Laplace's Holiday

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Introduction

There is a circular area of a beach where I plan to enjoy a sunbath this summer. I aim to set up several sunbeds via considering the temperature distribution over this region. According to the conservation law, the temperature, ϕ satisfy the Laplace's Equation in polar coordinates:

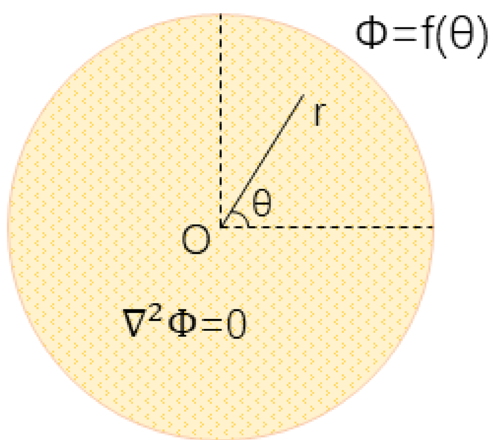
$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0$$

r for the radius, θ is the angle. I use two different methods to draw a conclusion by comparing.

Boundary Conditions

Generally, the heat equation satisfy three boundary conditions:

1. Radius of the beach is α in meters.
2. Periodicity: $\phi(\alpha,\pi) = \phi(\alpha,-\pi)$,$\frac{\partial\phi}{\partial t}(\alpha,\pi) = \frac{\partial\phi}{\partial t}(\alpha,-\pi)$.
3. Dirichlet's boundary condition: $\phi(\alpha,\theta)=f(\theta)$



In this problem, I set $\alpha=100$**,** $\phi(\alpha,\theta)=\sin(\theta)$.

Separation of Variables

By separation of variables, we write ϕ as

$$\phi(r,\theta)=R(r)T(t)$$

we substitute it into the Laplace's Equation, we obtain a set of two ordinary differential equations:

$$\frac{d^2T}{d\theta^2}=-\lambda T(\theta) \quad (1)$$

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}=\lambda R(r) \quad (2)$$

Some interesting technique are included while solving these ODEs. First of all, boundary conditions are important to consider. Additionally, Equation (2) is a very typical example of Cauchy Euler Equation. By substituting $r=\exp z$, we can obtain a simple ODE with constant coefficients. To compute the coefficients in the solution, we need to apply orthogonality and write A_n, B_n in Fourier Series form.

Specified Solution

When $\lambda>0, n=\sqrt[2]{\lambda}\in Z$, the general solution is:

$$\phi(r,\theta)=\frac{B_0}{2}+\sum_{1}^{\infty}[A_nr^n\sin(n\theta)+B_nr^n\cos(n\theta)]$$

$$A_n=\frac{1}{\pi\alpha^n}\int_{-\pi}^{\pi}f(\theta)\sin(n\theta)\,d\theta,\; B_n=\frac{1}{\pi\alpha^n}\int_{-\pi}^{\pi}f(\theta)\cos(n\theta)\,d\theta$$

Specifically, $\phi(\alpha,\theta)=\sin(\theta)$ **, the temperature equation is:**

$$\phi(r,\theta)=\sum_{1}^{\infty}B_nr^n\cos(n\theta) \qquad B_n=\frac{1}{100^n\pi}\int_{-\pi}^{\pi}\sin(\theta)\cos(n\theta)\,d\theta$$

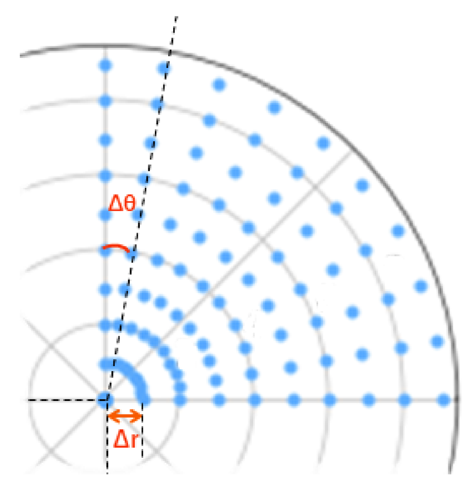
Finite Difference Method

We take partitions of r and θ: $r_i=i\Delta r,\theta_j=j\Delta\theta,\phi(i,j)=\phi(r_i,\theta_j)=\phi_{i,j}$. By mean value theorem, the derivatives can be approximated by "neighbour" points, for instance:

$$\frac{\partial\phi}{\partial r}(i,j)=\frac{\phi(i+\frac{1}{2},j)-\phi(i-\frac{1}{2},j)}{\Delta r},\;\frac{\partial\phi}{\partial\theta}(i,j)=\frac{\phi(i,j+\frac{1}{2})-\phi(i,j-\frac{1}{2})}{\Delta\theta}$$

. Assume $\Delta r=\Delta\theta$, substitute the central difference of r and θ into the Laplace's Equation and simplify:

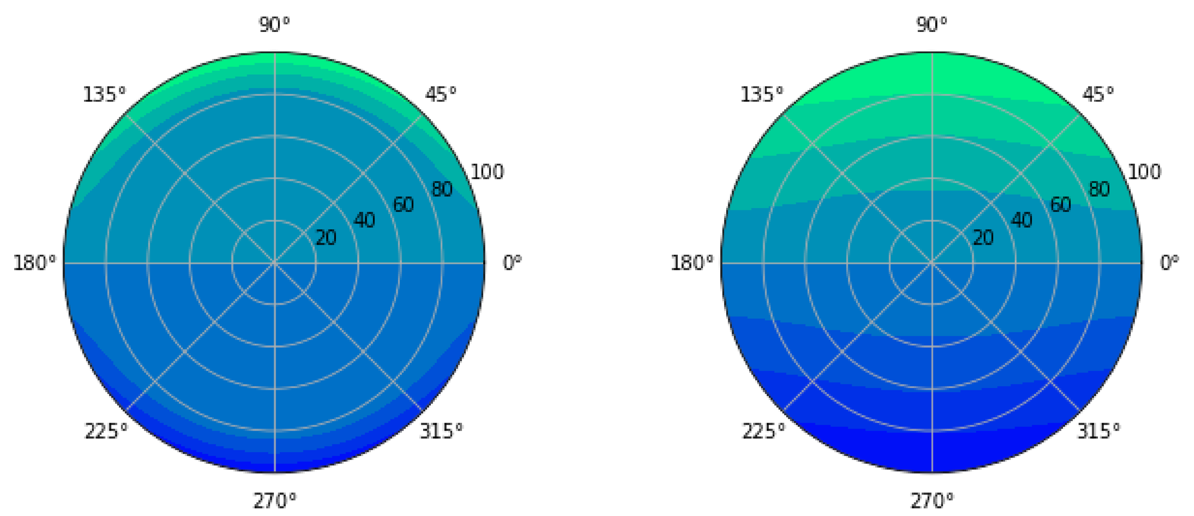
$$\phi(i,j)=\frac{1}{2(i^2+1)}[(i^2+\frac{i}{2})\phi(i+1,j)+(i^2-\frac{i}{2})\phi(i-1,j)+\phi(i,j-1)+\phi(i,j+1)]$$



which is quite useful in python to iterate the value of ϕ.

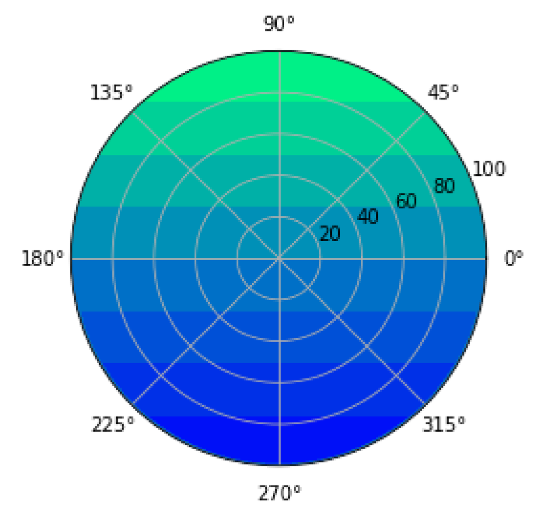
Graphs of Analytical and Numerical Methods

Finite difference method:



100 iterations $\Longrightarrow$ 1000 iterations

By separation of variable:



B_n take up to n=5

As the iteration times of ϕ increases, the inner values are interpreted more accurately. These two methods cooperate well to convince the temperature distribution of this area.

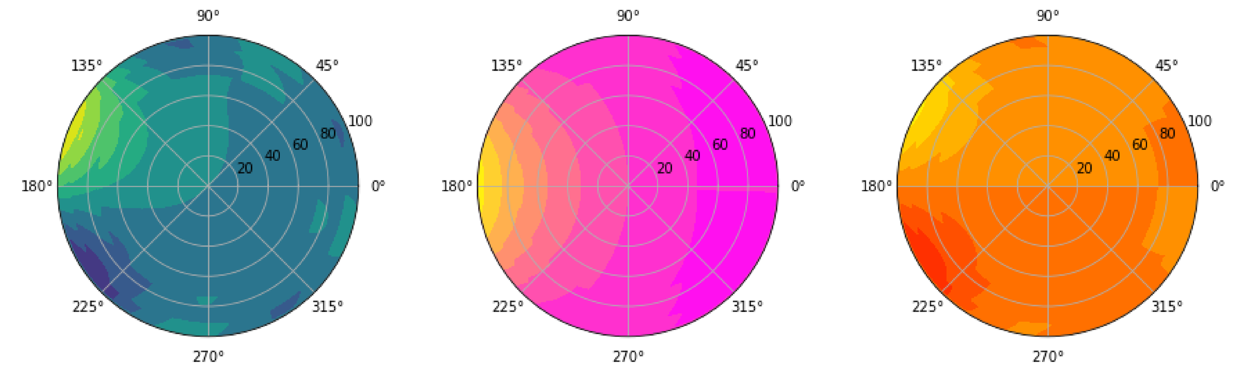
Conclusion

The graphs are **perfectly fit** to the boundary. When $\theta=\frac{\pi}{2}$, ϕ reaches a maximum, a minimum shows at $\theta=\frac{3\pi}{4}$, and it is periodic. Furthermore, the graph illustrates the nature of heat transfer: It locates at the north of the area, facing south (if we set east as $\theta=0$), and heat lost is directly proportional to the displacement transfered.

In conclusion, the **best spot** to place my sunbed is above the dark blue area, heading north as the basic temperature decreases.

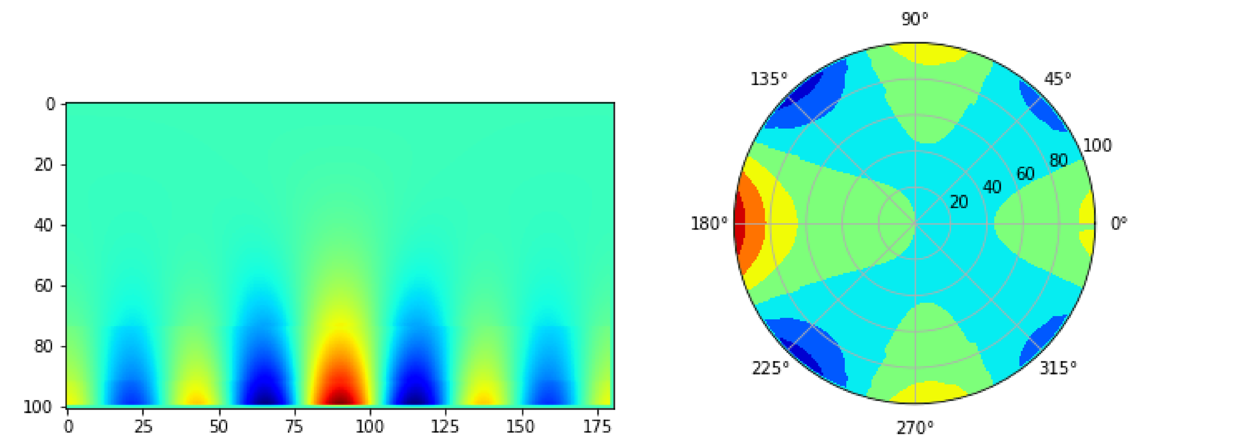
More Graphs

According to the General solution to this Dirichlet's boundary condition $\phi(\alpha,\theta)=f(\theta)$, I refine my python code to explore more graphs.



$f(\theta)=\exp(\theta)$ $\quad$ $f(\theta)=\theta^2$ $\quad$ $f(\theta)=\theta$

Moreover, my codes can show the corresponding temperature distribution in a rectangular shape when $f(\theta)=\theta\sin 5\theta$:



Findings

1. By setting Dirichlet's or Newman conditions we can not only visualize the temperature distribution, but also interpret the situation of heat source, such as location and direction.
2. Normally in the graphics, borders tend to have higher temperature than the centrals, which is true as well in real life.
3. Laplace's Equation can be applied into many different aspects which aims at restoring or approximating the complete picture by some boundary conditions.

References

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