GR5241_Homework1_xz2735

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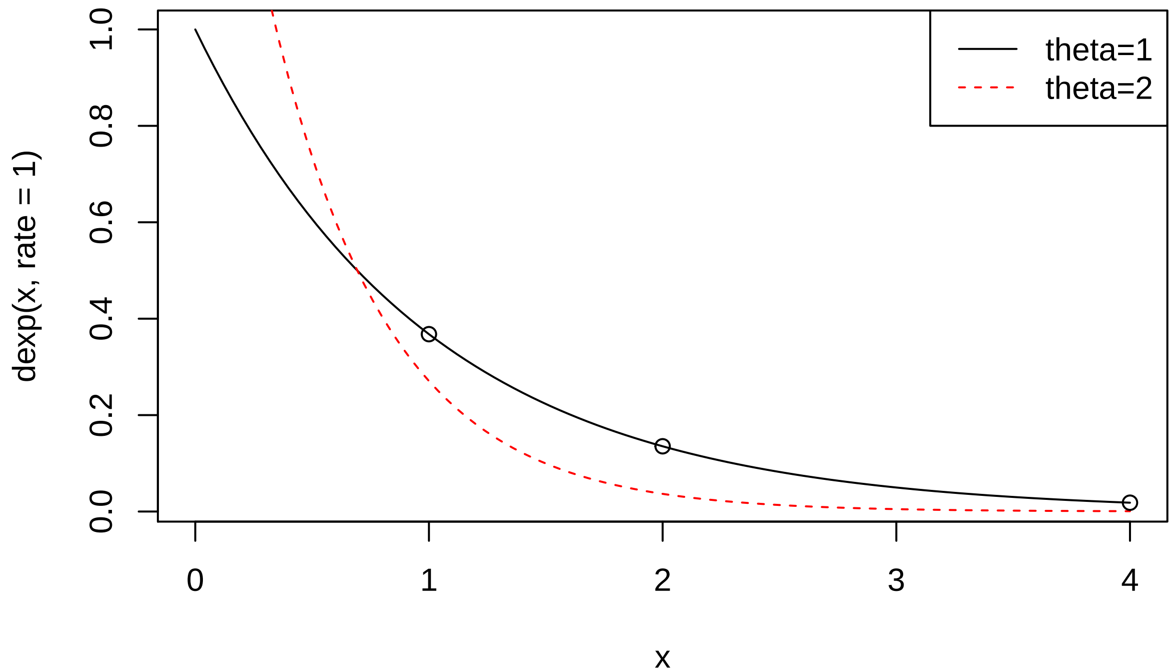
Problem 1.

1&2&3.

```
curve(dexp(x,rate=1),from=0,to=4,lty=1)
curve(dexp(x,rate=2),from=0,to=4,add=TRUE,col="red",lty=2)
legend("topright",legend=c("theta=1","theta=2"),col = c("black","red"),lty=1:2 )
points(1,exp(-1))
points(2,exp(-2))
points(4,exp(-4))
```



So the higher rate decreases the likelihood of of sample that is bigger than approximate 1.7.

Question 1

As

$$f_n(x|\theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$$

The posterior is

$$\xi(\theta|x) \propto \theta^n e^{-\theta \sum_{i=1}^{n} x_i} \cdot \theta^{\alpha-1} e^{-\beta\theta} = \theta^{n+\alpha-1} e^{-(\beta+\sum_{i=1}^{n} x_i)\theta}$$

Thereore, $\theta|X_n = x \sim Gamma(n+\alpha, \beta + \sum_{i=1}^{n} x_i)$.

Question 2

a.

As

$$\prod(\theta|X_{1:n}) = x \sim Gamma(n+\alpha, \beta + \sum_{i=1}^{n} x_i)$$

obtained above so

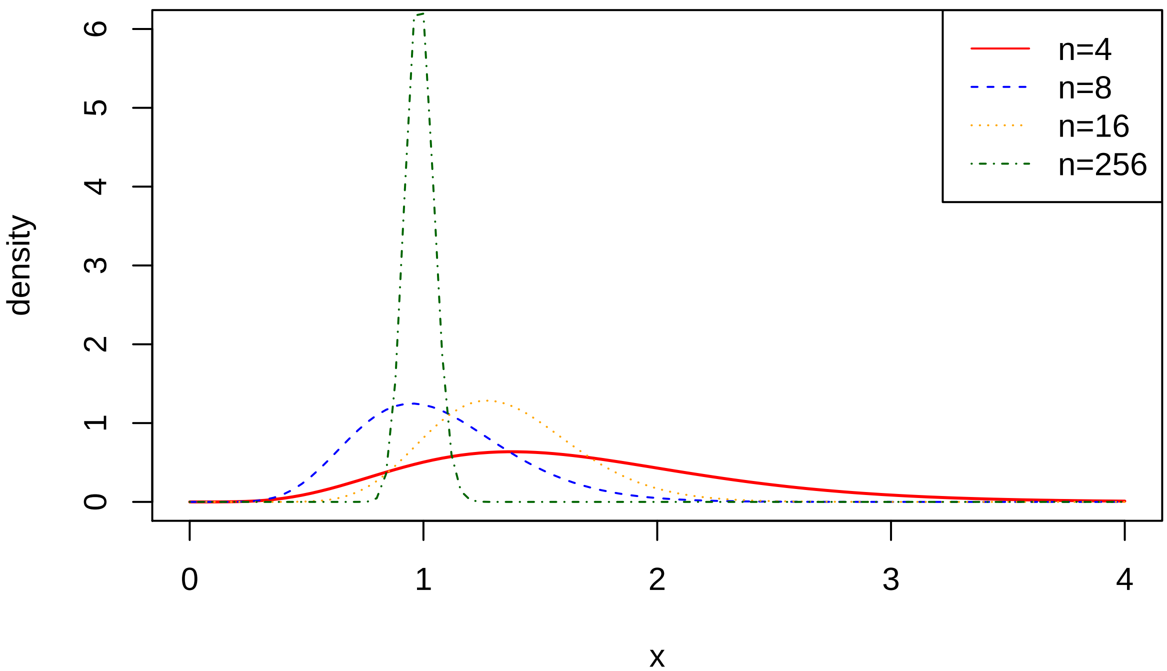
$$\widetilde{q}(\theta) = \prod(\theta|X_{1:n-1}) = x \sim Gamma(n-1+\alpha, \beta + \sum_{i=1}^{n-1} x_i)$$

. By Induction,

$$\xi(\theta|X) \propto \theta^n e^{-\theta \sum_{i=1}^{n} x_i} \cdot \theta^{\alpha-1} e^{-\beta\theta} = \theta^{n+\alpha-1} e^{-(\beta+\sum_{i=1}^{n} x_i)\theta} = \theta^{n-1+\alpha-1} e^{-(\beta+\sum_{i=1}^{n-1} x_i)\theta} \cdot \theta e^{-x_n\theta} = \widetilde{q}(\theta) \cdot \prod(\theta|X_n)$$

b.

```
a = rexp(256,rate=1)
curve(dgamma(x,shape=4+2,rate=0.2+sum(a[1:4])),from=0,to=4,col="red",ylim=c(0,6),lwd=1.5,ylab="density"
curve(dgamma(x,shape=8+2,rate=0.2+sum(a[1:8])),from=0,to=4,add=TRUE,col="blue",lty=2)
curve(dgamma(x,shape=16+2,rate=0.2+sum(a[1:16])),from=0,to=4,add=TRUE,col="orange",lty=3)
curve(dgamma(x,shape=256+2,rate=0.2+sum(a)),from=0,to=4,add=TRUE,col="dark green",lty=4)
legend("topright",legend=c("n=4","n=8","n=16","n=256"),col=c("red","blue","orange","dark green"),lty = 
```



As n increases, the scale of posterior distribution shrinks and the peak becomes larger.

Problem 2

T_i follows $Bernoulli(0.5)$, likelihood of T_i is $\frac{1}{2}^n$, Now, we know that $Y^{T_1} \sim Bernoulli(\pi^1)$ and $Y^{T_2} \sim Bernoulli(\pi^2)$. The number of patients who received treatment one is $n_1 = \sum_{i=1}^{n} I(T_i = 1)$, The number of patients who received other treatment is $n_2 = n - n_1 = n - \sum_{i=1}^n I(T_i = 1)$, where $I(T_i = 1)$ is an indicator. Here,

$$f(Y^t, T|\pi^1, \pi^2) = \prod_{i=1}^n (\pi^1)^{Y_i^1}(1-\pi^1)^{1-(Y_i^1)}(\pi^2)^{Y_i^2}(1-\pi^2)^{1-(Y_i^2)}$$

$$= (\pi^1)^{\sum_{i=1}^n Y_i^1}(1-\pi^1)^{n_1-\sum_{i=1}^n Y_i^1}(\pi^2)^{\sum_{i=1}^n Y_i^2}(1-\pi^2)^{n_2-\sum_{i=2}^n Y_i^2}$$

Thus,

$$\xi((\pi^1, \pi^2)|Y_1^{T_1}, ..., Y_n^{T_n}, T_1, ..., T_n) \propto \frac{1}{2}^n f(Y^t, T|\pi^1, \pi^2) \cdot 1$$

$$\propto f(Y^t, T|\pi^1, \pi^2) = (\pi^1)^{\sum_{i=1}^n Y_i^1}(1-\pi^1)^{n_1-\sum_{i=1}^n Y_i^1}(\pi^2)^{\sum_{i=1}^n Y_i^2}(1-\pi^2)^{n_2-\sum_{i=2}^n Y_i^2}$$

So,

$$\xi(\pi^1|Y_1^{T_1}, ..., Y_n^{T_n}, T_1, ..., T_n) \propto (\pi^1)^{\sum_{i=1}^n Y_i^1}(1-\pi^1)^{n_1-\sum_{i=1}^n Y_i^1}$$

,

$$\xi(\pi^2|Y_1^{T_1}, ..., Y_n^{T_n}, T_1, ..., T_n) \propto (\pi^2)^{\sum_{i=1}^n Y_i^2}(1-\pi^2)^{n_2-\sum_{i=1}^n Y_i^2}$$

, Finally,

$$\pi^1|Y_1^{T_1}, ..., Y_n^{T_n}, T_1, ..., T_n \sim Beta(\sum_{i=1}^n (Y_i^1) + 1, \sum_{i=1}^n I(T_i = 1) - \sum_{i=1}^n (Y_i^1) + 1)$$

,

$$\pi^2|Y_1^{T_1}, ..., Y_n^{T_n}, T_1, ..., T_n \sim Beta(\sum_{i=1}^n (Y_i^2) + 1, n - \sum_{i=1}^n I(T_i = 1) - \sum_{i=1}^n (Y_i^2) + 1)$$

.

Problem 3

(a)

$$E(\bar{X}) = E(\frac{\sum_{i=1}^n X_i}{n}) = \frac{1}{n}E(\sum_{i=1}^n X_i) = \frac{1}{n}\sum_{i=1}^n E(X_i) = \frac{1}{n}n\lambda = \lambda$$

So $\bar{X}$ isan unbiased estimator of λ.

(b)

$$E(T_n-\lambda)^2 = E(T_n-\bar{X}+\bar{X}-\lambda)^2 = E(T_n-\bar{X})^2+E(\bar{X}-\lambda)^2+2E(T_n-\bar{X})E(\bar{X}-\lambda) = E(T_n-\bar{X})^2+E(\bar{X}-\lambda)^2 >= E(\bar{X}-\lambda)^2$$

, as$E(\bar{X} - \lambda) = 0$. When $T_n = \bar{X}$, the both sides are equal. So $\bar{X}$ is optimal unbiased estimator among all unbiased estimator.