Stochastic Simulation (MIE1613H) - Homework 4

Due: April 10th, 2022

- Submit your homework on Quercus as a single PDF file by the deadline. Late submissions are penalized 10% each day the homework is late.
- At the top of your homework include your name, student number, department, and program.
- You may discuss the assignment with other students in general terms, but each student must solve the problems, write the code and the solutions individually.
- The simulation models must be programmed in Python. You must include both the source code (including comments to make it easy to follow) and the output of the simulation in your submission.
- **It is very important to explain your answers and solution approach clearly and not just provide the code/results**. Full marks will only be given to correct solutions that are fully and clearly explained.

Problem 1. A company would like to decide how many items of a certain product it should keep in inventory for each of the next T months (T is a fixed input parameter). The times between demands are IID exponential random variables with a mean of 0.1 month. The sizes of the demands, D, are IID random variables (independent of when the demands occur), where

$$D = \begin{cases} 1 & \text{w.p.}1/6 \\ 2 & \text{w.p.}1/3 \\ 3 & \text{w.p.}1/3 \\ 4 & \text{w.p.}1/6 \end{cases}$$

where w.p.is short for "with probability".

At the beginning of each month, the company reviews the inventory level and decides how many items to order from its supplier. If the company orders Z items, it incurs a cost of $K + iZ$, where $K = \$32$ is the setup cost and $i = \$3$ is the incremental cost per item ordered. (If $Z = 0$, no cost is incurred.) When an order is placed, the time required for it to arrive (called the delivery lag or lead time) is a random variable that is distributed according to a 2-stage Erlang distribution with mean 0.75 month.

The company uses a (s, S) policy to decide how many to order, i.e.,

$$Z = \begin{cases} S - I, & \text{if } I < s, \\ 0 & \text{if } I \geq s, \end{cases}$$

where I is the inventory level at the beginning of the month.

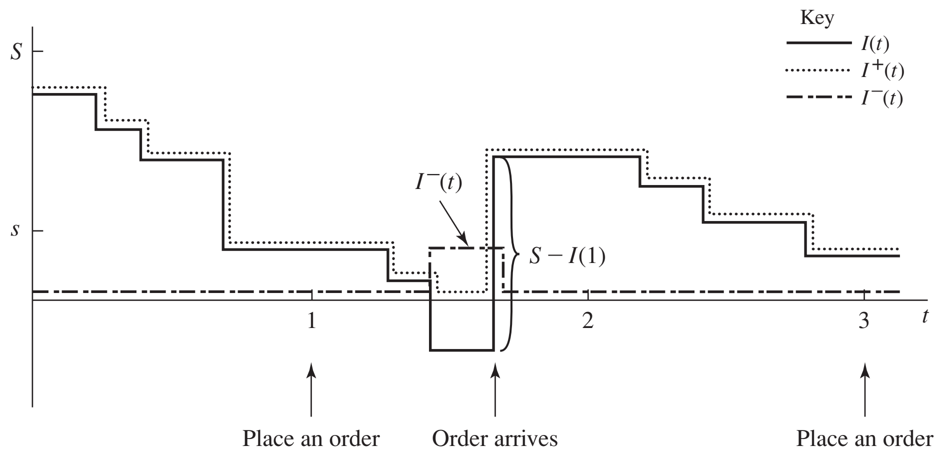

Figure 1: A sample path of $I(t), I^{+}(t)$, and $I^{-}(t)$.

When a demand occurs, it is satisfied immediately if the inventory level is at least as large as the demand. If the demand exceeds the inventory level, the excess of demand over supply is backlogged and satisfied by future deliveries. (In this case, the new inventory level is equal to the old inventory level minus the demand size, resulting in a negative inventory level.) When an order arrives, it is first used to eliminate as much of the backlog (if any) as possible; the remainder of the order (if any) is added to the inventory.

Let $I(t)$ be the inventory level at time t (note that $I(t)$ could be positive, negative, or zero); let $I^{+}(t) = \max\{I(t), 0\}$ be the number of items physically on hand in the inventory at time t (note that $I^{+}(t) \geq 0$) and let $I^{-}(t) = \max\{-I(t), 0\}$ be the backlog at time t (note that $I^{-}(t) \geq 0$). A sample path of $I(t), I^{+}(t)$ and $I^{-}(t)$ is show in Figure 1.

Assume that the company incurs a holding cost of $h = \$1$ per item per month held in (positive) inventory. The holding cost includes such costs as warehouse rental, insurance, taxes, and maintenance, as well as the opportunity cost of having capital tied up in inventory rather than invested elsewhere. Since $I^{+}(t)$ is the number of items held in inventory at time t, the time-average (per month) number of items held in inventory for the T-month period is

$$\bar{I}^{+} = \frac{\int_0^T I^{+}(t)dt}{T},$$

so the average holding cost per month is $h\bar{I}^{+}$.

Similarly, suppose that the company incurs a backlog cost of $\pi = \$5$ per item per month in backlog; this accounts for the cost of extra record keeping when a backlog exists, as well as loss of customers' goodwill. The time-average number of items in backlog is

$$\bar{I}^{-} = \frac{\int_0^T I^{-}(t)dt}{T},$$

so the average backlog cost per month is $\pi\bar{I}^{-}$.

Assume that the initial inventory level is $I(0) = 60$ and that no order is out-standing. We are interested in comparing the following six inventory policies with respect to the total expected costs over $T = 120$ months (which is the sum of the average ordering cost per month, the average holding cost per month, and the average backlog cost per month):

s	20	20	20	30	30	30
S	50	60	70	60	80	100

(a) (15 Pts.) Develop a simulation model in PythonSim to estimate the expected cost of a given (s, S) policy.

(b) (5 Pts.) Provide an estimate (with a 95% CI) of the expected average cost of an (s, S) policy with $s = 20$ and $S = 50$.

(c) (20 Pts.) Implement the subset selection method to identify a subset of "good" policies among the above 6 policies. Use $\alpha = 0.05$ and report your results with (1) 100 and (2) 500 replications.

Note: Use must use Common Random Numbers (CRN) when comparing the six designs (see Page 246 of the textbook). Note that re-calling the method SimRNG.InitializeRNSeed() in PythonSim does not re-initialize the random number generator. To implement CRN, you must either run the scenarios separately and save the outputs; generate the random numbers at the beginning of the simulation from different streams; or make you own modification to the code to reset the random number generator. This is possible but may require some programming experience.

Problem 2. Consider the continuous version of the SAN simulation optimization problem with $\tau_j = c_j = 1$, $l_j = 0.5$ for $j = 1, 2, 3, 4, 5$, and $b = 1$.

(a) (20 Pts.) Estimate the gradient of $E[Y(\mathbf{x})]$ at $\mathbf{x} = (0.5, 1, 0.7, 1, 1)$ using the Finite Difference (FD) method. Compare you estimate (together with a 95% confidence interval) for $\Delta x = 0.1, 0.05, 0.01$ to illustrate the Bias-Variance tradeoff. **Note:** Remember that you must always use Common Random Numbers (CRN) when estimating the gradient using the FD approach.

(b) (15 Pts.) Estimate the same gradient from part **(a)** using the Infinitesimal Perturbation Analysis (IPA) approach.

(c) (+8 Bonus Pts.) Implement a stochastic approximation search to find the optimal activity mean times. Use $\mathbf{x} = (1, 1, 1, 1, 1)$ as the starting scenario and at least 50 replications for your estimations.

Problem 3. In the single-period Newsvendor problem, a decision maker needs to decide on the order size (denoted by x) to satisfy a random demand (denoted by D) with cdf $F(\cdot)$. For each unit of extra inventory left after satisfying the demand, the decision maker incurs a cost of c_o, and for each unit of lost demand (i.e., inventory less than demand) the decision maker incurs a cost of c_u. The objective is to find the order size that minimizes the expected total cost given by:

$$\theta(x) = E[c_o \ \max(x - D, 0) + c_u \ \max(D - x, 0)].$$

(a) (5 Pts.) Assume that the cdf of the demand D is given by,

$$F(x) = \begin{cases} 1 - (\frac{5}{x})^2, & x \geq 5, \\ 0, & x < 5. \end{cases}$$

Propose an inversion algorithm to generate samples from this distribution.

(b) (10 Pts.) Propose an Infinitesimal Perturbation Analysis (IPA) estimator for $d\theta(x)/dx$.

(c) (10 Pts.) Write down a linear program (LP) formulation of the sample average approximation method using n replications to find the optimal order size x assuming that x can take any real value between 0 and 50. **Hint**: First write down the sample average approximation formulation and then re-formulate it as a linear program.

(d) (+ 8 Bonus Pts.) Solve the LP using a solver (e.g., Gurobi) and compare your solution with the exact solution $F^{-1}(\frac{c_u}{c_o + c_u})$.