**CS 134: Networks** — Practice Midterm 2

Prof. Yaron Singer — **Not to be turned in.**

**Instructions.** This exam consists of five questions. You have to answer the first two questions, and *any two of the last three questions* (in case you answer all three, please indicate clearly which two you want us to grade or we will grade the first two).

Unless otherwise stated, the following assumptions are satisfied by all graphs: They are undirected. There are never multiple edges between the same pair of nodes $\{i, j\}$. There are also no self-edges (ones starting and ending at the same node).

**1. Definitions (25 points).**

**a. (10 points)** Consider a three player game with players A, B, and C where each player respectively has two strategies: up and down; left and right; in and out. Their utilities, denoted via $(u_A, u_B, u_C)$, are given as follows:

| | L | R |
|---|---|---|
| U | (34, 25, 41) | (32, 32, 36) |
| D | (32, 30, 38) | (33, 31, 36) |

I

| | L | R |
|---|---|---|
| U | (34, 29, 37) | (38, 32, 30) |
| D | (35, 38, 27) | (36, 39, 25) |

O

What are the Nash equilibria of this game?

**b. (5 points)** Write down the transition matrix for the undirected graph shown in Figure 1 assuming that a random walk at node $v$ goes to any of $v$'s neighbors with equal probability.

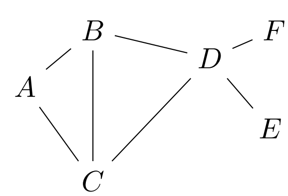



Figure 1: Graph for 1-b and 1-c

**c. (5 points)** For the graph in Figure 1, what is the probability of a random walk starting at node $D$ reaching node $A$ in 2 steps?

**d. (5 points)** In the $G(n, p)$ model, what's the probability of getting a graph with exactly two edges when $n = 5$ and $p = 1/2$?

**Solution:**

**a.** just one: $(D, R, I)$

**b.** Where the first row and column correspond to $A$, second to $B$, etc.:

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

**c.** We square the above matrix and look at the entry corresponding from going from $D$ to $A$ to get $\frac{1}{6}$.

**d.** There are $\binom{5}{2} = 10$ possible edges, so our overall probability is $\binom{10}{2}(.5)^2(.5)^8$

**2. True or False (25 points).** For each statement, indicate whether it is true or false. If you indicate it is true, nothing else is required. If you indicate it is false, correct the statement or give a counterexample.

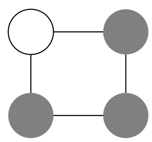
- **a. (10 points)** Aliens are applying to Harvard's new satellite campus on Mars. Each alien must choose a specific department to apply to. Suppose that green and blue aliens only apply to Computer Science, Applied Math, and Economics, and within each department, a higher percentage of green aliens that applied to that department are admitted than for blue aliens that applied. Then, overall, green aliens have a higher admittance rate than blue aliens.
- **b. (5 points)** Given a bipartite graph with equal number of nodes in each partition, if there is no augmenting path, then there is necessarily a constricted set.
- **c. (5 points)** Consider the stochastic voter model on an arbitrary network where nodes have opinions either 0 or 1. The number of people holding opinion 1 stays constant as the opinions update according to the model.
- **d. (5 points)** Consider a random walk on an undirected graph with $n$ nodes. If the PageRank algorithm converges, then there is a cycle of length $n$ in the network.

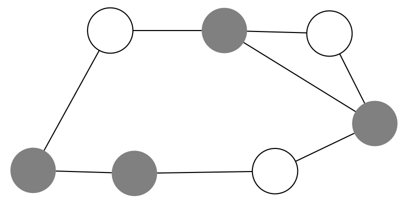
**Solution:**

- **a.** False; we could have Simpson's paradox.
- **b.** False; there could be a perfect matching.
- **c.** False; a path of three nodes with the middle node initially holding opinion 1 and the two other nodes initially holding opinion 0. In the next round, we could have that all nodes flip opinions.
- **d.** False; $G = (\{a, b, c, d, e\}, \{ab, ac, ad, ae, bc, de\})$

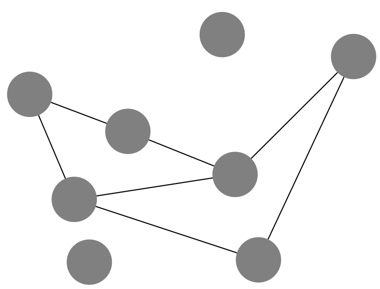
**3. Conformists and Rebels (25 points).** Each person in town can choose to wear a hat (action 1) or not to wear a hat (action 0). There are two types of people in the community: *conformists* and *rebels*. A conformist prefers to do what the majority of his friends do, e.g. wear a hat if a majority of friends wear hats; if exactly half of his friends wear hats, he is indifferent. A rebel prefers to do the opposite of the majority of her friends do, e.g. she likes not to wear a hat if the majority of her friends do; if exactly half of her friends wear hats, she is indifferent.

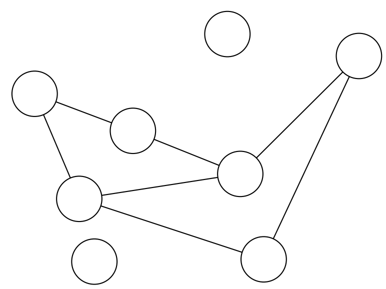
**a.** (**5 points**) Find a Nash equilibrium in each of the two networks shown below. Filled circles are conformists, hollow circles are rebels.





**b.** (**5 points**) In each of the following networks, what is the maximum number of people wearing hats that can be achieved in any equilibrium?





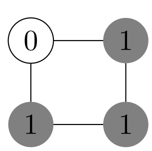
**c.** (**5 points**) How many equilibria are there in the left-hand network in part (b)?

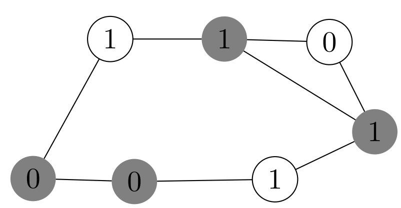
**d.** (**5 points**) Find an example of a network in which a (pure-strategy) Nash equilibrium does not exist. If this is not possible, explain why.

**e.** (**5 points**) Find an example of a social network *with conformists only* such that a Nash equilibrium does not exist. If this is not possible, explain why.

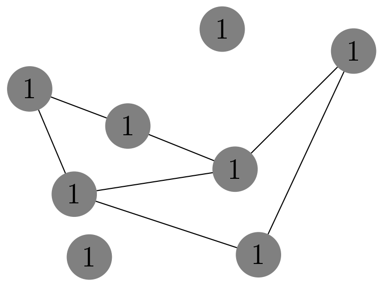
**Solution:**

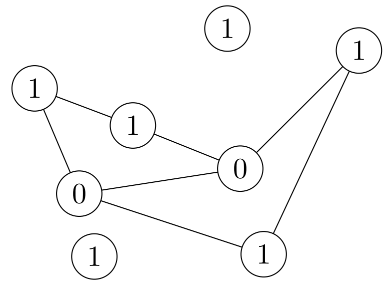
**a.** For the smaller graph, all conformists play 1 and the rebel plays 0.



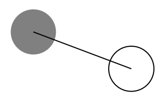

**b.** For the all-conformist graph, 8. For the all-rebel graph, 6.




**c.** 32

**d.** A graph of one conformist and one rebel connected to each other:



**e.** Not possible; if there are all conformists, everyone wearing hats is always a Nash equilibrium.

**4. Influence in Networks. (25 points)** Consider an influence model which is similar to, but different from, the voter model as taught in lecture. As in the voter model, nodes in an undirected network initially all have opinion 0 or 1. At time step $t+1$, a node adopts opinion 1 if at least half of its neighbors hold opinion 1 at time $t$ and 0 otherwise. Isolated nodes always adopt opinion 1. The graph does not have self loops.

Provide a concise justification for your answers.

**a. (5 points)** Is the evolution of opinions in this model deterministic or random? Give the answer for the voter model as well.

**b. (10 points)** Denote by $f(S)$ the expected number of nodes holding opinion 1 after $k$ time steps if the set of nodes which initially hold opinion 1 is $S$. We call such an influence function additive if

$$f(S) = \sum_{i \in S} f(\{i\}).$$

Is the influence function for this influence model additive? Does this differ from the voter model?

**c. (10 points)** Construct an undirected network and select initial opinions such that at least half of the nodes initally hold opinion 1 and such that after some number of steps, all nodes have opinion 0 for the model described above.

**Solution:**

**a. (5 points)** The voter model is random; this model is deterministic.

**b. (10 points)** Consider a clique of 4 nodes. If we choose our initial set to be a single node $v$, then $f(\{v\}) = 0$. However, if we choose any two nodes $u, v$ as our initial set, then $f(\{u, v\}) = 4$. Thus, the influence function is not additive for this model.

The influence function in the voter model is additive because the in adding any node to our initial seed set, we can just add their expected number of nodes influenced since there are no influence interactions in the voter model.

**c. (10 points)** Consider a graph constructed as follows: we have a clique of four nodes. Each node in this clique also has an edge to a "spoke" node whose only edge is to that clique node. Then we have a total of 8 nodes and choose our initial set of 1-players as the four "spoke nodes". They are unable to influence the clique and they themselves adopt opinion 0.

**5. Graph Coloring (25 points).** Consider a graph $G = (V, E)$ for which we want to assign each vertex in $G$ a color, red or blue. To do so, we use the following algorithm:

| **ALG: Color Graph** |
| --- |
| **input:** $G = (V, E)$ |
| **initialize**: Randomly choose $\mu_r, \mu_b$ from $V$ u.a.r. |
| **repeat until convergence** |
| set $S_r = \{v \in V : d(v, \mu_r) \leq d(v, \mu_b)\}$, $S_b = V \setminus S_r$ |
| $\mu_r = \arg\min_{v \in S_r} \sum_{u \in S_r} d(v, u)$ |
| $\mu_b = \arg\min_{v \in S_b} \sum_{u \in S_b} d(v, u)$ |
| **return** $S_r, S_b$ |

where $d(u, v)$ is the length of the shortest path between nodes $u$ and $v$. Prove that the *dispersion* of the clusters is non-increasing between consecutive iterations of the algorithm, where dispersion is defined as

$$disp(S_r, S_b, \mu_r, \mu_b) = \sum_{v \in S_r} d(v, \mu_r) + \sum_{v \in S_b} d(v, \mu_b)$$

**Solution:** We observe that this algorithm is just $k$-means with $k = 2$ and the dispersion is exactly the $k$-means objective function, albeit with a more general distance function. Then the same proof of the non-increasing-ness of the $k$-means objective holds for non-increasing-ness of dispersion.