Probability Questions

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1. **Hat matching problem** Suppose that each of three men at a party throws his hat into the center of the room. The hats are first mixed up and then each man randomly selects a hat. What is the probability that none of them selects his own hat?

   Solution Denote E = (none of them selects his own hat). The complement of the event E is E^c= (at least one man selects his own hat). We shall solve this problem by first calculating $P(E^c)$. Let E_i = (the event that the ith man selects his own hat), i = 1, 2, 3.

$$\begin{aligned} P(E^c) &= P(E_1 \cup E_2 \cup E_3) \\ &= \sum_i P(E_i) - \sum_{i<j} P(E_i E_j) + \sum_{i<j<k} P(E_i E_j E_k) \end{aligned} \tag{1}$$

   We need to compute $P(E_i)$, $P(E_i E_j)$, and $P(E_1 E_2 E_3)$. ◀

2. **Hat matching problem** At a party N men throw their hats into the center of a room. The hats are mixed up and each man randomly selects one. Find the expected number of men who select their own hat. What's the variance?

   Solution Let X denote the number of men that select their own hats. Define

$$X_i = \begin{cases} 1 \text{ , if the ith man selects his own hat} \\ 0 \text{ , otherwise} \end{cases} \tag{2}$$

   Hence, $X = X_1 + X_2 + \cdots + X_N$. $E(X_i) = 1/N$. Thus, $E(X) = 1$. That is, no matter how many people at the party, on the average just

one of them will select his own hat.

$$Var(X) = Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j}\sum Cov(X_i, X_j) \quad (3)$$

Recall that since each person is equally likely to select any of the N hats $P(X_i = 1) = \frac{1}{N}$. Hence,

$$E(X_i) = 1\frac{1}{N} + 0(1 - \frac{1}{N}) = \frac{1}{N} \quad (4)$$

and

$$E(X_i^2) = 1^2\frac{1}{N} + 0^2(1 - \frac{1}{N}) = \frac{1}{N} \quad (5)$$

$$Var(X_i) = E(X_i^2) - (E(X_i))^2 = \frac{N-1}{N^2} \quad (6)$$

$$Cov(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j) \quad (7)$$

$$X_iX_j = \begin{cases} 1 \text{ , if both person i and j chose their own hat} \\ 0 \text{ , otherwise} \end{cases} \quad (8)$$

$$\begin{aligned} P(X_i = 1, X_j = 1) &= P(X_i = 1 \mid X_j = 1)P(X_j = 1) = \frac{1}{N}\frac{1}{N-1} \\ E(X_iX_j) &= 1\frac{1}{N}\frac{1}{N-1} + 0(1 - \frac{1}{N}\frac{1}{N-1}) = \frac{1}{N}\frac{1}{N-1} \\ Cov(X_iX_j) &= \frac{1}{N}\frac{1}{N-1} - (\frac{1}{N})^2 = \frac{1}{N^2(N-1)} \\ Var(X) &= \frac{N-1}{N} + 2\binom{N}{2}\frac{1}{N^2(N-1)} = \frac{N-1}{N} + \frac{1}{N} = 1 \end{aligned} \quad (9)$$

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3. **Hat matching problem.** Suppose that each of N men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat. What is the probability that none of the men select their own hat? What is the probability that exactly k of the men select their own hats? reference

   Solution We first calculate the complementary probability of at least one man's selecting his own hat. Let us denote by $E_i, i = 1, 2, \ldots, N$

the event that the ith man selects his own hat. Now, $P\left(\bigcup_{i=1}^{N} E_i\right)$, the probability that at least one of the men selects his own hat, is given by

$$\begin{aligned} P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \cdots \\ + (-1)^{n+1} \sum_{i_1<i_2\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n}) \\ + \cdots + (-1)^{N+1} P(E_1E_2\cdots E_N) \end{aligned} \tag{10}$$

If we regard the outcome of this experiment as a vector of N numbers, where the ith element is the number of the hat drawn by the ith man, then there are N! possible outcomes. [The outcome $(1, 2, 3, \ldots, N)$ means, for example, that each man selects his own hat.] Furthermore, $E_{i_1}E_{i_2}\ldots E_{i_n}$, the event that each of the n men $i_1, i_2, \ldots, i_n$ selects his own hat, can select any of N-n hats, the second can then select any of N-(n+1) hats, and so on. Hence, assuming that all N! possible outcomes are equally likely, we see that

$$P(E_{i_1}E_{i_2}\cdots E_{i_n}) = \frac{(N-n)!}{N!} \tag{11}$$

Also, as there are $\binom{N}{n}$ terms in $\sum_{i_1<i_2\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n})$, we see that

$$\sum_{i_1<i_2\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_n}) = \frac{N!(N-n)!}{(N-n)!n!N!} = \frac{1}{n!} \tag{12}$$

and thus

$$P\left(\bigcup_{i=1}^{N} E_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{N+1}\frac{1}{N!} \tag{13}$$

Hence the probability that none of the men selects his own hat is

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^N}{N!} \tag{14}$$

which for N large is approximately equal to $e^{-1} \approx 0.36788$. In other words, for N large, the probability that none of the men selects his own hat is approximately 0.37.

To obtain the probability that exactly k of the N men select their own hats, we first fix attention on a particular set of k men. The number of ways in which these and only these k men can select their own hats is equal to the number of ways in which the other N-k men can select among their hats in such a way that none of them selects his own hat. But, as

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!} \tag{15}$$

is the probability that not one of N-k men, selecting among their hats, selects his own, it follows that the number of ways in which the set of men selecting their own hats corresponds to the set of k men under consideration is

$$(N-k)!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}\right] \tag{16}$$

Hence, as there are $\binom{N}{k}$ possible selections of a group of k men, it follows that there are

$$\binom{N}{k}(N-k)!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}\right] \tag{17}$$

ways in which exactly k of the men select their own hats. The desired probability is thus

$$\frac{\binom{N}{k}(N-k)!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdot + \frac{(-1)^{N-k}}{(N-k)!}\right]}{N!} = \frac{1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}}{k!} \tag{18}$$

which for N large is approximately $e^{-1}/k!$. ◀

4. **Birthday problem.** What is the probability that at least two people in this class share the same birthday?

   Solution Assume there are 365 days in a year (no leap year), and each day of the year has an equal chance of being somebodies' birthday.

Denote
E = no two people share a birthday
E^C = at least two people share a birthday
$P(E^C) = 1 - P(E)$
The first person can have any birthday. The second person's birthday has to be different. There are 364 different days to choose from, so the chance that two people have different birthdays is 364/365. That leaves 363 birthdays out of 365 open for the third person. To find the probability that both the second person and the third person will have different birthdays, we have to multiply: (365/365) * (364/365) * (363/365) = 0.9918.
A formula for the probability that n people have different birthdays is $((365-1)/365) * ((365-2)/365) * ((365-3)/365) * \cdots * ((365-n+1)/365) = 365!/((365-n)! * 365^n)$. ◀

5. **Three prisoner problem**. Three prisoners, A, B and C, are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the others who is going to be executed. "If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C."

   The warden tells A that B is to be executed. Prisoner A is pleased because he believes that his probability of surviving has gone up from 1/3 to 1/2, as it is now between him and C. Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of 1/3 to be the pardoned one, but his chance has gone up to 2/3. What is the correct answer? See reference

   Solution Call A, B and C the events that the corresponding prisoner will be pardoned, and b the event that the warden mentions prisoner B as the one not being pardoned, then, using Bayes' formula, the posterior probability of A being pardoned, is:

$$\begin{aligned} P(A|b) &= \frac{P(b|A)P(A)}{P(b|A)P(A) + P(b|B)P(B) + P(b|C)P(C)} \\ &= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{1}{3}. \end{aligned} \tag{19}$$

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6. **Monty Hall Problem.** There are 3 doors, behind one of which is a prize. The host asks you to pick any door. Say you pick door A. The host then opens door B and shows there is nothing behind door B. He then gives you the choice of either sticking with your original choice of door A, or switching to door C. Should you switch?

   Solution The a priori probability that the prize is behind door X, P(X) = 1/3.
   Let,
   A = prize behind A
   B = prize behind B
   C = prize behind C
   E = Host opens B
   The probability that the host opens door B if the prize were behind A, $P(E \mid A) = 1/2$.
   The probability that the host opens door B if the prize were behind B, $P(E \mid B) = 0$.
   The probability that the host opens door B if the prize were behind C, $P(E \mid C) = 1$.
   The probability that Monty opens door B is therefore,

$$\begin{aligned} p(E) &= p(A) * p(E|A) + p(B) * p(E \mid B) + p(C) * p(E \mid C) \\ &= 1/6 + 0 + 1/3 = 1/2 \end{aligned} \tag{20}$$

   Then, by Bayes' Theorem,
   $P(A \mid E) = p(A) * p(E \mid A)/p(E) = (1/6)/(1/2) = 1/3$ and
   $P(C \mid E) = p(C) * p(E \mid C)/p(E) = (1/3)/(1/2) = 2/3$.
   In other words, the probability that the prize is behind door C is higher when the host opens door B, and you SHOULD switch! ◀

7. Suppose there are 4 different types of coupons and suppose that each time one obtains a coupon, it is equally likely to be any one of the 4 types. Compute the expected number of different types that are contained in a set of 10 coupons.

   Solution Define

$$X_i = \begin{cases} 1, \text{ if at least one type-i coupon is in the set of 10} \\ 0, \text{ otherwise.} \end{cases} \tag{21}$$

Hence $X = X_1 + X_2 + X_3 + X_4$. Now,

$$\begin{aligned} E(X_i) &= P(X_i = 1) \\ &= P(\text{at least one type-i coupon is in the set of 10}) \\ &= 1 - P(\text{no type-i coupons are in the set of 10}) \\ &= 1 - (\frac{3}{4})^{10} \end{aligned} \tag{22}$$

$E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4) = 4(1 - (\frac{3}{4})^{10})$ ◀

8. Suppose that the number of car arrive at a gas station each day is Poisson random variable with mean λ. Suppose further that each car that arrives is, independently, two-door with probability p and four-door with probability $(1 - p)$. Find the joint probability that exactly n two-door car, and m four-door car visit the gas station.

   Solution Let N denote the total number of cars, let N_2 and N_4 be the number of cars with two doors and four doors respectively.

$$\begin{aligned} P(N_2 = n, N_4 = m) &= \sum_{j=0}^{\infty} P(N_2 = n, N_4 = m | N = j) P(N = j) \\ &= P(N_2 = n, N_4 = m | N = n + m) P(N = n + m) \\ &= \binom{n+m}{n} p^n (1-p)^m e^{-\lambda} \frac{\lambda^{n+m}}{(n+m)!} \end{aligned} \tag{23}$$

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9. Box 1 contains two white balls and three blue balls, while Box 2 contains three white and four blue balls. A ball is drawn at random from Box 1 and put into Box 2, and then a ball is picked at random from Box 2 and examined. What is the probability it is blue?

   Solution A = ball chosen from Box 1 is blue.
   B = ball chosen from Box 2 is blue.
   $P(B) = P(B \mid A)P(A) + P(B \mid A^C)P(A^C)$
   $P(B \mid A) = 5/8$
   $P(A) = 3/5$
   $P(B \mid A^C) = 4/8$

$P(A^C) = 2/5$
P(B) = 5/8*3/5 + 4/8*2/5 = 23/40 ◀

10. There are three machines (A, B and C) at a factory that are used to make a certain product. Machine A makes 25% of the products, B 35% and C 40%. Of the products that A makes, 5% are defective, while for B 4% are defective and for C 2%. The products from the different machines are mixed up and sent to the customer. (a) What is the probability that a customer receives a defective product? (b) What is the probability that a randomly chosen product was made by A, given the fact that it is defective?

    Solution E = product defective
    F1 = product comes from A
    F2 = product comes from B
    F3 = product comes from C
    (a) $P(E) = P(E \mid F1)P(F1) + P(E \mid F2)P(F2) + P(E \mid F3)P(F3) = 0.05 * 0.25 + 0.04 * 0.35 + 0.02 * 0.40 = 0.0345$
    (b) $P(F1|E) = 0.25 * 0.5/0.0345 = 0.36$ ◀

11. A plane is missing and it is asumed that it has crashed in any of three possible regions with equal probability. Assume that the probability of finding the plane in Region 1, if in fact the plane is located in there, is 0.8. What is the probability the plane is in the ith region given that the search of Region 1 is unsuccessful?

    Solution R1 = plane is in region 1
    R2 = plane is in region 2
    R3 = plane is in region 3
    E = search of region 1 unsuccessful.

$$
\begin{aligned}
P(R1 \mid E) &= \frac{P(E \mid R1)P(R1)}{\sum P(E \mid Ri)P(Ri)} \\
P(Ri) &= 1/3 \\
P(E \mid R1) &= 1 \\
P(E \mid R3) &= 1 \\
P(R1 \mid E) &= (1/5 * 1/3)/(1/5 * 1/3 + 1/3 + 1/3) = (1/15)/(11/15) = 1/11 \\
P(R2 \mid E) &= (1 * 1/3)/(1/5 * 1/3 + 1/3 + 1/3) = (1/3)/(11/15) = 5/11 \\
P(R3 \mid E) &= (1 * 1/3)/(1/5 * 1/3 + 1/3 + 1/3) = (1/3)/(11/15) = 5/11
\end{aligned} \tag{24}
$$

Original hypothesis: P(R1) = P(R2) = P(R3)
New hypothesis: P(R1) ¡ P(R2) = P(R3) ◀

12. A series of independent trials are performed, which result in success with probability p and failure with probability 1-p. What is the probability that exactly n successes occur before m failures?

    Solution In order for exactly n successes to occur before m failures, it is equivalent to that there are exactly n successes in the first m+n-1 trials. Hence,
    $P = (m + n - 1n)p^n(1 - p)^(m - 1)(1 - p)$. ◀

13. Independent trials, consisting of rolling a pair of fair dice, are performed. What is the probability that an outcome of 5 appears before an outcome of 7 when the outcome of a roll is the sum of the dice.

    Solution Let N = neither 5 or 7
    5, N5, NN5, NNN5, ...
    Let, E_n= no 5 or 7 appears on the first n-1 trials and a 5 appears on the nth trial. The desired probability is,
    $P(\cup E_n) = \sum_{n=1}^{\infty} P(E_n)$.
    $P(5) = P((1,4),(2,3),(3,2),(4,1)) = 4/36$
    $P(7) = P((1,6),(2,5),(3,4),(4,3),(5,2),(6,1)) = 6/36$
    $P(\text{neither 5 or 7}) = 1 - (4/36 + 6/36) = 26/36 = 13/18$
    $P(E_n) = (3/18)^(n - 1) * 4/36$

$$
\begin{aligned}
P(\cup E_n) &= \sum_{n=1}^{\inf}(3/18)^(n-1) * 4/36 \\
&= 1/9 \sum_{j=0}^{\inf}(13/18)^(j) \\
&= 1/9 * (1/(1-13/18)) \\
&= 1/9 * 1/(5/18) = 2/5
\end{aligned} \tag{25}
$$

We used the formula for the geometric series:

$$
\sum_{k=0}^{n} t^k = \frac{1-t^(n+1)}{1-t} fort \neq 1 \sum_{k=0}^{\inf} t^k = \frac{1}{1-t} for|t| < 1 \tag{26}
$$

To remember this, just recall the trick:
$\sum_{k=0}^{n} t^k - t\sum_{k=0}^{n} t^k = 1 - t^{n+1}$,
because the sum telescopes. Let, F_n= no 5 or 7 appears on the first n-1 trials and a 7 appears on the nth trial. ◀

14. A game consists of 5 matches and two players, A and B. Player A wins a match with probability 2/3. Player B wins a match with probability 1/3. The player who first wins three matches wins the game. Assume the matches are independent. What is the probability that A wins the game?

    Solution Player A wins 3 matches or more out of the 5 matches. (A wins at least 3 matches) A wins three and B two OR A wins four and B one OR A wins five and B zero
    $P(\text{A wins}) = \binom{5}{3}(\frac{2}{3})^3(\frac{1}{3})^2 + \binom{5}{4}(\frac{2}{3})^4(\frac{1}{3}) + \binom{5}{5}(\frac{2}{3})^5 = \frac{64}{81}$ ◀

15. A man and woman decide to meet at a certain location. If each person independently arrives at a time uniformly distributed between 2 and 3 PM, find the probability that the man arrives at least 10 minutes before the woman.

    Solution X = number of minutes past 2 the man arrives
    Y = number of minutes past 2 the woman arrives
    X and Y are independent random variables each uniformly distributed over (0,60).

$$
\begin{aligned}
P(X+10<Y) &= \iint_{x+10<y} f(x,y)dxdy \\
&= \iint_{x+10<y} f_X(x)f_Y(y)dxdy \\
&= \int_{10}^{60}\int_{0}^{y-10} (\frac{1}{60})^2 dxdy \\
&= (\frac{1}{60})^2 \int_{10} 60(y-10)dy \\
&= \frac{1}{3600}(\frac{y^2}{2} - 10y)_{10}^{60} \\
&= \frac{1250}{3600}
\end{aligned} \tag{27}
$$

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16. Twenty people, consisting of 10 married couples, are to be seated at five different tables, with four people at each table. If the seating is done at random, what is the expected number of married couples that are seated at the same table?

    Solution Let X = the number of married couples at the same table. Number then couples from 1 to 10 and let,

$$
X_i = \begin{cases} 1, \text{ if couple i is seated at the same table} \\ 0, \text{ otherwise.} \end{cases} \tag{28}
$$

    Then $X = X_1 + X_2 + \cdots + X_{10}$. To calculate $E(X)$ we need to know $E(X_i)$.Consider the table where husband i is sitting. There is room for three other people at his table. There are a total of 19 possible people which could be seated at his table.

$$
P(X_i = 1) = \frac{\binom{1}{1}\binom{18}{2}}{\binom{19}{3}} = \frac{3}{19} E(X_i) = 1\frac{3}{19} + 0\frac{16}{19} = \frac{3}{19} \tag{29}
$$

    Hence, $E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = 10\frac{3}{19} = \frac{30}{19}$ ◀

17. Bowl B1 contains 2 white chips and 1 red, bowl B2 contains two red chips. The probabilities of selecting bowl B1, B2 are 1/3, 2/3 respectively. A chip is drawn at random from a randomly selected bowl. Find P(B1 and White chip). (Use conditional probability)

Solution

$$\begin{aligned} P(\text{B1 and White chip}) &= P(\text{White chip—B1 })P(B1) \\ &= (2/3)(1/3) = 2/9. \end{aligned} \tag{30}$$

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References

- Degroot and Schervish, 2012. Probability and Statistics.