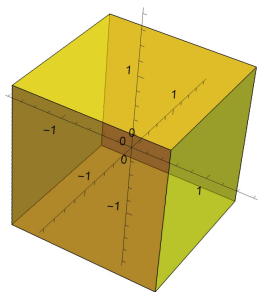
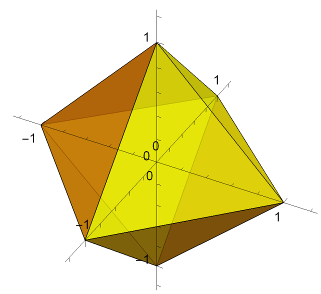
Homework 2: More linear programs

Due date: 11:00pm on Friday February 8, 2019
See the course website for instructions and submission details.

1. **Polyhedron modeling.** We saw that the set of x such that $Ax \le b$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ describes a polyhedron. For each polyhedron below, find a matrix A and vector b such that $Ax \le b$ describes the polyhedron. **Hint:** since each inequality describes a different face, m should be equal to the number of faces. Make sure the inequalities go the right way!



(a) Regular cube centered at the origin with vertices at $(\pm 1, \pm 1, \pm 1)$.



(b) Regular octahedron centered at the origin with vertices at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$.

2. **Standard form with equality constraints.** Rather than using the standard LP form we saw in class, some prefer using a form where all variables are nonnegative, all constraints are equality constraints, and the cost function is a minimization. So a general LP would look like:

$$\begin{aligned} \text{minimize} \quad & c^\mathsf{T} x \\ \text{subject to:} \quad & Ax = b \\ & x \ge 0 \end{aligned} \tag{1}$$

Consider the following LP:

$$\begin{array}{rrcrcrcrcr} \underset{z_1,z_2,z_3,z_4}{\text{maximize}} & 3z_1 & - & z_2 & & & & & & \\ \text{subject to:} & -z_1 & + & 6z_2 & - & z_3 & + & z_4 & \ge & -3 \\ & & & 7z_2 & & & + & z_4 & = & 5 \\ & & & & & z_3 & + & z_4 & \le & 2 \end{array}$$
$$-1 \le z_2 \le 5, \quad -1 \le z_3 \le 5, \quad -2 \le z_4 \le 2$$

a) Transform the above LP into the equality-constrained standard form of (1). What are A, b, c, and x? Be sure to explain how the decision variables of your transformed LP relate to those of the original LP.

b) Solve both versions of the LP using JuMP and show that you can recover the optimal z and objective value by solving your transformed version of the LP.

3. Road Lighting. [Solve this problem without using Julia (you can submit either handwritten solutions or solutions typed up as text in the notebook).] Consider a road divided into n segments illuminated by m lamps. Let p_j be the power of the jth lamp. The illumination I_i on the ith segment is determined by $I_i = \sum_{j=1}^{m} a_{ij} p_j$, where a_{ij} are known, non-negative parameters that describe the relationship between the amount of power and the strength of illumination. Let I_i^{min} be the minimum required illumination of road i. We are interested in choosing the lamp powers p_j which minimize the total power usage along the road.

a) Write down a reasonable linear programming formulation (there may be more than one!). Explain what your variables and constraints represent.

b) What happens if you try to maximize the use of power instead of minimizing? Would you obtain a solution to the problem?

c) If you change all the inequalities to strict inequality in your problem formulation from part **a)**, will you get the same solution? Is it still a standard linear program?

4. Stigler's diet. True story! In 1945, American economist (and future Nobel laureate) George Stigler published a paper investigating the composition of an *optimal* diet; minimizing total cost while meeting the recommended daily allowance (RDA) of several nutrients. To answer this question, Stigler tabulated a list of 77 foods and their nutrient content for 9 nutrients: calories, protein, calcium, iron, vitamin A, thiamine, riboflavin, niacin, and ascorbic acid. Here is what the first few rows and columns of his table looked like:

	Calories (1000)	Protein (g)	Calcium (g)	Iron (mg)	...
Wheat Flour (Enriched)	44.7	1411	2	365	...
Macaroni	11.6	418	0.7	54	...
Wheat Cereal (Enriched)	11.8	377	14.4	175	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

To make calculations easier, Stigler normalized his data so each row shows the nutrients contained in $1's worth of the given food item. Back then, $1 could buy you quite a lot!

When Stigler posed his diet problem, the simplex method had not yet been invented. In his paper, he wrote: "...the procedure is experimental because there does not appear to be any direct method of finding the minimum of a linear function subject to linear conditions." Nevertheless, through a combination of what he called "trial and error, mathematical insight, and agility", he eventually arrived at a solution: a diet costing only $39.93 per year. Though he confessed: "There is no reason to believe that the cheapest combination was found, for only a handful of the [many] possible combinations of commodities were examined."

a) Formulate Stigler's diet problem as an LP and solve it. To get you started, Stigler's original data is provided in `stigler.csv`, and the IJulia notebook `stigler.ipynb` imports the data and puts it into a convenient array format. How does your cheapest diet compare in annual cost to Stigler's? What foods make up your optimal diet?

b) Suppose we wanted to find the cheapest diet that was also vegetarian. How much would that cost per year, and what foods would be used?