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# Optimal portfolio allocations with tracking error volatility and stochastic hedging constraints

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The performance of mutual fund or pension fund managers is often evaluated by comparing the returns of managed portfolios with those of a benchmark. As most portfolio managers use dynamic rules for rebalancing their portfolios, we use a dynamic framework to study the optimization of the tracking error–return trade-off. Following these observations, we assume that the manager minimizes the tracking error under an expected return goal (or, equivalently, maximizes the information ratio). Moreover, we assume that he/she complies with a stochastic hedging constraint whereby the terminal value of the portfolio is (almost surely) higher than a given stochastic payoff. This general setting includes the case of a minimum wealth level at the horizon date and the case of a performance constraint on terminal wealth as measured by the benchmark (i.e. terminal portfolio wealth should be at least equal to a given proportion of the index). When the manager cares about absolute returns and relative returns as well, the risk–return trade-off acquires an extra dimension since risk comprises two components. This extra risk dimension substantially modifies the characteristics of portfolio strategies. The optimal solutions involve pricing and duplication of spread options. Optimal terminal wealth profiles are derived in a general setting, and optimal strategies are determined when security prices follow geometric Brownian motions and interest rates remain constant. A numerical example illustrates the type of strategies generated by the model.

*Keywords*: Asset management; Asset allocation; Continuous time finance; Portfolio optimization; Stochastic hedging constraints; Tracking error

*JEL Classification*: G1, G11

## 1. Introduction

The performance of money managers is often measured against a benchmark. The choice of the benchmark generally depends on the investor's preferences. Conservative investors tend to choose a fixed-income investment alternative, while more aggressive investors would be closer to a market index. Once the benchmark is chosen, the investor chooses an active or a passive asset manager. For this manager, the benchmark is given exogenously, but he/she may be allowed to deviate more or less from it. Active portfolio managers seek over-performance and are thus going to deviate from this benchmark. However, this deviation is usually constrained, implicitly or explicitly, by an upper bound on the tracking error volatility defined as the standard deviation of the excess return (the difference between the portfolio and the benchmark returns).

An overwhelming proportion of the literature on portfolio management is based on a static framework, *à la* Markowitz, without benchmarking. Some papers have introduced an additional dimension, whereby the performance of the portfolio is measured against a benchmark. In particular, Roll (1992) solved the optimal asset allocation problem, generated from the minimization of the tracking error volatility (TEV hereafter) for a given expected portfolio return. He showed that if the benchmark is not efficient, the corresponding optimal portfolio is not mean–variance efficient. It is somewhat difficult, therefore, to understand why the industry maintains such an emphasis on the tracking error

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approach, although some attempts to rationalize benchmarking practices relying on non-conventional arguments can be found in the literature.† The purpose of this paper is not to rationalize benchmarking. We acknowledge the fact that the relative performance evaluation of portfolios is widely practiced in the asset management industry. Moreover, the standard measures of performance relate average returns to tracking errors. For instance, the 'information ratio' (the ratio of the difference between the expected returns of the portfolio and the benchmark over the tracking error) is currently a standard measure of performance for benchmarked funds, although some problems have recently been identified.‡ Since their performance is assessed using this measure, it is natural that practitioners manage the 'tracking error–return' trade-off.

As it turns out, implementing a framework involving a tracking error–return trade-off is very similar to using a quadratic utility function (mean–variance optimization) calculated on the basis of excess returns rather than actual portfolio returns.

Another widespread characteristic of asset management practices is rebalancing. Indeed, dynamic asset allocation advice is commonly provided by most brokerage firms and financial advisors, and active portfolio managers often carry out dynamic asset allocation strategies (especially for long investment horizons). Properly defining the related dynamic optimal rules is therefore central to portfolio management. Consider some of the problems raised by active portfolio management: what should the manager's reaction to a stock market movement be? Should a momentum or a contrarian strategy be followed? How should the weight on stocks be modified as time passes and the horizon gets closer? How should the bond-to-stock ratio vary with respect to the investor's risk aversion?, i.e. the Canner *et al.* (1998) puzzle.§ These fundamental asset allocation issues can only be addressed in dynamic frameworks such as those developed by Merton (1971, 1973). Bajeux-Besnainou and Portait (1998), BP hereafter, in a paper related to the present one, derive closed-form solutions for optimal wealth and dynamic portfolio policies assuming a dynamic mean–variance framework without benchmarking. Recognizing that benchmarked asset allocation generally relies on a mean-TEV criterion while simultaneously implementing dynamic trading rules, this paper links these two features.

Recently, the issue of benchmarking in a dynamic setting was addressed in a continuous-time setting: Tepla (2001) considers a HARA-utility investor with the constraint that the investor's wealth should be larger than that generated by a stochastic payoff interpreted as a benchmark; Basak *et al.* (2005) use a similar objective function with a different constraint, similar to a value-at-risk constraint written on the difference between terminal wealth and a benchmark. The benchmark, which appears in the downside constraint but not on the objective function, is a combination of the stock market and the risk-free asset. They show that a risk-averse portfolio manager under- or over-performs the benchmark depending on economic conditions, on his/her attitude towards risk, and on the choice of the benchmark. A recent paper by Basak and Chabakauri (2009b) examines mean–variance time-consistent continuous-time allocations and considers benchmarking as a special case of hedging without imposing a constraint on the terminal portfolio value. In contrast, our paper focuses on the dynamic return–tracking error trade-off (the benchmark appears in the objective function) with a downside hedging constraint. More precisely, the hedging constraint takes the general form $W_T \geq C_T$, where $C_T$ is a stochastic payoff. The unconstrained case is found as a particular case, when $C_T$ goes to negative infinity. In the mean–variance framework, as well as in the excess return–TEV framework, the unconstrained model yields an optimal terminal wealth unbounded from below, which is an unsatisfactory property, even if the probability of a negative or low terminal wealth is very small. Expected utility maximization of the terminal wealth (or of the excess return) would be an alternative to excess return–TEV trade-off that prevents unrestricted losses for utility functions satisfying the Inada conditions. A drawback of expected utility maximization is that it does not match the criteria on which the manager's performance is typically assessed (alpha, IR). In this respect, expected utility maximization is less relevant for most managers than the excess return–TEV trade-off.

Introducing a downside hedging constraint is one way to take advantage of the mean–variance framework while avoiding the shortcoming of unrestricted losses. Another advantage of the downside constraint is that it is flexible enough to represent other types of restrictions often imposed on fund managers, like a minimum performance, or minimum management fees, as explained in section 4. Note that, in a continuous-time framework without benchmarking, hedging constraints have already been studied, for instance by Nguyen and Portait (2002) and El Karoui *et al.* (2005). The solution of the constrained problem is written as a linear combination of the solution of the unconstrained problem and a put option on the unconstrained optimal portfolio value.

†Clarke *et al.* (1994) argue that the tracking error model should be developed from the concept of aversion to regret (regret comes when the portfolio deviates from the benchmark in the wrong direction). Jorion (2003) shows that additional constraints can mitigate the inefficiency of benchmarking. Another way to rationalize benchmarking, as argued by Wagner (2001), is to acknowledge the fact that there might be a lack of information on the assessment of the return distributions of the traded assets. Rudolf *et al.* (1999) show that linear models (as opposed to quadratic models) for minimizing tracking errors are consistent with expected utility maximization.

‡It has been proved by Goetzmann *et al.* (2007) that IR can be manipulated. These manipulation strategies can be either static or dynamic, but the effects of dynamic manipulation schemes are often more dramatic.

§Bajeux-Besnainou *et al.* (2001) provided a theoretical explanation of the Canner *et al.* (1997) puzzle. As a key argument to solve the puzzle, they use the ability of investors to continuously manage their positions.

However, contrary to the Nguyen and Portait (2002) and El Karoui *et al*. (2005) models, due to the benchmarked objective function, the put option is a non-standard spread option whose price and duplication strategies have no closed-form solution even in the standard Black and Scholes framework.† In the present paper, the main point of interest is the profile of the portfolio strategies defined as their weights on traded assets, and for this endeavour we implement numerical examples.

In order to derive the optimal strategies, the payoff $C_T$ is assumed linear affine in the benchmark ($C_T = W^{\min} + pI_T$). For $p=0$, the wealth constraint ($W_T \geq W^{\min}$) is the standard portfolio insurance constraint; for $W^{\min}=0$, a minimum performance is required ($W_T \geq pI_T$, where $p$ is between 0 and 1 and represents a performance parameter). Other useful interpretations of this constraint in terms of minimum fees are discussed in section 4.

The optimization programs are described and interpreted in section 2. The optimal portfolio value at terminal date $T$ is derived in section 3. In section 4, we characterize the optimal portfolio value at any intermediate date $t$ and the optimal weights. Section 5 provides numerical illustrations and comparative static results. We conclude in section 6.

## 2. Optimization programs

### *2.1. Hypothesis, notation and setting*

We assume that markets are continuously open between an initial date 0 and a terminal date $T$, that portfolios are rebalanced without transaction costs, and that markets are dynamically complete, which implies that any random cash flow is attainable by a dynamic self-financing strategy.

The value at date $T$ of the growth optimal portfolio G (also called the log-optimal portfolio or the numeraire portfolio) is denoted $G_T$. The properties of G have been studied extensively, in particular by Long (1990), Merton (1992), and BP (1998). If the initial value of the growth optimal portfolio is 1, $1/G_t$ is the pricing kernel (or stochastic discount factor for time $t$). Therefore, any self-financing strategy yielding a random payoff at date $T$ can be priced under the true probability using $1/G_T$ as a deflator.

The portfolio value (investor's wealth) at an intermediate date $t$ is $W_t$ and $I_t$ represents the value of the benchmark. The benchmark is arbitrarily defined as a self-financing portfolio strategy (as a total return index).‡ The initial values are normalized to unity ($I_0 = W_0 = \$1$), which implies that $I_T$ and $W_T$ represent returns and $Y_T = W_T - I_T$ is the 'excess return' over the period $(0, T)$.

The Tracking Error Volatility ($TEV$) is the standard deviation of the excess return:

$$TEV = SD(W_T - I_T). \tag{1}$$

The Information Ratio ($IR$) is defined as the ratio of the expected excess return over $TEV$:

$$IR = \frac{E(W_T - I_T)}{SD(W_T - I_T)}. \tag{2}$$

A stochastic hedging constraint forces the terminal wealth at maturity (return) to be greater than or equal to the stochastic payoff $C_T$ (minimum return):

$$W_T \geq C_T, \text{ almost surely.} \tag{3}$$

This stochastic payoff $C_T$ is attainable by a self-financing portfolio strategy (with an initial investment equal to $E(C_T/G_T)$) since the markets are complete. Three particular cases are analysed: no budget constraint, $C_T$ equals a constant (portfolio insurance), and $C_T$ equals a given proportion of the benchmark (performance constraint).

We consider a manager trying to maximize the expected return with a constraint on the tracking error ($TEV \leq v$; Roll's (1992) problem). When the manager accounts for the budget and the hedging constraints, he/she writes the optimization program:§,¶

$$(\mathrm{P}_0)\ \max_{W_T}\ E(W_T),$$

subject to

$$TEV^2 \leq v^2, \quad E\left(\frac{W_T}{G_T}\right) = 1 (= W_0), \quad W_T \geq C_T, \text{ a.s.}$$

Since the terminal value $I_T$ of the benchmark is exogenous, this program can be written in terms of the

†Besides, El Karoui *et al*. (2005) express the portfolio optimal value in a general form and are not interested in the actual duplication of the put. Since the put is not actually traded, the actual implementation of the portfolio strategy involves its duplication.

‡For simplicity, we do not consider intermediary dividend payments. In practice, many fund managers measure their performance relative to a total return benchmark with dividends reinvested (e.g. MSCI and FTSE for equity). Taking dividends into account is a straightforward technical exercise.

§Since the quadratic program is not scale invariant, the risk tolerance parameter $a$ must be defined as a proportion of the initial wealth $W_0$: when $W_0$ is different from 1, in order to obtain decision rules independent of the initial wealth $W_0$, the objective function is written

$$\max_{W_T}\left[E(W_T) - \frac{1}{2aW_0}E(W_T^2)\right] \Leftrightarrow \max_{W_T/W_0}\left[E\left(\frac{W_T}{W_0}\right) - \frac{1}{2a}E\left(\left(\frac{W_T}{W_0}\right)^2\right)\right].$$

¶Some technical 'feasibility' conditions have to be satisfied for this optimization program to have a solution. In particular, the stochastic payoff $C_T$ should be attainable through an initial investment equal to or less than \$1($E(C_T/G_T) \leq 1$), otherwise this hedging constraint could never be met with an initial investment of \$1.

excess return ($Y = W - I$), and we consider its Lagrangian form (P):

$$\text{(P)} \ \max_{Y_T} \ E(Y_T) - \frac{1}{2a}\sigma^2(Y_T),$$

subject to

$$E(Y_T/G_T) = 0, \quad Y_T \geq C_T - I_T, \quad \text{with } a > 0.$$

In (P), $(1/2a)$ is exogenous and positive and can be interpreted as the Lagrange multiplier of the first constraint of ($P_0$). Moreover, for any solution of (P), the tracking error constraint of ($P_0$) is binding (since the corresponding multiplier is positive). Hence, (P) is equivalent to maximizing the Information Ratio (IR) (also equal to $E(Y_T)/\nu$):

$$\text{(IR)} \quad \max_{W_T} \ IR,$$

subject to

$$TEV = \nu, \quad W_0 = I_0 = \$1, \quad W_T \geq C_T.$$

As far as active asset allocation under benchmarking is concerned, the relevance of the mean-TEV trade-off as expressed by ($P_0$), (P) and (IR) can be understood from either the investor's viewpoint or the manager's perspective.

(P) can be interpreted as the optimization program of the investor in a 'behavioral framework'. In this case, $a$ is a measure of the *investor's* tolerance to regret and becomes a given parameter for the manager whose role is purely technical (his or her activity consists only of program trading).

Alternatively, (P) or (IR) can be considered as the optimization program of the money manager when his/her performance is assessed against the benchmark, through an 'alpha'† or an 'IR', when his/her remuneration is a linear function of the excess return $Y_T$. In this case, $a$ can be interpreted as the *manager's* risk tolerance. Section 2.2 provides justifications for (P) based on these ideas.

Under both interpretations, $a$ denotes a level of tolerance towards a potential deviation from the benchmark. This tolerance level is chosen by the investor under the first interpretation and by the manager under the second. As far as the optimization model is concerned, the choice between the two interpretations is irrelevant. A purely passive strategy (i.e. aimed at replicating the benchmark) would be characterized by a zero value for $a$, whereas an active strategy (i.e. aimed at beating the index) would be characterized by a positive value for $a$.

### 2.2. *Alternative interpretations of programs ($P_0$), (P) or (IR) from the management viewpoint*

As previously discussed, from the manager's viewpoint, there are different ways to understand the optimization programs ($P_0$), (P) and (IR). In this section, we provide a thorough discussion of these interpretations.

- An active management tries to generate excess returns while generally committed to keep the 'distance' to the benchmark (the TEV) at or under an explicit or implicit limit $\nu$.‡ A typical asset allocation process involves two steps. In the first step, the manager determines the strategic asset allocation to stocks, bonds, real estate and other asset classes. In the second step, the fund's investors delegate the active management of parts of the portfolio (each part is assessed against a specific benchmark) to the fund manager. These 'peer-fund' managers typically make a trade-off between tracking error and expected excess returns relative to their benchmark as written in programs ($P_0$), (P) and (IR).
- In practice, IR often measures the manager's performance. The intent is then to measure the manager's ability to generate expected excess returns.§ Therefore, he/she tries to maximize IR under a TEV constraint and a hedging constraint imposed by the risk manager, which is (IR).
- A quadratic manager whose performance fees are linear affine on the excess return $Y_T$ also solves ($P_0$), (P) and (IR). Note that such performance fees are symmetric, hence government regulations often penalize managers when $Y_T$ is negative, for instance in the US (Amendment to the Investment Advisors Act of 1940, passed by Congress in 1970), and in several European countries. This regulation

†The empirical IR is directly linked to the $t$-statistics of the test $alpha > 0$.

‡This TEV constraint may be imposed by the funds prospectus. Besides, the benchmarked funds are classified by their empirical TEV and excessive deviations may cause declassification. For instance, funds tracking a benchmark are divided into three classes by the IOSCO (International Organization of Securities Commissions) and by regulators, for instance the French AMF (Autorité des Marchés Financiers): Indexed ($TEV \leq 1\%$), tilted ($TEV$ around 2%) or active funds (with a $TEV$ often specified in the prospectus). Once a fund is in a given class, the $TEV$ is constrained by the regulator or/and by the risk manager.

§Empirical and theoretical arguments have been formulated in favor of the IR as a performance measure:

- the empirical IR is linked to the $t$-statistics of the test $\alpha > 0$ ($\alpha$ is defined with respect to the benchmark);
- the Sharpe ratio of a portfolio (with respect to the usually unobservable tangent portfolio) is equal to the sum of its IR and of the Sharpe ratio of the benchmark (the latter is usually suboptimal, since the benchmark is presumably inefficient, which justifies the active manager's attempt to outperform it). IR maximization thus implies Sharpe ratio maximization.

prohibits mutual funds, pension funds, and other publicly registered investment firms to apply an asymmetric (bonus only) compensation scheme.† Indeed, an asymmetric performance fee (like an option payoff on the portfolio) is an incentive for the manager to take excessive risk‡ and such an undesirable incentive is often given as a justification of this regulation. In fact, the limitation on asymmetry applies to the contract between the investment firm and its outside investors. However, internally, the investment firm usually gives its employees—the portfolio managers of specific funds—asymmetric compensation. Hence, the actual portfolio manager may have an incentive to take risk while the management company does not. We do not address this potential agency problem in our model, and we assume that the 'manager's' objective coincides with the 'management company's' objective. The literature on optimal contracts has extensively studied and rationalized linear compensation fees. One strand of the principal-agent literature provides justifications for the linearity of the compensation schemes based on different models, frameworks and assumptions. We can cite Holmström and Milgrom (1987), in discrete time, and Schätler and Sung (1993), Sung (1995), Ou-Yang (2003) and Cadenillas *et al.* (2007), in continuous time. Another strand of the literature takes the linearity of the contract as given and derives either the two optimal parameters of the contract (see, for instance, among the numerous papers taking this approach, Golec (1992) or Kapur and Timmermann (2005)), or the optimal benchmark (for instance, Lioui and Poncet 2007).

- Management fees (different from performance fees) are commonly a percentage of the asset value that depends, at the end of the current period, on the net inflow of assets during this period (0, $T$). Therefore, if the flow is linear on the excess return $Y_T$, ($P_0$), (P) or (IR) is the relevant optimization program for the management. Whether a linear function of excess returns is an accurate explanation of flows or not is mainly an empirical issue. It seems that such a function may reasonably well explain the flows for pension funds while it poorly explains those of mutual funds. Indeed, there is some evidence that, while the response of mutual funds flows to excess returns is roughly symmetrical, in the periods following poor performances, mutual funds do not lose as many clients as they gain in the periods following good performances (see, for instance, del Guercio and Tkac 2002). Our model is flexible enough to constrain the level of fees downward through the downward hedging constraint, as explained in section 4. A related and interesting theoretical question, addressed by Basak *et al.* (2007), beyond the scope of this paper, is the possible undesirable incentive given to the management by such a positive flow–performance relationship.

## 3. Optimal terminal wealth

$G_T$ represents the value at date $T$ of the log-optimal or numeraire portfolio. As markets are complete, $1/G_T$ is attainable, and the required initial investment is $E(1/G_T^2)$. The corresponding self-financing strategy L (whose value at date $t$ is denoted $L_t$), with an initial investment of \$1, yields a payoff $L_T$ at $T$ of

$$L_T = 1/(G_T E(1/G_T^2)). \tag{4}$$

Portfolio L is an inefficient portfolio, negatively correlated with G and with the market, and has been characterized by BP (1998) as the 'minimum norm portfolio' since it minimizes the second moment of the return.§

Using the deflator $1/G_T$ to write the budget constraint $Y_0 = 0$, we solve the quadratic program (P)¶ whose solutions also solve (for a particular value of $\nu$) the mean-tracking error trade-off optimization program ($P_0$) of section 2.1:⊥

$$\text{(P)} \quad \max_{Y_T} \; E(Y_T) - \frac{1}{2a} E(Y_T^2),$$

subject to

$$E(Y_T/G_T) = 0, \quad Y_T \geq C_T - I_T.$$

†However, performance fees are not necessarily linear (convex schemes exist) and hedge funds are free to apply asymmetric schemes. Nevertheless, the majority of performance fees are linear affine in excess returns and the literature on optimal contracting has focused on such schemes.

‡This argument has recently been discussed and studied extensively in the literature. In particular, Carpenter (2000) shows that a manager compensated with an asymmetric fee does not always take excessive risk. He shows that a manager compensated with a call option on the assets under management may either take dramatic risks (when her option is out of the money), or be surprisingly conservative (when her option is in the money).

§It solves min $E((X_T)^2)$, s.t. $E(X_T/G_T)=1$.

¶We are assuming here that the initial wealth $W_0$ equals \$1, which allows us to use parameter $a$ instead of $aW_0$. In this case, the dimension of the parameter $a$ is a \$ amount (see footnote 6).

⊥Note that such a mean–variance program is not 'time consistent', because its solution at time 0 does 'not match' the solution at $t > 0$ (see, for instance, Li and Ng (2000), Lim and Zhou (2002) and Basak and Chabakauri (2009a,b)). However, the solution at time 0 ('pre-committed' mean–variance optimization) coincides with the solution of the quadratic program (P) defined in the sequel, which is 'time consistent', and which is the optimization program considered throughout this paper.

(P) is solved in the appendix and its solution is given in proposition 3.1.

**Proposition 3.1:** *Under complete markets:*

(i) *The optimal terminal wealth solution* $W_T^*$ *of (P) is*

$$W_T^* = C_T + \max(0, a - C_T - \kappa L_T + I_T), \tag{5}$$

*where* $L_T$ *is defined by (4) and the constant* $\kappa$ *is implicitly defined by*

$$E\left(\frac{W_T^*}{G_T}\right) = E\left(\frac{C_T}{G_T}\right) + E\left[\frac{\max(0, a - C_T + I_T - \kappa L_T)}{G_T}\right] = 1.$$

(i) *The corresponding optimal strategy is a buy and hold combination*† *of*:

- *a fund yielding* $C_T$ *at time* $T$ *(in general, a continuously managed portfolio);*
- *a put spread option on the terminal payoff* $(C_T + \kappa L_T - I_T)$*, with a strike price equal to* $a$.

The optimal terminal wealth when there is no hedging constraint is called the 'unconstrained solution' and is denoted $W_T^{\mathrm{u}}$. It can be derived from $W_T^*$ as provided in (5). We make $C_T$ go to infinity and derive

$$W_T^{\mathrm{u}} = a + I_T - aB_T(0)L_T. \tag{6}$$

Note that, in (6), the multiplier $\kappa$ of (5) (which generally depends on $C_T$) is equal to $aB_T(0)$.

It follows from equation (6) that the optimal unconstrained strategy yielding $W_T^{\mathrm{u}}$ can be characterized as a buy-and-hold combination of three funds. At time 0, the initial position (corresponding to an initial investment of \$1) can be characterized as follows:

- \$1 invested in the self-financing index fund $I$;
- \$$aB_T(0)$ invested in the zero-coupon bond yielding \$$a$ at the investor's horizon (where $B_T(0)$ is the price at date 0 of the zero-coupon maturity $T$);
- a short position of \$–$aB_T(0)$ in the minimum norm portfolio L.

Note that these three funds are identical for all investors with the same horizon $T$ and the same benchmark and that the last one is not a traded security but the dynamically managed portfolio L previously characterized.

The optimal unconstrained terminal wealth as derived in (6) can be compared to the optimal Tracking Error wealth in a static framework (allowing only buy-and-hold strategies). In the static framework, assuming that the zero-coupon bond maturity $T$ is traded (there is a risk-free asset), the optimal static terminal wealth denoted $W_T^{\mathrm{s}}$ is given by $W_T^{\mathrm{s}} = I_T + \theta(M_T - 1/B_T(0))$, where $M_T$ is a Static Markowitz tangent efficient portfolio (Roll 1992, Jorion 2003, Bajeux-Besnainou *et al.* 2011)‡ and $\theta$ is a parameter associated with the risk tolerance of the investor. As noted by Roll (1992), since $M_T$ is efficient, $W_T^{\mathrm{s}}$ is efficient if and only if the benchmark $I_T$ is efficient. The excess return is denoted $Y_T^{\mathrm{s}}$ and is defined as $Y_T^{\mathrm{s}} = W_T^{\mathrm{s}} - I_T$. The corresponding efficient frontier is denoted TESEF (Tracking Error Static Efficient Frontier) and is represented in the Mean Excess return-Tracking Error space ($E(Y_T^{\mathrm{s}})$ as a function of $\mathrm{SD}(Y_T^{\mathrm{s}})$). TESEF is a semi-straight line starting from the origin, and its slope is $[E(M_T) - 1/B_T(0)]/\sigma(M_T)$.

In the dynamic counter-part of this analysis, as shown in equation (6), the excess return $W_T^{\mathrm{u}} - I_T = Y_T^{\mathrm{u}} = (1 - B_T(0)L_T)$ drives the improvement of dynamic trading over static trading in the absence of a downward constraint. From this excess return equation, we can derive the equation for the TEDEF (Tracking Error Dynamic Efficient Frontier) in the Mean Excess return-Tracking Error space and, in particular, its slope. Specifically, we obtain

$$a = \frac{E(1/G_T^2)}{\sigma^2(1/G_T)} E(Y_T^{\mathrm{u}}),$$

hence

$$E(Y_T^{\mathrm{u}}) = \frac{\sigma(1/G_T)}{B_T(0)} \sigma(Y_T^{\mathrm{u}}).$$

These results are analogous to those of BP (1998) in the case of a continuous mean–variance optimization with no reference to a benchmark. In a similar way to proposition 4.2, we can prove that the ratio of the slopes of TESEF by TEDEF is equal to $\mathrm{corr}(M_T, L_T)$, the correlation coefficient between the standard Markowitz efficient tangent portfolio $M_T$ and portfolio $L_T$. This correlation coefficient, being strictly less than 1, TEDEF dominates TESEF. Therefore, optimal dynamic strategies are always superior to optimal static strategies when we assume a tracking error and no stochastic hedging constraint. For example, and according to BP (1998, section 3.1.2), assuming a geometric Brownian motion for the static Markowitz portfolio M and a constant risk-free rate, assuming the 'market price of risk' (the ratio of the difference between the drift of M and the risk-free rate by its volatility) equals 20%, the volatility of M equals 20%, and assuming a five-year horizon, TEDEF's slope is 22% larger than TESEF's slope.

Note finally that the solution of the constrained program given by (5) can also be written as the sum of the solution $W'_T$u of the unconstrained program (starting with an initial investment smaller then 1§) and a 'put' option (spread option) on $W'_T$u with a stochastic strike

†Cox and Huang (1989) first introduced the option-like decomposition of the optimal strategy in the particular case of positive wealth constraints.

‡In this case, the benchmark $I_T$ must be attainable through a buy and hold strategy (which is the case, for instance, for value weighted indices).

§Calling $P_0$ the price of the put at date 0, the initial investment in the unconstrained strategy is 1–$P_0$.

price equal to $C_T$:

$$W^*_T = W'_T\mathrm{u} + \max(0, C_T - W'_T\mathrm{u}), \tag{7}$$

A similar result prevails in the absence of benchmarking and in the presence of constraints (see, for instance, Nguyen and Portait 2002, El Karoui *et al.* 2005; the interpretations are analogous in both cases (adding the put enforces the satisfaction of the hedging constraint)).

## 4. Optimal intermediary wealth

We make the following assumptions.

**Assumption (A):** *Linear affine stochastic hedging. The minimum return level $C_T$ is a linear affine function of the benchmark:*

$$C_T = W^{\min} + pI_T. \tag{8}$$

*Equation (8) implies that the hedging constraint is given by $Y_T \geq W^{\min} + (p-1)I_T$. The only technical requirement is that $B_T(0)W^{\min} + p \leq 1$, which is the feasibility condition: $E(C_T/G_T \leq 1)$).*

Although restrictive, this assumption leads to useful interpretations. $W^{\min}$ can represent a minimum level of wealth (or return) and $p$, typically between 0 and 1, aiming to a minimum performance. Particular cases are standard portfolio insurance: $p=0$, hence $W_T \geq W^{\min}$; performance constraint: $W^{\min}=0$, $W_T \geq pI_T$; and shortfall constraint: $p=1$ and $W^{\min}<0$, hence $Y_T \geq W^{\min}$ (a fixed shortfall $W^{\min}$ from the benchmark is allowed).

Alternatively, the model and its hedging constraint may be interpreted in terms of performance fees. For instance, assume that the fees $Z_T$ are linear in the excess return: $Z_T = a + bY_T$. Recall, as noted previously, that a quadratic manager facing such fees maximizes the objective function of (P). Besides, the hedging constraint (8) can be interpreted as a constraint on the minimum level of performance fee. For instance, the shortfall constraint $Y_T \geq W^{\min}$ implies $Z_T \geq a + bW^{\min}$: fees are bounded from below at any arbitrary level $c = a + bW^{\min}$. Note that, as far as the manager's incentives are concerned, such a compensation scheme bounded from below through a hedging constraint is radically different to the seemingly analogous option-type compensation $\max(Z_T, c)$. In the option-type compensation, an incentive may be given to the manager to take downside risk (on the excess return),† while in the case of a hedging constraint $Y_T \geq W^{\min}$, such downside risk is forbidden.

An interesting generalization (not implemented in this paper)‡ yields further interpretations and more flexibility: assuming that the fees are linear combinations of both $W_T$ and $Y_T$ (both management and performance fees), the quadratic objective function is similar to that of Chow (1995) and depends on the return, the variance, and the TEV.

**Assumption (AA):** *Constant investment opportunity set. The instantaneous risk-free rate r is constant; N risky assets are traded and their instantaneous returns follow a multivariate geometric Brownian motion.*

It is well known that, under (AA), the weights of the tangent portfolio M are constant. However, in general, the composition of the benchmark I is different from that of M. Under AA, as derived by Merton (1992), in the absence of benchmarking, two fund separation prevails: the risk-free asset and portfolio M span all optimal portfolios. Within our optimization program involving benchmark I, a third fund yielding $I_T$ and called 'the index' is necessary because we consider the general case where I and M are imperfectly locally correlated. Therefore, the simplest framework§ sufficient for our analysis involves the risk-free asset and two risky funds:

- the index is assumed to follow a geometric Brownian motion:

$$\mathrm{d}I_t/I_t = \mu_I \mathrm{d}t + \sigma \mathrm{d}w_1. \tag{9}$$

- the tangent portfolio whose value $M_t$ follows a geometric Brownian motion is imperfectly correlated with I:

$$\mathrm{d}M_t/M_t = \mu_M \mathrm{d}t + \sigma_{1M}\mathrm{d}w_1 + \sigma_{2M}\mathrm{d}w_2, \tag{10}$$

where $w_1$ and $w_2$ are two standard non-correlated Brownian motions.

The following lemma presents several technical results used in the paper. (i) and (ii) have been proved by others, for instance Merton (1992) and BP (1998), and (iii) by BP (1998).

**Lemma 4.1: Under (AA):**

(i) *The growth optimal portfolio G is a constant weights combination of the tangent portfolio M and the risk-free asset (the weights on the N risky assets of G and M are homothetic).*

(ii) *The minimum norm portfolio L is a constant weights strategy with weight −1 in G and +2 in the risk-free asset (with an initial investment of \$1). Therefore, its value follows a geometric Brownian motion:*

$$\mathrm{d}L_t/L_t = \mu_L \mathrm{d}t + \sigma_1 \mathrm{d}w_1 + \sigma_2 dw_2, \tag{11}$$

(iii) *The value $L_t$ of the minimum norm portfolio is derived from the value $M_t$ of the tangent portfolio and is given by*

$$L_t = \psi(t)(M_t)^{-\beta},$$

†The TEV constraint mitigates this incentive.

‡The solution and the results are very similar to those of this paper.

§Since the market portfolio has by definition a zero weight in the index I, 'compatibility' conditions require $(\mu_I - r)(\sigma^2_{1M} + \sigma^2_{2M}) = (\mu_M - r)\sigma\sigma_{1M}$.

*with*

$$\beta = \frac{\mu_M - r}{\sigma_{1M}^2 + \sigma_{2M}^2}, \quad \psi(t) = \exp(-(1+\beta)(\mu_M - r)t/2) \times \exp((1+\beta)(r - \beta\sigma_M\sigma_M'/2)t).$$

*In addition, the dynamics of L (as expressed by equation (11)) are linked to those of M (as expressed by equation (10)) by* $\mu_L = r - \beta(\mu_M - r)$, $\sigma_1 = -\beta\sigma_{1M}$, $\sigma_2 = -\beta\sigma_{2M}$.

The correspondence between L and M provided by (iii) is convenient since many results are simpler to derive as a function of L while M is an interpretable and standard reference. For instance, we can express the optimal wealth using equation (5) as a function of portfolio M: $W_T^* = C_T + \max\{0,\ a - C_T - (\kappa'\psi(T)M_T^{-\beta} - I_T)\}$.

Based on equation (9), the return $J_T(t)$ of the benchmark between $t$ and $T$ is given by

$$I_T/I_t = J_T(t) = \exp[(r - \sigma^2/2)(T-t) + (w_1(T) - w_1(t))]. \tag{12}$$

The 'return' $J_T(t)$, viewed from time $t$, is random and is independent of the current benchmark level $I_t$. Following Shimko (1994), the valuation of the put spread option can be implemented in two steps, as derived in proposition 4.2.

**Proposition 4.2:** *Assuming (A) and (AA), the optimal portfolio value can be computed at any time t:*

(i) *Let $P_1(t, I_T)$ be the price at date t, for a given $I_T$, of a put on $\kappa L_T$ with a strike price equal to $(a - W^{\min} + (1-p)I_T)$, then*

$$P_1(t, I_T) = (a - W^{\min} + (1-p)I_tJ_T(t))e^{-r(T-t)}N(-d_2) - \kappa L_t e^{-(T-t)d(J_T(t))}N(-d_1), \tag{13}$$

*with*

$$d_1 = \frac{\left\{\begin{array}{c}\log(\kappa L_t/[a - W^{\min} + (1-p)I_tJ_T(t)]) \\ +(r - d(J_T(t)) + (\sigma^2/2))(T-t)\end{array}\right\}}{\sigma_2\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma_2\sqrt{T-t},$$

*and*

$$d(J_T) = \sigma_1\left[\frac{\sigma_1 - \sigma}{2} + \frac{1}{\sigma}\left(r - \frac{\log(J_T)}{T-t}\right)\right]. \tag{14}$$

*Then the price $P_t$ at date t of the put-spread option defining the second component of the optimal strategy yielding $W_T^*$ is computed as*

$$P_t = G_tE_t[P_1(t, I_T)/G_T], \tag{15}$$

*where $E_t$ is the expectation provided the information available at time t.*

(i) *Therefore, the optimal wealth at any intermediate date t is*

$$W_t^* = W^{\min}e^{-r(T-t)} + pI_t + P_t, \tag{16}$$

The derivation of this proposition is given in the appendix and follows Shimko's (1994) approach for the pricing of options on futures spreads.

Note that the computation of the price $P_t$ of the put spread option involves two steps. In the first step, $I_T$ is given, yields a closed-form formula for $P_1(t, I_T)$, and relies on the Black–Scholes–Merton formula ($d(J_T)$ is interpreted as a continuous dividend stream). The second step involves the numerical computation of an integral as given in (15).

Note also that all equations of proposition 4.2 can be written as a function of $M_t$ (in lieu of $L_t$) using lemma 4.1(iii).

In one particular case, $p=1$ and $W^{\min}$ negative (shortfall constraint), the standard Black and Sholes formula applies. Indeed, equation (5) is $W_T^* = W^{\min} + I_T + \max(0, (a - W^{\min}) - \kappa L_T)$, which involves the pricing of a European put option on $\kappa L_T$ (which is log-normal) with a strike equal to $a - W^{\min}$.

In a second particular case, $C_T = W^{\min} = a$, an explicit formula *a la* Margrabe (1978) for the pricing of an exchange option between $I_T$ and $\kappa L_T$ can be obtained.

The characterization of the optimal strategies relies on the valuation of the put spread option of proposition 4.2; it is presented in proposition 4.3.

**Proposition 4.3:** *Assuming (A) and (AA), the optimal strategy is defined by its weights, $x_I$ and $x_M$, in I and M, respectively, and can be computed, at any time t:*

$$x_I(t) = \frac{I_t}{W_t^*}\left(\frac{\partial P_t}{\partial I_t}\right), \quad x_M(t) = \frac{M_t}{W_t^*}\left(\frac{\partial P_t}{\partial M_t}\right), \tag{17}$$

$$\frac{\partial P_t}{\partial I_t} = \underset{t}{E}\frac{\partial P_1}{\partial I_t}, \quad \frac{\partial P_t}{\partial M_t} = \underset{t}{E}\frac{\partial P_1}{\partial M_t}, \tag{18}$$

$$\frac{\partial P_1}{\partial I_t} = e^{-r(T-t)}\left[N(-d_2) + \frac{N'(-d_2)}{\sigma_2\sqrt{T-t}}\right](1-p)J_T(t) - \frac{\kappa L_t e^{-(T-t)d(J_T(t))}N'(-d_1)}{\sigma_2\sqrt{T-t}(a - W^{\min} + (1-p)I_T)}(1-p)J_T(t), \tag{19}$$

$$\frac{\partial P_1}{\partial M_t} = \kappa\exp[-(T-t)d(J_T(t))]N(-d_1)\beta\psi(t)M_t^{-(\beta+1)}. \tag{20}$$

Note that the partial derivatives of $P_t$, and hence the optimal weights, involve an expectation that can be computed numerically, as in section 5.

## 5. Comparative static results and interpretations on a numerical example

In this section, we develop further interpretations and comparative static results through a numerical example in the case of portfolio insurance ($p=0$, $W_T \geq W^{\min}$).

In addition to a risk-free security (whose rate $r$ is equal to 4%), we assume that two risky funds are continuously traded, the first fund being the benchmark I and the

Table 1. Different capital protection levels for $a = 0.25$ and time to maturity $T - t = 5$ years.

| Capital protection ($W^{\min}$) | $x_{\mathrm{M}}$ | $x_{\mathrm{I}}$ | $x_{\mathrm{r}}$ | $\Gamma_{\mathrm{M/M}}$ | $\Gamma_{\mathrm{I/I}}$ | $\Gamma_{\mathrm{I/M}}$ | $\Gamma_{\mathrm{M/I}}$ |
|---|---|---|---|---|---|---|---|
| 121% | 3% | 3% | 94% | 0.08% | 0.10% | 0.10% | 0.10% |
| 110% | 17% | 27% | 57% | 0.09% | 0.54% | 0.35% | 0.35% |
| 100% | 22% | 44% | 35% | −0.02% | 0.67% | 0.34% | 0.34% |
| 90% | 24% | 58% | 18% | −0.12% | 0.67% | 0.28% | 0.28% |
| 80% | 26% | 69% | 5% | −0.20% | 0.60% | 0.20% | 0.20% |
| 70% | 27% | 78% | −5% | −0.26% | 0.49% | 0.12% | 0.12% |
| 50% | 29% | 89% | −19% | −0.36% | 0.27% | −0.03% | −0.03% |
| 0% | 36% | 99% | −34% | −0.63% | 0.04% | −0.28% | −0.28% |
| None | 41% | 100% | −41% | −0.97% | 0.00% | −0.40% | −0.41% |

second the tangent portfolio M. Their dynamics follow equations (9) and (10), and we set their parameters to $\mu_{\mathrm{I}} = 11.5\%$, $\sigma = 25\%$, $\mu_{\mathrm{M}} = 9.625\%$, $\sigma_{1\mathrm{M}} = 15\%$ and $\sigma_{2\mathrm{M}} = 7.5\%$.

We analyse the effect of changes in the values of several variables and parameters (capital protection level, absolute and deviation risk, risk tolerance level, time to maturity) on the optimal initial allocation (at date $t = 0$) for time to maturity $T$. Portfolio strategies are represented by two sets of parameters:

- the weights $x_{\mathrm{I}}$, $x_{\mathrm{M}}$ and $x_{\mathrm{r}}$ on the benchmark I, the tangent portfolio M and the riskless asset. Note that these weights are also the sensitivities of the portfolio value to market prices:

$$x_I = \frac{\partial W_t/W_t}{\partial I_t/I_t}, \quad x_{\mathrm{M}} = \frac{\partial W_t/W_t}{\partial M_t/M_t};$$

- the variations of these weights for a 1% variation of the fund values, called in this analysis the Gammas and defined as

$$\Gamma_{\mathrm{M/M}} = \frac{\partial x_{\mathrm{M}}}{\partial M_t/M_t}, \quad \Gamma_{\mathrm{I/I}} = \frac{\partial x_{\mathrm{I}}}{\partial I_t/I_t},$$
$$\Gamma_{\mathrm{I/M}} = \frac{\partial x_I}{\partial M_t/M_t}, \quad \Gamma_{\mathrm{M/I}} = \frac{\partial x_{\mathrm{M}}}{\partial I_t/I_t}.$$

According to our adopted terminology,† positive Gammas correspond to momentum or convex (portfolio insurance) strategies, while negative Gammas correspond to contrarian or concave strategies. Thereafter, we consider a trade-off between the expected return and two different risks.

- The risk of deviations from the benchmark, called deviation risk with a corresponding risk aversion of $1/a$. Note that this risk implies concave strategies since the utility function is quadratic.
- The risk of small absolute returns $W_T$, called the total return risk. This risk triggers portfolio insurance strategies, which are convex. In that case, the Lagrange multiplier of the wealth constraint (which increases with $W^{\min}$) can be interpreted as the corresponding risk aversion.

These two contradictory risks enhance the complexity and the interest of this analysis.

### *5.1. Capital protection, absolute and deviation risk*

Table 1 presents the weights and the Gammas, for $a = 0.25$ and $T - t = 5$, as a function of the capital protection $W^{\min}$ varying between 121% and 0% of the portfolio value. From this table we observe that the smaller the capital protection, the smaller the weight in the risk-free asset and the larger the weights on the other funds. Without capital protection the optimal strategy is a 100% investment in the benchmark I, combined with a position in M financed by cash borrowing. In fact, leverage is optimal only for low levels of capital protection. These results confirm the expected impact of the capital protection requirement on the weights $x_{\mathrm{I}}$, $x_{\mathrm{M}}$ and $x_{\mathrm{r}}$.

From convex to concave, we obtain a wide spectrum of strategic behavior. More precisely, for large values of $W^{\min}$, the strategy has a pure convex profile (all Gammas are positive) with almost no equity holdings; it is close to a 100% bond portfolio. When the capital protection $W^{\min}$ decreases, the allocation converges to the solution of the non-constrained problem, which is a 100% concave strategy (all Gammas are negative); this strategy has an active 'overlay' (active) position corresponding to a leverage long tangent portfolio/short risk-free asset.

The interpretation of the Gammas from table 1 involves both total return risk and deviation risk effects. These two risks may or may not have contradictory effects on the allocations in M and I, as shown in table 2.

As noted previously, the wealth constraint (portfolio insurance) implies convexity, which means that risky funds are bought and the risk-free asset is sold when prices (and hence wealth) increase. There is an inverse effect when prices decrease.

On the contrary, the quadratic objective function yields concave strategies in the case without benchmarking. This means that positive (negative) returns imply selling (buying) risky assets. This is due to the increasing risk aversion characterizing the quadratic utility (when wealth goes up the quadratic investor reduces the weight in risky

†Note that these momentum or contrarian strategies are defined, depending on the authors, either according to variations of weights as done here, or according to variations of $ amounts invested in the funds when the values of these funds vary.

Table 2. Effects of total return and deviation risks on allocations in M and I.

| | | $x_M$ | $x_I$ | $x_r$ |
|---|---|---|---|---|
| M increases | Tracking error with no wealth constraint effect | Down | Down | Up |
| | Portfolio insurance effect | Up | Up | Down |
| | Total effect | ? | ? | ? |
| I increases | Tracking error with no wealth constraint effect | Down | No change | Up |
| | Portfolio insurance effect | Up | Up | Down |
| | Total effect | ? | Up | ? |

Table 3a. Varying risk tolerance levels with 80% capital protection and horizon $T = 5$ years.

| Tolerance level | $x_M$ | $x_I$ | $x_r$ | $\Gamma_{M/M}$ | $\Gamma_{I/I}$ | $\Gamma_{I/M}$ | $\Gamma_{M/I}$ |
|---|---|---|---|---|---|---|---|
| 0.01 | 10% | 75% | 15% | −0.14% | 0.64% | 0.10% | 0.10% |
| 0.10 | 17% | 72% | 11% | −0.19% | 0.62% | 0.15% | 0.15% |
| 0.25 | 26% | 69% | 5% | −0.20% | 0.60% | 0.20% | 0.20% |
| 0.50 | 38% | 64% | −2% | −0.15% | 0.57% | 0.26% | 0.26% |
| 0.80 | 50% | 59% | −9% | −0.04% | 0.54% | 0.31% | 0.31% |
| 1.00 | 57% | 57% | −13% | 0.05% | 0.53% | 0.33% | 0.33% |
| 1.50 | 70% | 52% | −22% | 0.29% | 0.49% | 0.36% | 0.36% |
| 2.00 | 81% | 48% | −29% | 0.53% | 0.46% | 0.38% | 0.38% |

Table 3b. Varying risk tolerance levels with no capital protection, no wealth constraint and horizon $T = 5$ years.

| Tolerance level | $x_M$ | $x_I$ | $x_r$ | $\Gamma_{M/M}$ | $\Gamma_{I/I}$ | $\Gamma_{I/M}$ | $\Gamma_{M/I}$ |
|---|---|---|---|---|---|---|---|
| 0.01 | 2% | 100% | −2% | −0.03% | 0.00% | −0.02% | −0.02% |
| 0.10 | 16% | 100% | −16% | −0.35% | 0.00% | −0.16% | −0.16% |
| 0.25 | 41% | 100% | −41% | −0.97% | 0.00% | −0.40% | −0.41% |
| 0.50 | 82% | 100% | −82% | −2.26% | 0.00% | −0.80% | −0.81% |
| 0.80 | 131% | 100% | −131% | −4.22% | 0.00% | −1.27% | −1.30% |
| 1.00 | 164% | 100% | −164% | −5.77% | 0.00% | −1.59% | −1.62% |
| 1.50 | 246% | 100% | −246% | −10.53% | 0.00% | −2.36% | −2.43% |
| 2.00 | 327% | 100% | −327% | −16.49% | 0.00% | −3.13% | −3.24% |

assets). When using a quadratic objective function on the deviation $Y_t = W_t - I_t$, concavity means that positive deviations from the benchmark imply increasing the weight in the risk-free asset and decreasing the weight in the tangent portfolio M (and negative deviations the opposite). Consequently, an M increment triggers positive deviations and the two opposite effects on the weights $x_I$ and $x_M$ as documented in table 1. When I increases, the absolute return effect (portfolio insurance) has a positive impact on $x_M$ and $x_I$, but the deviation effect has a negative effect on $x_M$ and none on $x_I$. From equation (6), in the absence of capital protection, the weight on I is constant and equal to 100% (hence $\Gamma_{I/I}$ is equal to 0).

From table 2 (and as confirmed in table 1), we expect the following.

- $\Gamma_{M/M}$ increases with the level of capital protection as portfolio insurance becomes predominant. In the numerical example of table 1, for a capital protection of less than 100%, $\Gamma_{M/M}$ is negative and the deviation effect (concavity) prevails. For a capital protection greater than 100%, $\Gamma_{M/M}$ is positive and convexity prevails.
- $\Gamma_{I/I}$ is always positive. Moreover, with increasing levels of capital protection $W^{\min}$, $\Gamma_{I/I}$ first increases as portfolio insurance becomes more important and then decreases towards 0 as the strategy converges to the risk-less strategy.
- The values $\Gamma_{M/I}$ and $\Gamma_{I/M}$ are very close; they both decrease starting from a positive value (characterizing a convex profile) for a large value of $W^{\min}$ to a negative value for a small or negative value of $W^{\min}$ (concave profile).

### 5.2. *Risk tolerance*

The weights and Gammas for an horizon $T = 5$ years and a capital protection $W^{\min} = 80\%$ as a function of the deviation-risk tolerance $a$ varying between 0.01 and 2 are presented in table 3a. As expected, $x_I$ decreases for increasing risk tolerance to deviations at the expense of $x_M$. These features are in contrast to those displayed for the pure concave strategy (with no wealth constraint) where $x_I$ remains at 100% while $x_M$ increases (table 3b). In table 3a, $x_r$, the weight in the risk-free security,

Table 4a. Weights and Gammas with varying horizons when $W^{\min} = 80\%$ and $a = 0.25$

| Investment horizon (in years) | $x_M$ | $x_I$ | $x_r$ | $\Gamma_{M/M}$ | $\Gamma_{I/I}$ | $\Gamma_{I/M}$ | $\Gamma_{M/I}$ |
|---|---|---|---|---|---|---|---|
| 0.08 | 49% | 99% | −48% | −1.07% | 0.27% | −0.28% | −0.29% |
| 0.25 | 44% | 91% | −35% | −0.62% | 0.91% | 0.21% | 0.20% |
| 1.00 | 34% | 76% | −10% | −0.29% | 1.01% | 0.39% | 0.39% |
| 3.00 | 28% | 69% | 3% | −0.21% | 0.73% | 0.27% | 0.27% |
| 5.00 | 26% | 69% | 5% | −0.20% | 0.60% | 0.20% | 0.20% |
| 10.00 | 22% | 74% | 5% | −0.18% | 0.43% | 0.11% | 0.11% |

Table 4b. Weights and Gammas with varying horizons, no capital protection and $a = 0.25$.

| Investment horizon (in years) | $x_M$ | $x_I$ | $x_r$ | $\Gamma_{M/M}$ | $\Gamma_{I/I}$ | $\Gamma_{I/M}$ | $\Gamma_{M/I}$ |
|---|---|---|---|---|---|---|---|
| 0.08 | 50% | 100% | −50% | −1.22% | 0.00% | −0.49% | −0.49% |
| 0.50 | 49% | 100% | −49% | −1.20% | 0.00% | −0.48% | −0.49% |
| 1.00 | 48% | 100% | −48% | −1.17% | 0.00% | −0.47% | −0.48% |
| 3.00 | 44% | 100% | −44% | −1.06% | 0.00% | −0.43% | −0.44% |
| 5.00 | 41% | 100% | −41% | −0.97% | 0.00% | −0.40% | −0.41% |
| 10.00 | 34% | 100% | −34% | −0.77% | 0.00% | −0.33% | −0.33% |

decreases when the tolerance to deviations increases. Indeed, for higher risk tolerance levels, the manager is more willing to take more risk and therefore engages in a more leveraged position in equity assets (which is even more the case when there is no wealth constraint, as shown in table 3b).

### 5.3. *Time to maturity*

Table 4a presents the values of the weights and Gammas, for $a = 0.25$ and $W^{\min} = 80\%$, when the horizon $T$ varies between 1 month and 10 years. As a comparison, the case of no wealth constraint is presented in table 4b. In contradiction with the conventional wisdom, the weight on the risk-free asset *increases* with the time to maturity. This can be intuitively explained as the capital protection requirement of 80% being easily met for short horizons (when the stock price processes are continuous, a loss greater than 20% is extremely unlikely in the following month, even for mildly leveraged positions). On the contrary, important losses are possible in the following three years if there is no compliance today with the wealth constraint (the variance of the stock prices is 36 times higher over three years than over one month). In fact, holding $W^{\min} = 80\%$ constant over time corresponds to a particular path (that can be considered bearish) on which the value of the portfolio remains constant, which would be very peculiar.

On an actual path over five years, the passage of time and the wealth and substitution effects combine, and tables 1 and 4 help us to understand the corresponding dynamic portfolio strategy: on bullish paths, $W^{\min}$ decreases, portfolio insurance becomes irrelevant and the policy is explained by concavity on the deviation $W_t - I_t$. On such paths, the portfolio ends with leverage. On bearish paths, the contrary applies, portfolio insurance considerations become the predominant effect as time passes, and the risk-free asset is increasingly used to comply with the wealth constraint.

It is interesting to note that, in alternative settings, the reason why an investment strategy may be contrarian relies on mean reversion. In this paper, although the stock prices are not mean reverting (they follow simple geometric Brownian motions), the strategy balances between contrarian and momentum depending on the relative strength of the hedging constraint: when the hedging constraint is binding, convexity prevails; when the portfolio value is way above its minimum value, the 'quadratic behavior' prevails and the strategy is concave (contrarian).

It is also interesting to note that most portfolio managers (employees of the fund management company) have short-term incentives (typically one-year pay contracts) while the management company is interested in a longer term. Therefore, the objectives of a fund manager and of a fund management company may be different (the manager is willing to take more risk), which enhances the potential agency problem pointed out in our comments about management compensation.

## 6. Conclusion

The model developed in this paper integrates three widespread features of the asset management industry: benchmarking, rebalancing and hedging constraints. Its aim is to provide allocation rules and to study the characteristics of their implied strategies. When the manager cares about absolute returns and relative returns as well, the risk–return trade-off acquires an extra dimension since two components of risk matter. This extra risk dimension substantially modifies the characteristics of portfolio strategies. In this paper we obtain separation results for the optimal policies under such a context, analyse the conditions under which these strategies are either contrarian or momentum, and describe how optimal allocations should vary over time.

This work could be extended by considering different goals to those studied in this paper. Although real-world management styles can be characterized by different goals and constraints with respect to portfolio policies, only two of them (minimum value and minimum performance) have been thoroughly studied in this paper. To analyse a wider variety of asset management styles, we could consider two different constraints (performance and floor value) in place of one combined constraint (which would lead to the pricing and the duplication of a compound option), introduce a performance constraint using an index other than that used as a benchmark, or impose a value-at-risk or a conditional value-at-risk constraint. The methodology developed in this paper could be adapted to address some of these issues. Another interesting research question would be to introduce interest rate uncertainty. This extension would allow for an important distinction between short-term bills and long-term bonds, and would provide an adequate tool for choosing between bills, bonds and stocks in the context of a benchmarked strategic asset allocation.

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## Appendix

**Proof of proposition 3.1:** In a discrete state space $\Omega$ the Lagrangian of (P) can be written explicitly as

$$\sum_{\omega\in\Omega} P(\omega)Y_T(\omega) - \frac{1}{2a}\sum_{\omega\in\Omega} P(\omega)Y_T^2(\omega) - \kappa_1\sum_{\omega\in\Omega} P(\omega)\frac{Y_T(\omega)}{G_T(\omega)} - \sum_{\omega\in\Omega}\kappa_2(\omega)(Y_T(\omega)+I_T(\omega)-C_T(\omega)),$$

where $Y_T(\omega)$ are the control variables. The first-order conditions are

$$\frac{\partial L}{\partial Y_T(\omega)} = P(\omega) - \frac{1}{a}P(\omega)Y_T^*(\omega) - \frac{\kappa_1 P(\omega)}{G_T(\omega)} - \kappa_2(\omega) = 0, \quad \text{for all} \quad \omega\in\Omega.$$

If, in state $\omega$, the inequality constraint is binding: $\kappa_2(\omega) > 0$ and $Y_T(\omega) = C_T(\omega) - I_T(\omega)$. If it is non-binding, then $\kappa_2(\omega) = 0$ and, based on the first-order condition:

$$1 - \frac{1}{a}Y_T(\omega) - \frac{\kappa_1}{G_T(\omega)} = 0.$$

This implies in all states that

$$Y_T^*(\omega) = \max\left(C_T(\omega) - I_T(\omega), a - a\frac{\kappa_1}{G_T(\omega)}\right),$$

hence

$$Y_T^* = \max\left(C_T - I_T, a - a\frac{\kappa_1}{G_T}\right) = C_T - I_T + \max\left[0, a - \frac{a\kappa_1}{G_T} - C_T + I_T\right],$$

where $\kappa_1$ is implicitly derived from the budget constraint $E(Y_T^*/G_T) = 0$ or, equivalently, $E(W_T^*/G_T) = 1$. Denoting $\kappa = a\kappa_1 E(1/G_T^2)$, we obtain equation (5).

**Proof of proposition 4.2:** The price $P_t$ of a put spread option with a terminal payoff equal to $W_T^* = \max[0, K + (1-p)I_T - \kappa L_T]^+$ with $K = a - W^{\min}$ needs to be derived. $P_t$ is given by

$$P_t = E_t\left[\frac{\max(0, K + (1-p)I_T - \kappa L_T)}{G_T}\right],$$

where $E_t$ is the conditional expectation to the information available at $t$. $P_t$ can also be written as

$$P_t = E_t\left[E_t\left(\frac{\max(0, K + (1-p)I_T - \kappa L_T)}{G_T}/I_T\right)\right].$$

The first expectation, $E_t(\cdot/I_T)$, is conditional on the information available at $t$, and on a given value $I_T$. The second expectation is conditional on the information available at $t$.

We define $P_1(t, I_T)$ as the price at time $t$ of a put option yielding $\kappa' L_T$ with a strike price equal to $K + (1-p)I_T$ (where $I_T$ is given); $P_1(t, I_T)$ is given by

$$P_1(t, I_T) = E_t\left(\frac{\max(0, K + (1-p)I_T - \kappa L_T)}{G_T}/I_T\right).$$

Note that we alternatively use (depending on computational convenience) the pricing with the growth optimal portfolio G in the historical probability or with the risk-neutral probability (in which case the expectation is denoted $E^*$) and discounting at the risk-free rate. The Black/Scholes formula can be applied, but not in a straightforward way. I follows, in the risk-neutral world, a geometric Brownian motion with a drift equal to $r$. We thus determine the distribution of $L_T$ conditional on $I_T$. In the risk-neutral probability, the equations governing $I_t$ and $L_t$ are

$$\frac{\mathrm{d}I_t}{I_t} = r\mathrm{d}t + \sigma\mathrm{d}w_1,$$
$$\frac{\mathrm{d}L_t}{L_t} = r\mathrm{d}t + \sigma_1\mathrm{d}w_1 + \sigma_2\mathrm{d}w_2,$$

with

$$\mathrm{d}w_1\mathrm{d}w_2 = 0, \qquad \frac{\mathrm{d}I_t}{I_t}\frac{\mathrm{d}L_t}{L_t} = \sigma\sigma_1\mathrm{d}t.$$

This implies that

$$\log\left(\frac{I_T}{I_t}\right) = \log(J_T(t)) = \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(w_1(T) - w_1(t)),$$

which follows a normal distribution

$$\mathrm{N}\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t),\ \sigma\sqrt{T-t}\right).$$

We then have

$$\log\left(\frac{L_T}{L_t}\right) \sim \mathrm{N}\left(\left(r - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2\right)(T-t); \sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{T-t}\right),$$

and the conditional distribution

$$\log\left(\frac{L_T}{L_t}\right)\Big/\log(J_T(t)) \sim \mathrm{N}\left(\left(\left(r - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2\right)\right)(T-t) + \sigma_1(w_1(T) - w_1(t)), \sigma_2\sqrt{T-t}\right).$$

Finally, $\log J_T(t) = (r - (\sigma^2/2))(T-t) + \sigma(w_1(T) - w_1(t))$ implies that

$$w_1(T) - w_1(t) = \frac{1}{\sigma}\left[\log J_T(t) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right].$$

The parameters of the conditional distribution can then be rewritten as

$$\log\left(\frac{L_T}{L_t}\right)/\log(J_T(t)) \sim \mathrm{N}\left(r - \frac{\sigma_1^2+\sigma_2^2}{2}\right)(T-t) + \frac{\sigma_1}{\sigma}\left(\log J_T(t) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right), \sigma_2\sqrt{T-t}$$
$$\sim \mathrm{N}\left(\left(r - \frac{\sigma_1^2 - \sigma_1\sigma}{2} + \frac{\sigma_1}{\sigma}(r - \log J_T(t)/(T-t))\right) \times (T-t) - \frac{\sigma_2^2}{2}(T-t), \sigma_2\sqrt{T-t}\right).$$

Letting

$$d(J_T(t)) = \sigma_1\left[\frac{\sigma_1 - \sigma}{2} + \frac{1}{\sigma}\left(r - \frac{\log(J_T(t))}{T-t}\right)\right],$$

we obtain

$$\log\left(\frac{L_T}{L_t}\right)\Big/\log(J_T(t)) \sim \mathrm{N}\left(\left(r - d(J_T(t)) - \frac{\sigma_2^2}{2}\right)(T-t), \sigma_2\sqrt{T-t}\right),$$

which is the distribution of the payoff at time $T$ for a security yielding a dividend rate of $d(J_T(t))$ (which is constant when $I_T$, and hence $J_T$, is given).

The put on '$\kappa L_T$ conditional on $(1-p)I_T$' can thus be priced by the Black–Scholes–Merton formula, valid when the underlying asset yields a dividend stream $d(J_T(t))$:

$$P_1(t, I_T) = \exp(-r(T-t))E^*((K + (1-p)I_T - \kappa L_T)^+/I_T)$$
$$= (K + (1-p)I_T)\exp(-r(T-t))N(-d_2) - \kappa L_t \times \exp(-(T-t)d(J_T(t)))N(-d_1),$$

with

$$d_1 = \frac{\log(\kappa L_t/[K + (1-p)I_T]) + (r - d(J_T(t)) + \sigma_2^2/2)(T-t)}{\sigma_2\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma_2\sqrt{T-t},$$

which are the formulas for $d_1$ and $d_2$ in proposition 4.2. The put spread value can thus be written with the risk-neutral expectation at date $t$ of the value $P_1(t, I_T)$:

$$P_t = E_t^*[(K + (1-p)I_T)\exp(-r(T-t))N(-d_2) - \kappa L_t \exp(-(T-t)d(J_T(t)))N(-d_1)].$$

**Proof of proposition 4.3—Derivation of the optimal strategies:** Let us define more explicitly $P_1(t, I_T)$ as a function of $P_1(t, I_tJ_T(t), L_t)$ of $t$, $I_T$ $(=I_tJ_T(t))$ and the underlying asset value $L_t$:

$$P_1(t, I_tJ_T(t), L_t) = E_t^Q[(K + (1-p)I_tJ_T(t))\exp(-r(T-t)) \times N(-d_2) - k'L_t \exp(-r(T-t)d(J_T(t))N(-d_1)].$$

The price of the put spread option can be calculated using the risk-neutral expectation:

$$P_t = E_t^Q(P_1(t, I_tJ_T(t), L_t)) = \int P_1(t, I_tJ_T(t), L_t)\mathrm{d}Q$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} P_1\big(t, I_t\exp\big(\big(r - \sigma^2/2\big)(T-t) + \sigma\sqrt{T-tu}\big), L_t\big)e^{-u^2/2}\,\mathrm{d}u,$$

where $J_T(t)$ is defined by (12).