Calculating the sum of n squares

October 12, 2017

The sum of squares of the first n integers can be simplified with:

$$1^2 + 2^2 + ... + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

We start from the simple cube of binomium:

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1$$

Moving the terms slightly we get:

$$3k^2 + 3k + 1 = (k+1)^3 - k^3$$

Assigning $k = 1, 2, ...n-1$ the formulas become:

$$3 \cdot 1^2 + 3 \cdot 1 + 1 = 2^3 - 1^3$$

$$3 \cdot 2^2 + 3 \cdot 2 + 1 = 3^3 - 2^3$$

$$3(n-1)^2 + 3(n-1) + 1 = n^3 - (n-1)^3$$

Adding the three formulas and canceling:

$$3[1^2 + 2^2 + ... + (n-1)^2] + 3[1 + 2 + ... + (n-1)] + (n-1) = n^3 - 1^3$$

The second sum on the left is a succession the sum of which is $\frac{n(n-1)}{2}$.

$$3[1^2 + 2^2 + ... + (n-1)^2] + \frac{3}{2}n(n-1) + (n-1) = n^3 - 1^3$$

We can cancel -1 on both sides:

$$3[1^2 + 2^2 + ... + (n-1)^2] + \frac{3}{2}n(n-1) + n = n^3$$

Moving the terms to the right:

$$3[1^2 + 2^2 + ... + (n-1)^2] = n^3 - \frac{3}{2}n^2 + \frac{3}{2}n - n$$

Finally:

$$1^2 + 2^2 + ... + (n-1)^2 = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$$

Adding n on both sides:

$$1^2 + 2^2 + ... + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$