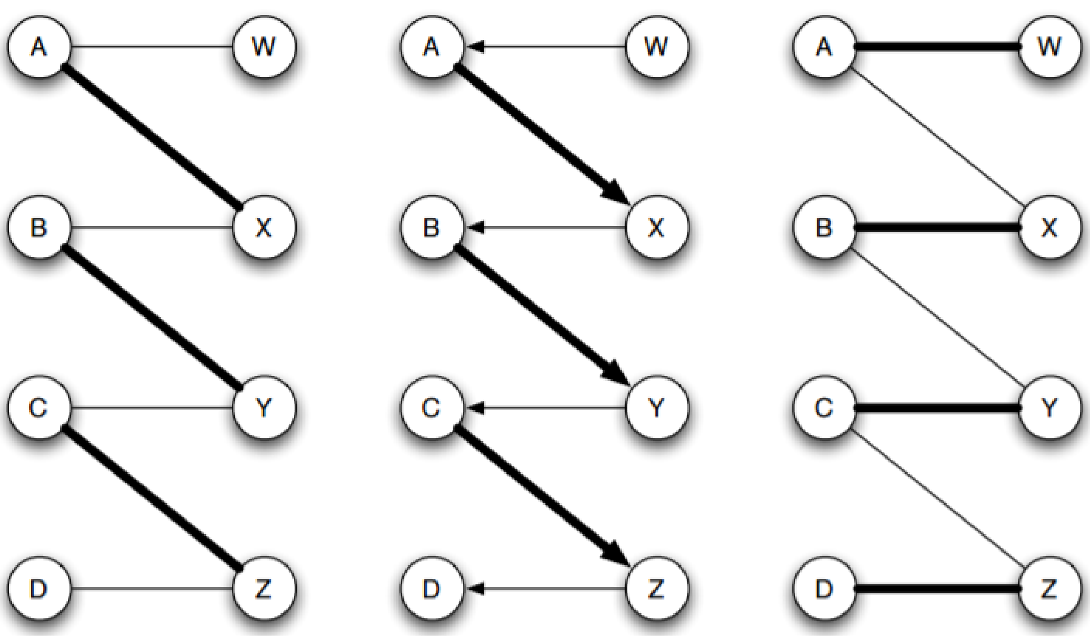
## 3.2 Images for Proof

Relevant images for the proof are Figures 10.10, 10.11, 10.12, 10.13, and 10.14.



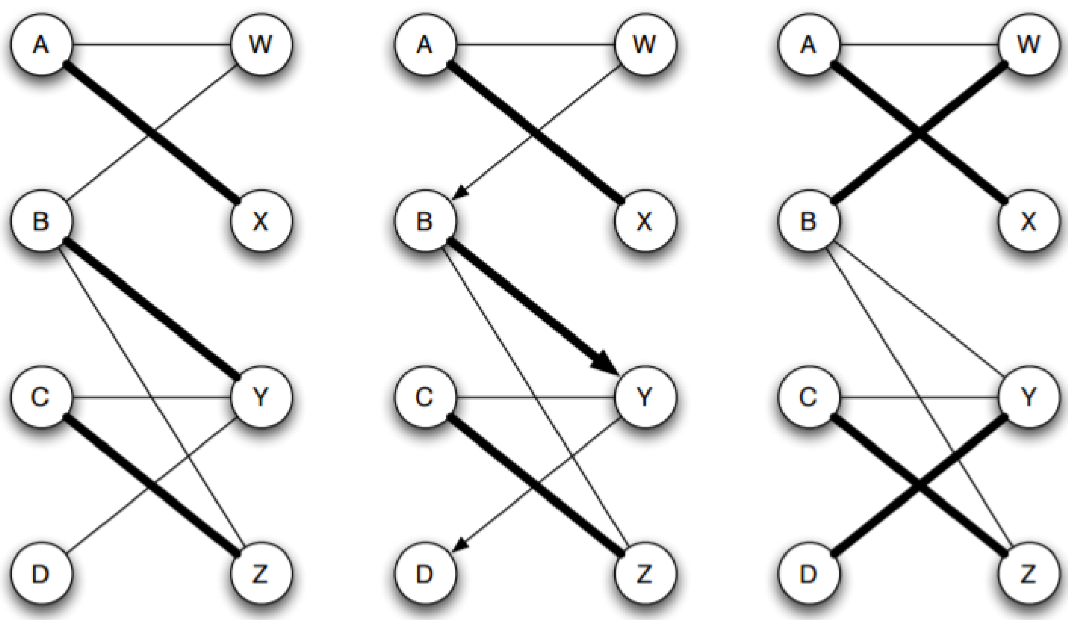

(a) *A matching that is not of maximum size* (b) *An augmenting path* (c) *A larger (perfect) matching*

Figure 10.8: (a) A matching that does not have maximum size. (b) What a matching does not have maximum size, we can try to find an *augmenting path* that connects unmatched nodes on opposite sides while alternating between non-matching and matching edges. (c) If we then swap the edges on this path — taking out the matching edges on the path and replacing them with the non-matching edges — then we obtain a larger matching.




(a) *A matching that is not of maximum size* (b) *An augmenting path* (c) *A larger (perfect) matching*

Figure 10.9: The principle used in Figure 10.8 can be applied to larger bipartite graphs as well, sometimes producing long augmenting paths.




(a) *A matching that is not of maximum size*

(b) *An augmenting path*

(c) *A larger (perfect) matching*

Figure 10.10: In more complex graphs, finding an augmenting path can require a more careful search, in which choices lead to "dead ends" while others connect two unmatched nodes.

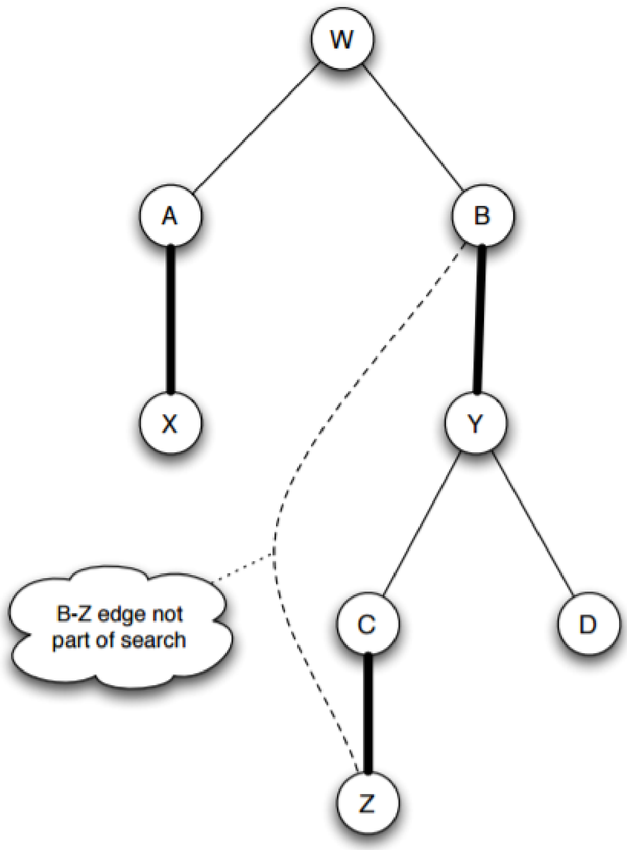



Figure 10.11: In an alternating breadth-first search, one constructs layers that alternately use non-matching and matching edges; if an unmatched node is ever reached, this results in an augmenting path.

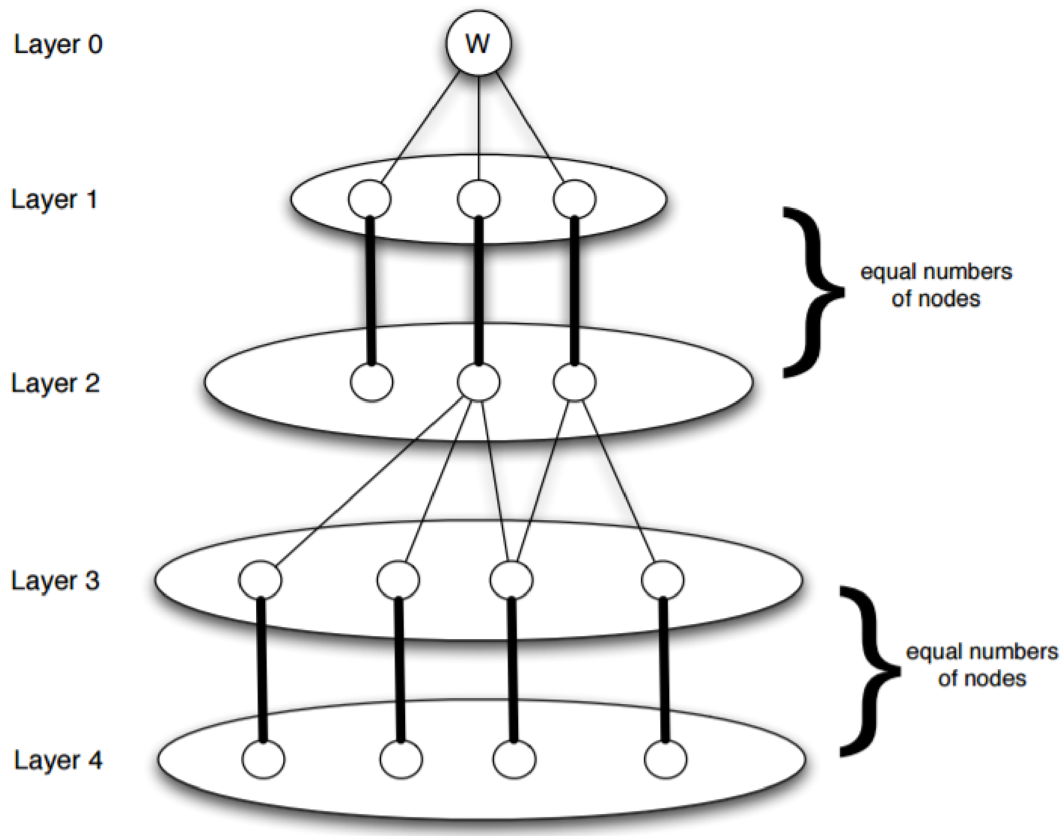
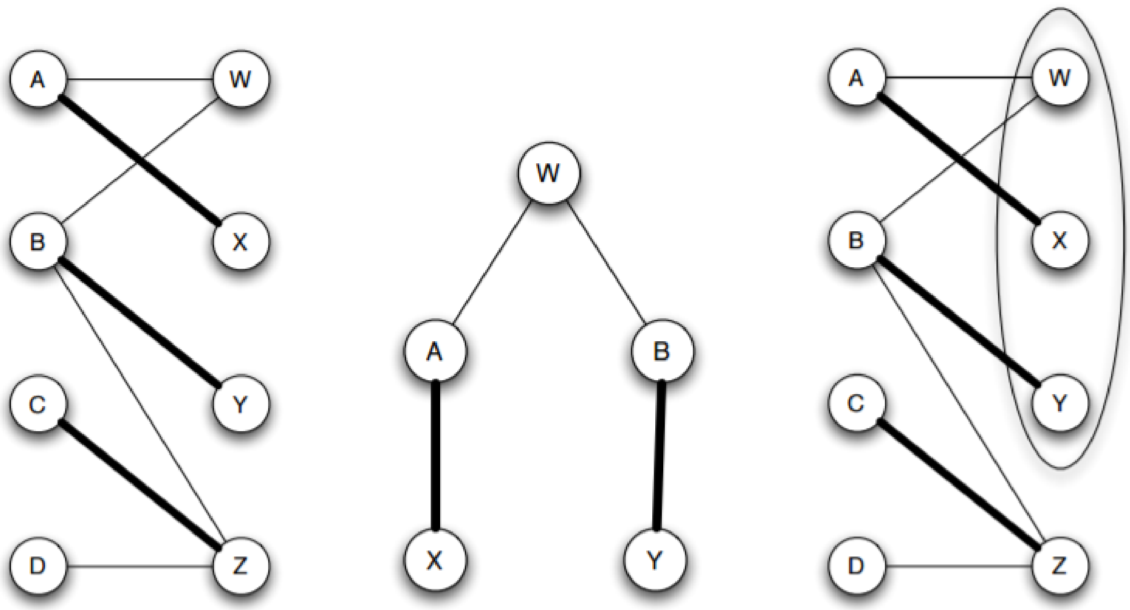



Figure 10.12: A schematic view of alternating breadth-first search, which produces pairs of layers of equal size.




(a) *A maximum matching that is not perfect*

(b) *A failed search for an augmenting path*

(c) *The resulting constricted set*

Figure 10.13: (a) A matching that has maximum size, but is not perfect. (b) For such a matching, the search for an augment path using alternating breadth-first search will fail. (c) The failure of this search exposes a constricted set: the set of nodes belonging to the even layers.

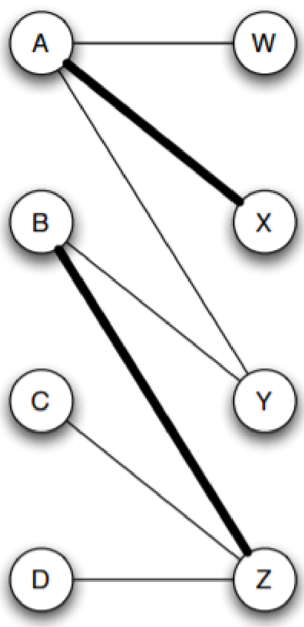



Figure 10.14: If the alternating breadth-first search fails from *any* node on the right-hand side, this is enough to expose a constricted set and hence prove there is no perfect matching. However, it is still possible that an alternating breadth-first search could still succeed from some other node. (In this case, the search from $W$ would fail, but the search from $Y$ would succeed.)