Fact 41. *Let (p, q) be a point. Its reflection in the line $ay + bx + c = 0$ is the point*

$$\left(\frac{p(a^2 - b^2) - 2b(aq + c)}{a^2 + b^2}, \frac{q(b^2 - a^2) - 2a(bp + c)}{a^2 + b^2} \right).$$

Proof. Consider the line that is perpendicular to the line of reflection and which contains (p, q). It can be written as $-by + ax + d = 0$, where d is an unknown. Since (p, q) is on this line, we have $-bq + ap + d = 0$, so that $d = bq - ap$. So the perpendicular line is $-by + ax + bq - ap = 0$.

The intersection of the line of reflection and the perpendicular line we just found is given by the system of equations:

$$\begin{aligned} ay + bx + c &\stackrel{1}{=} 0, && \text{(equation of line of reflection)}, \\ -by + ax + bq - ap &\stackrel{2}{=} 0 && \text{(equation of perpendicular line)}. \end{aligned}$$

Take $b \times \stackrel{1}{=}$ plus $a \times \stackrel{2}{=}$ and do the algebra to get

$$x = \frac{a^2 p - b(aq + c)}{a^2 + b^2}.$$

Similarly, take $a \times \stackrel{1}{=}$ minus $b \times \stackrel{2}{=}$ and do the algebra to get

$$y = \frac{b^2 q - a(bp + c)}{a^2 + b^2}.$$

(x, y) is the midpoint between (p, q) and the reflection point we are looking for. Thus, our reflection point has x-coordinate

$$\begin{aligned} 2x - p &= 2\frac{a^2 p - b(aq + c)}{a^2 + b^2} - p \\ &= \frac{p\left(a^2 - b^2\right) - 2b(aq + c)}{a^2 + b^2}, \end{aligned}$$

and y-coordinate

$$\begin{aligned} 2y - q &= 2\frac{b^2 q - a(bp + c)}{a^2 + b^2} - q \\ &= \frac{q\left(b^2 - a^2\right) - 2a(bp + c)}{a^2 + b^2}. \end{aligned}$$

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