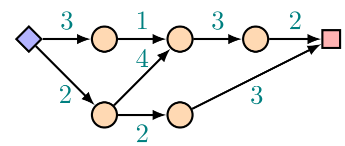
Homework 4: Duality

Due date: 11:00pm on Friday February 22, 2019
See the course website for instructions and submission details.

1. Max-flow to min-cost. Consider the following graph (where blue nodes are sources, yellow nodes are relays, red nodes are sinks, and the edge capacity is labeled on each edge):



We wish to maximize the flow from the source to the sink nodes. Using the trick learned in lecture 5, you will formulate this problem as a min-cost problem. **DO NOT use Julia to solve the problem.** Simply state the answers to the questions.

a) Recall the min-cost model from class:

$$\begin{aligned} \max_{x} \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & p \le x \le q \end{aligned}$$

Remember that A (the incidence matrix) is a property of the graph, not the specific problem. Find A for this graph. What is x? What are p and q?

b) Modify the graph using the trick to formulate a min-cost problem. What is your new x, p, and q? What are c and b?

Remember from lecture 7 that the dual problem of this problem is the minimum cut problem.

c) What is the minimum cut of the this graph (you can just look at the graph to determine the minimum cut, and either give the solution either graphically or as a list of the edges in the cut)? What can you say about the values of the dual variables corresponding to the capacity constraints λ_{ij} and the nodal balance constraints μ_i?

2. The chess problem. A small joinery makes two different sizes of boxwood chess sets. The small set requires 3 hours of machining on a lathe, and the large set requires 2 hours. There are four lathes with skilled operators who each work a 40 hour week, so we have 160 lathe-hours per week. The small chess set requires 1 kg of boxwood, and the large set requires 4 kg. Unfortunately, boxwood is scarce and only 200 kg per week can be obtained. When sold, each of the large chess sets yields a profit of \$8, and one of the small chess set has a profit of \$5. The problem is to decide how many sets of each kind should be made each week so as to maximize profit.

a) Write out the primal LP. Plot the feasible set and solve the LP graphically. Be sure to label the axes and indicate units. Label the optimal point and find the optimal objective.

b) Repeat all the same steps as in part **a)** but for the dual LP this time. Verify that the optimal dual objective is the same as the optimal objective of part **a)**.

3. Stigler's supplement. Consider Stigler's diet problem from Homework 2. To help further lower the cost of your diet, a friend offers to sell you calcium supplements. Each calcium pill contains 500 mg of calcium.

a) What is the most you would be willing to pay per pill? **Hint:** use duality!

b) Suppose you can buy calcium pills at a cost \$0.01 each. What is your new optimal diet? How much money does it save compared to the original optimal diet that didn't have access to the calcium supplement?