CS 134: Networks	Lecture 4 **outline**, Wednesday, 2/1
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1 Overview

Desirable properties of social network models:

1. The *average degree* of the nodes in the graph is **constant**;
2. The *clustering coefficient* of the graph is **large** (constant, bounded from 0);
3. The *diameter* of the graph is **small** (poly-logarithmic in size of graph).

2 Square Lattices, Expansion, and Regular Random Graphs

Definition. *In a* ***square lattice*** *on $n = r^2$ nodes and parameter k (see figure below), nodes are positioned in an $r \times r$ lattice and are connected to nodes at a lattice distance at most k away on the lattice for some given parameter $k \in \mathbb{N}$. The* ***lattice distance*** *between two nodes on a lattice is the minimum number of horizontal and vertical steps between the two nodes.*

Question: *What is the longest distance between two nodes in a square lattice?*

Definition. *A d-regular random graph is a graph where all nodes have degree d and every two nodes have the same likelihood to be connected. That is, fix some number of vertices n, consider all graphs on n vertices that are d-regular, and pick one of these uniformly at random.*

Question: *Do d-regular random graphs have short distances?*

Definition. *The* expansion *of a graph $G = (V, E)$ is the minimum, over all cuts we can make (dividing the graph in two pieces), of the number of edges crossing the cut divided by the number of vertices in the smaller half of the cut. Formally, it is*

$$\alpha = \min_{S \subseteq V, 1 \leq |S| \leq \frac{|V|}{2}} \frac{|e(S)|}{|S|}$$

where $e(S)$ is the number of edges leaving the set S.

Theorem. *Suppose a graph G is d-regular, for some constant $d \geq 3$, and has constant expansion α. Then the diameter of G is $O(\frac{d}{\alpha} \log n)$.*

Question: *What is the clustering coefficient of d-regular random graphs?*

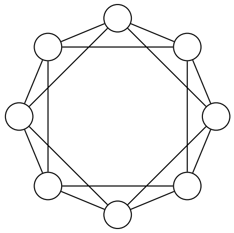

Figure 1: A ring lattice with $n = 8$ nodes and $k = 2$

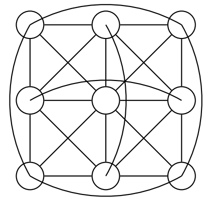

Figure 2: A square lattice with $r \times r = 3 \times 3 = 9$ nodes and $k = 2$.

3 Constructing a Small World Model

	diameter	clustering coefficient	maximum degree
Lattice	bad (too large)	good (constant)	good (constant)
d-regular random	good (logarithmic)	bad (too small)	good (constant)
Watts-Strogatz	good	good	good

4 Navigation in a Small World

Question: *How did people navigate the package towards the correct destination without knowing the entire social network?*

Definition. *The Kleinberg model is defined as follows:*

- *Nodes are positioned on a square lattice with n nodes.*
- *Each node is connected to the nodes that are at most a distance k away on the lattice, for some given parameter $k \in \mathbb{N}$.*
- *For some given $\ell \in \mathbb{N}$ each node u has ℓ* long range *edges which are randomly connected to other nodes in the network with the following probability:*

$$\Pr[u \to v] = \frac{\Delta(u,v)^{-2}}{\sum_{w \neq u} \Delta(u,w)^{-2}}$$

where

- *$u \to v$ is the event of a long range edge from u to v*
- *$\Delta(u,v)$ is the distance from u to v on the grid*

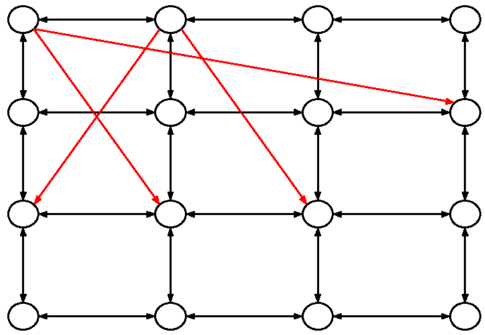

Figure 3: A partially constructed Kleinberg model with $k = 1$, $\ell = 2$, and $n = 16$. We start with a 4×4 square lattice. Connect all nodes with a lattice distance of 1 (black arrows). Then for each node u, add 2 random edges where the probability of adding edge (u, v) is proportional to $\Delta(u, v)^{-2}$. For clarity, the diagram above shows only a partial construction – only the first two nodes on the first row have been augmented with random edges (red arrows). Note also that Kleinberg's model is a random graph, so on a different realization the black arrows will stay the same but the red arrows will likely change.

4.1 Properties of Kleinberg's model

Lemma. *In Kleinberg's model,*

1. *for all $u \in V$, the number of nodes at lattice distance exactly d from u is $\Theta(d)$;*
2. *for all $u \in V$, the number of nodes at lattice distance at most d from u is $\Theta(d^2)$;*
3. $\Pr[u \to v] = \Theta(\frac{\Delta(u,v)^{-2}}{\log n})$.

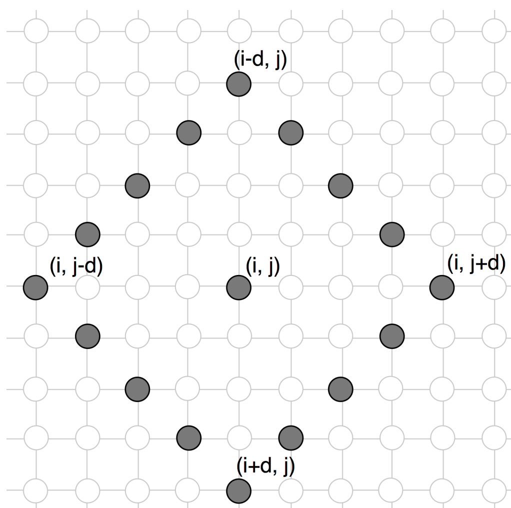

Figure 4: A square lattice near coordinate (i, j). The outer shaded vertices have a lattice distance exactly d from (i, j). The vertices inside this square have a lattice distance less than d from $(i.j)$.

4.2 The Main Result

Theorem. *For graphs generated according to Kleinberg's model with $k = \ell = 1$, when every node forwards information to the neighbor closest to the target, the information reaches its target after $O(\log^2 n)$ steps, in expectation.*

References

[1] Duncan J. Watts and Steven H. Strogatz. *Collective dynamics of small-world networks.* Nature, 393:440442, 1998.

[2] Jon Kleinberg. *The small-world phenomenon: An algorithmic perspective.* Proc. 32nd ACM Symposium on Theory of Computing, 2000.