ESC194

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Homogeneous 2nd order linear DE with constant coefficients:

$$y'' + ay' + by = 0$$

$$y = e^{rx} \to (r^2 + ar + b)e^{rx} = 0$$

$(r^2 + ar + b)$ is called the auxiliary equation, which can be solved:

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

1 Case 1

$$a^2 - 4b > 0 \therefore \quad r_1 \text{ and } r_2 \text{ are real and distinct}$$

$$\therefore y_1 = e^{r_1 x} \qquad y_2 = e^{r_2 x}$$

$$\therefore y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Example:

$$y'' + \frac{7}{3}y' - 2y = 0$$

Aux equation:

$$\to r^2 + \frac{7}{3}r - 2 = 0 \to (3 - \frac{2}{3})(r + 3 = 0)$$

$$r_1 = \frac{2}{3}, \qquad r_2 = -3$$

$$\therefore y = c_1 e^{\frac{2}{3}x} + c_2 e^{-3x}$$

2 Case 2

$$a^2 - 4b = 0 \quad \therefore r_1 = r_2 = -\frac{a}{2} = r$$
$$\therefore y_1 = e^{rx}, \quad y_2 = xe^{rx}$$

General solution:

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

Example:

$$y'' - 6y' + 9y = 0 \rightarrow \text{aux eqn:} \quad r^2 - 6r + 9 = (r-3)^2 = 0$$

$$r = 3$$

General solution:

$$\therefore y = c_1 e^{3x} + c_2 x e^{3x}$$

3 Case 3

$$a^2 - 4b < 0 \rightarrow \quad r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta$$

where:

$$\alpha = \frac{a}{2}, \quad \beta = \sqrt{\frac{4b - a^2}{2}}$$

and using :

$$e^{i\theta} = \cos(\theta)\sin(\theta)$$
$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$
$$= c_1 e^{\alpha x}(\cos(\beta x) + i\sin(\beta x)) + c_2 e^{\alpha x}(\cos(\beta x) - i\sin(\beta x))$$
$$= e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$$

Example:

$$y'' - 3y' + 4y = 0 \rightarrow \text{aux eqn: } r^2 - 3r + 4 = 0$$

$$\therefore r = \frac{3 \pm \sqrt{9 - 16}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

$$y = e^{\frac{3}{2}x}(A\cos(\frac{\sqrt{7}}{2}) + B\sin(\frac{\sqrt{7}}{2})x)$$

Initial value problems(for 2nd order DE):

$$y(x_0) = y_0$$

$$y'(x_0 = y_1)$$

Always have a solution in this class! **Boundary value problems (for 2nd order DE:)**

$$y(x_0) = y_0 \text{ and } y(x_1 = y_1)$$

or

$$y'(x_0) = y_0 \text{ and } y'(x_1) = y_1$$

Don't always have a solution!

Example:

$$y'' + y' - 2y = 0$$

$$y(0) = 2 \quad y'(0) = 5$$

$$\rightarrow \text{Aux Eqn:} \quad r^2 + r - 2 = 0 \rightarrow \quad r_1 = 2 \quad r_2 = -2$$

$$\therefore \text{General Solution: } y = c_1 e^x + c_2 e^{-2x}$$

$$y' = c_1 e^x - 2c_2 e^{-2x}$$

$$y(0) = 2 \quad y(0) = c_1 + c_2 = 2$$

$$y'(0) = 5 \quad y'(0) = c_1 - 2c_2 = 5$$

From these we ge that:

$$c_1 = 3 \quad c_2 = -2$$

$$\therefore y = 3e^x - e^{-2x}$$

Example

$$y'' + 4y' + 5y = 0$$

$$y(0) = 1$$

$$y(\frac{\pi}{2}) = 0$$

$$\rightarrow \text{Aux Eqn: } r^2 + 4r + 5 = 0 \rightarrow \quad r = -2 \pm i$$

General solution:

$$y = e^{-2x}(A\cos(x) + B\sin(x))$$

$$y(0) = 1(A \cdot 1 + b \cdot 0) = A = 1$$

$$y(\frac{\pi}{2}) = e^{-\pi}(A \cdot 0 + B \cdot 1) = Be^{-\pi} = 0 \rightarrow \quad B = 0$$

Final, specific solution:

$$\therefore y = e^{-2x}\cos(x)$$

Let's say:

$$y(0) = 1$$

$$y(\pi) = 0$$

First eqn the same, but the other one:

$$y(\pi) = e^{-2\pi}(A(-1) + B \cdot 0) = -Ae^{-2\pi} \rightarrow \quad A = 0$$

4 17.2 Homogeneous Linear Equations

$$y'' + ay' + by = \phi(x)$$

$$y'' + ay' + by = 0 \quad \rightarrow \quad \text{complementary equation}$$

Theorem: The general solution to a non-homogeneous 2nd order linear differential equation with constant coefficients is given by:

$$y(x) = y_p(x) + y_c(x)$$

where:

$$y_p(x)$$

is a particular solution of the complete differential equation, and

$$y_c(x)$$

is the general solution of the complementary y equation.
Proof: no

5 The Method of Undetermined Coefficients

Example:

$$y'' - 6y' + 8y = x^2 + 2x$$

$$\rightarrow \text{ Aux Eqn: } r^2 - 6r + 8 = 0 \quad \rightarrow \quad r_1 = 2 \qquad r_2 = 4$$

$$\therefore y_c = c_1 e^{2x} + c_2^{4x}$$

$$y_p = Ax^2 + Bx + C \rightarrow \quad y_p' = 2Ax + B \rightarrow \quad y_p'' = 2A$$

Basically plug in and see if we can solve for the coefficients, if we can, then it's a solution, if not, then not. (really sherlock).

$$2A - y(2Ax + B) + 8(Ax^2 + bx + c) = x^2 + 2x$$

Look at coefficients of each order term on both sides:

$$x^2 : 8A = 1$$

$$x : -12A + 8B = 2$$

$$1 : 2A - 6B + 8C = 0$$

"easy" to solve:

$$A = \frac{1}{8}$$

$$B = \frac{7}{16}$$

$$C = \frac{19}{64}$$

$$\therefore y = c_1 e^{2x} + c_2 e^{4x} + \frac{1}{8}x^2 + \frac{7}{16}x + \frac{19}{64}$$

$$\phi(x)e^{3x} \quad s \text{ try } y_p = Ae^{3x}$$

$$\phi(x) = C\cos(kx) \text{ or } C\sin(kx)$$

$$\rightarrow \text{try: } y_p = A\cos(kt) + B\sin(kx)$$

$$\pi(x) = x^2\sin(kx) \rightarrow \qquad y_p(Ax^2 + Bx + c)\sin(kx) + (Dx^2 + Ex + F)\cos(kx)$$