Report Project 2
Stabilization of a double pendulum on a cart

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1 Introduction

The goal of this project is to design a static controller and a dynamic controller to stabilize an upright double pendulum standing on a gliding cart, see figure 1. One type of the controller is a static feedback controller with feedback directly from the internal states of the system to the input. The other controller is based on an observer, which dynamically reconstructs the internal states from the system's output.
Physical constraints for the pendulums are such that all the movements take place in a vertical plane.
The system is highly unstable and nonlinear.
The designing steps to solve the problem follows the sequence of the headlines below.

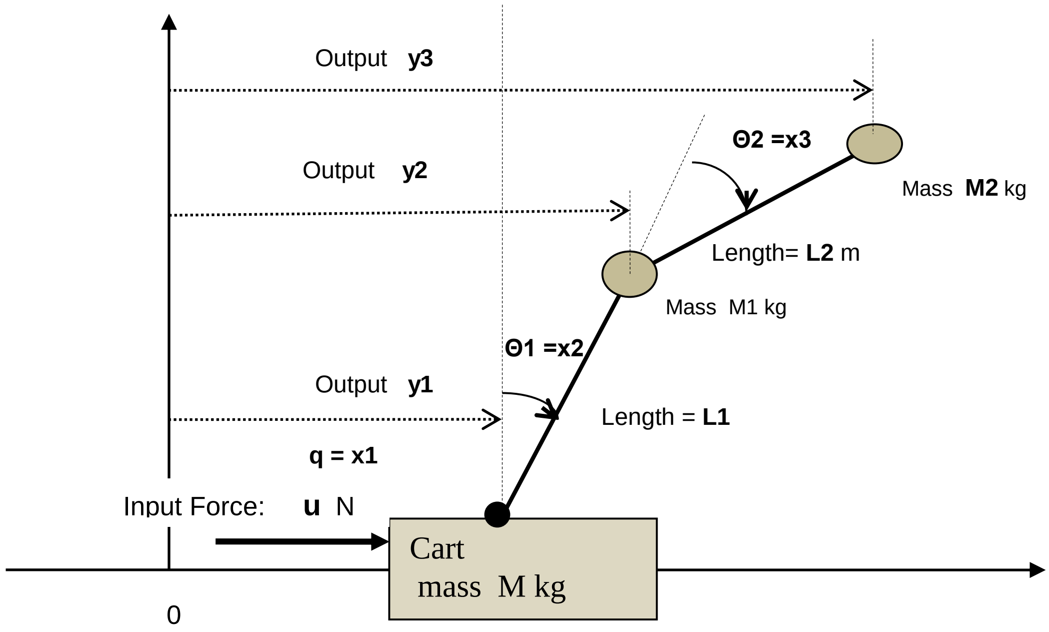

*Figure 1 shows the pendulums which are hinged upon the gliding Cart. The generalized coordinates **q, θ1, θ2** are also and the state variables **x1, x2 , x3.***

2 Modelling

We use the Lagrangian **L=K-P** , where P is the system's Potential energy and K is the Kinetic energy in the generalized coordinates ***q, θ1, θ2*** and input **u,** as shown below.

$$\begin{aligned} K(q,\theta_1,\theta_2,\dot{q},\dot{\theta}_1,\dot{\theta}_2) &= \frac{1}{2}M\dot{q}^2 + \frac{1}{2}M_1[(\dot{q}+L_1\dot{\theta}_1\cos\theta_1)^2 + (L_1\dot{\theta}_1\sin\theta_1)^2] \\ &+ \frac{1}{2}M_2[(\dot{q}+L_1\dot{\theta}_1\cos\theta_1 + L_2\dot{\theta}_2\cos(\theta_1+\theta_2))^2 \\ &+ (L_1\dot{\theta}_1\sin\theta_1 + L_2\dot{\theta}_2\sin(\theta_1+\theta_2))^2] \end{aligned}$$

$$P(q,\theta_1,\theta_2,\dot{q},\dot{\theta}_1,\dot{\theta}_2) = M_1 g L_1\cos\theta_1 + M_2 g[L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2)].$$

We set state variables

$$x_1 = q,\ x_2 = \theta_1,\ x_3 = \theta_2,\ x_4 = \dot{q}, x_5 = \dot{\theta}_1,\ x_6 = \dot{\theta}_2.$$

We get the equations of motion of the system in the form of:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q,\theta_1,\theta_2,\dot{q},\dot{\theta}_1,\dot{\theta}_2) - \frac{\partial L}{\partial q}q,\theta_1,\theta_2,\dot{q},\dot{\theta}_1,\dot{\theta}_2) = u$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1}(q,\theta_1,\theta_2,\dot{q},\dot{\theta}_1,\dot{\theta}_2) - \frac{\partial L}{\partial \theta_1}((q,\theta_1,\theta_2,\dot{q},\dot{\theta}_1,\dot{\theta}_2) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2}(q,\theta_1,\theta_2,\dot{q},\dot{\theta}_1,\dot{\theta}_2) - \frac{\partial L}{\partial \theta_2}(q,\theta_1,\theta_2,\dot{q},\dot{\theta}_1,\dot{\theta}_2) = 0$$

Which simplifies to:

$$-(L_1M_1+L_1M_2)\left(\frac{d\theta_1}{dt}\right)^2\sin\theta_1 - L_2M_2\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\sin(\theta_1+\theta_2) - L_2M_2\left(\frac{d\theta_2}{dt}\right)^2\sin(\theta_1+\theta_2)$$

$$+(M+M_1+M_2)\frac{d^2q}{dt^2} + (L_1M_1+L_1M_2)\frac{d^2\theta_1}{dt^2}\cos\theta_1 + L_2M_2\frac{d^2\theta_2}{dt^2}\cos(\theta_1+\theta_2) = u$$

$$-gL_1(M_1+M_2)\sin\theta_1 - gL_2M_2\sin(\theta_1+\theta_2) + L_2M_2\frac{dq}{dt}\frac{d\theta_2}{dt}\sin(\theta_1+\theta_2) - L_1L_2M_2\left(\frac{d\theta_2}{dt}\right)^2\sin\theta_2$$

$$+(L_1M_1+L_1M_2)\frac{d^2q}{dt^2}\cos\theta_1 + L_1^2(M_1+M_2)\frac{d^2\theta_1}{dt^2} + L_1L_2M_2\frac{d^2\theta_2}{dt^2}\cos\theta_2 = 0$$

$$-gL_2M_2\sin(\theta_1+\theta_2) - L_2M_2\frac{dq}{dt}\frac{d\theta_1}{dt}\sin(\theta_1+\theta_2) + L_2M_2\frac{d^2q}{dt^2}\cos(\theta_1+\theta_2)$$

$$+L_1L_2M_2\frac{d^2\theta_1}{dt^2}\cos\theta_2 + L_2^2M_2\frac{d^2\theta_2}{dt^2} = 0$$

After solving for the second derivatives d(x4)/dt, d(x5)/dt and d(x6)/dt using Mathematica, we obtain the last three components of the symbolic equations f(x,u,t), see Appendix 1.

The output y is found from state variables by simple geometric calculation and can be written in the standard form: y = h(x) which when written component wise gives

y1 = x1
y2 = x1 + L1 sin(x2)
y3 = x1 +L1 sin(x2) + L2 sin(x2+x3)

3 Linearization

To design a linear feedback, we linearize the system dx/dt= f(x,u,t) about its equilibrium points.

To find the equilibrium points x_0 we use Mathematica to solve dx/dt= f(x,u,t) = 0
The equilibrium points we obtained are:

$$x(0) = (q , 0, 0, 0, 0, 0)$$
$$x(0) = (q , \pi, 0, 0, 0, 0)$$
$$x(0) = (q , 0, \pi, 0, 0, 0)$$
$$x(0) = (q , \pi, \pi, 0, 0, 0)$$

We note that we can choose the state variable x1=q arbitrary.
The state variables θ1 =x2 and θ2=x3 can only be 0 or π and the physical interpretation for that is that the hanging modes of the pendulums can vary. They can stand upright or hang down.

Linear Standard Form

We want the linearized system to be in standard form:

$$d\Delta x/dt = A\Delta x + B\Delta u \ldots\ldots\ldots\ldots\ldots \text{ equ 1}$$
$$\Delta y = C\Delta x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ equ 2}$$

For convenience we from now onwards use x instead of Δx, y instead of Δy and u instead of Δu keeping in mind that the important equilibrium point is chosen to be **x**=0.

The matrices A, B and C in equ 1 and equ 2 are obtained by using the Taylor expansion about the equilibrium point, and dropping the higher order terms. That means the matrices A, B and C are Jacobian matrices evaluated at the equilibrium point.

We now have:

$$dx/dt = Ax + Bu\ldots\ldots\ldots\ldots\ldots\ldots\text{equ 3}$$
$$y= Cx\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{equ 4}$$

Where

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{g(L_1(M_1+M_2)+L_2M_2)}{L_1M} & -\frac{gL_2M_2}{L_1M} & 0 & 0 & 0 \\ 0 & \frac{g(L_1M_1(M+M_1+M_2)+L_2M_2(M+M_1))}{L_1^2MM_1} & \frac{gM_2(-L_1M+L_2(M+M_1))}{L_1^2MM_1} & 0 & 0 & 0 \\ 0 & -\frac{gM_2}{L_1M_1} & \frac{g(L_1(M_1+M_2)-L_2M_2}{L_1L_2M_1} & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{L_1M} \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & L_1 & 0 & 0 & 0 & 0 \\ 1 & L_1+L_2 & L_2 & 0 & 0 & 0 \end{pmatrix}$$

4 Analysis of the open system

Stability issues, eigenvalues of A

We limit our analysis to the equilibrium point 0, that is, when the pendulums are standing still in the upright position. To analyse the system stability we look at the eigenvalues of the matrix A. By using Mathematica we found out that A has six eigenvalues ($\lambda 1$, $\lambda 2$, $\lambda 3$, $\lambda 4$, $\lambda 5$, $\lambda 6$) see Appendix 2. We note that in the output from Mathematica some of the eigenvalues will always have positive real part, thus by Lyaponov Theorem, the nonlinear system and linear system are unstable. This can also be deduced from the fact that there are two pairs of eigenvalues where one has to be a negative of the other so at least two of the eigenvalues must always have a positive real part because none of the four eigenvalues can be zero by roof finding in Maathematica.

Controllability issues

Full controllability is important for the possibility to design a controller for the system. To check controllability we look at the Reachability matrix R = (B, AB, …, A^5B) and see if it has full row rank=6.

Mathematica rank function says that rank of R is 6. Therefore the system is controllable.

Observability issues

The full observability is important for implementation an observer. To check the observability we look at the observability matrix O=(C, CA, … $CA^5)^T$ and see if it has full column rank = 6. Mathematica says that the rank is 6, so an observer can be designed.

Henceforth we assume that the system parameters are fixed

The parameters are set to: M= 100 kg, M1=10 kg , M2=10 kg , L1=2 m , L2=1 m , g=10 m/sec^2

Those parameters is plugged into Equ 3 and Equ 4 gives (via Matlab):

$$A=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2.5 & -0.5 & 0 & 0 & 0 \\ 0 & 8.75 & -2.25 & 0 & 0 & 0 \\ 0 & -5 & 15 & 0 & 0 & 0 \end{bmatrix}$$

$$B = [0 \ \ 0 \ \ 0 \ \ 0.1 \ \ -0.005 \ \ 0]^T$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$D= 0$$

Eigenvalues of A is (via Matlab)

Eig(A) = [0 0 4.0570 -4.0570 2.7001 -2.7001]

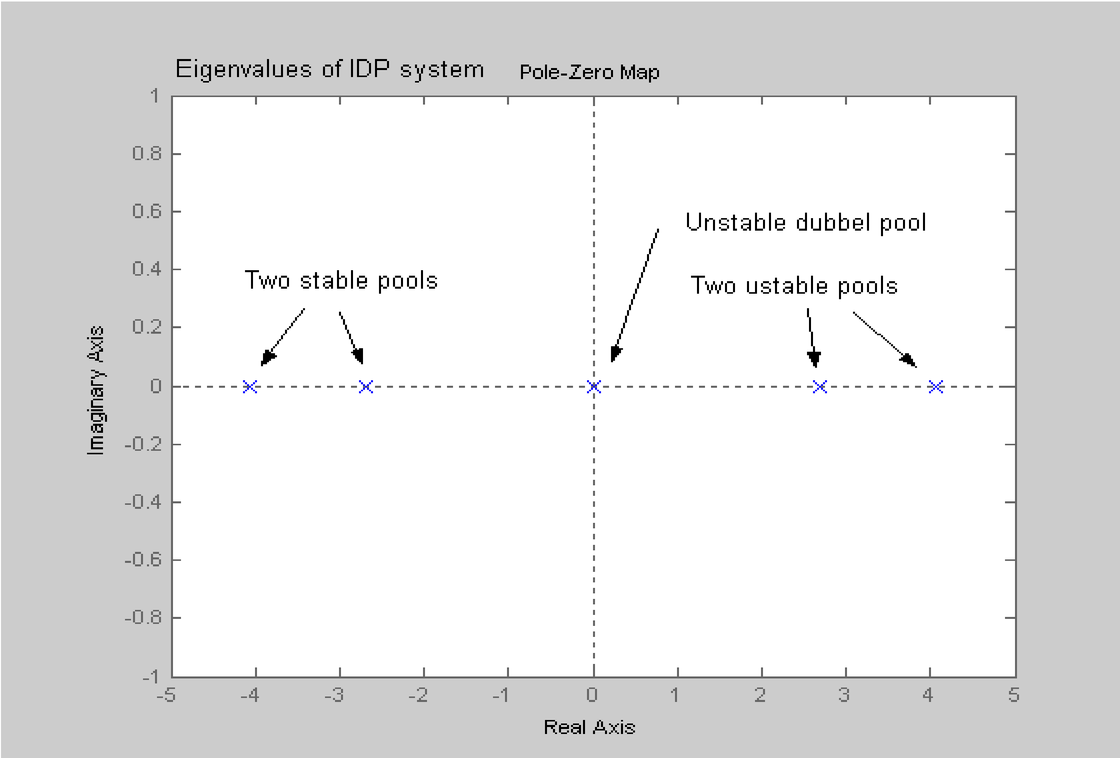

Figure 2 shows the plot in the complex plane of the eigenvalues of the open Pendulum system.
The plot that the system has poles with a real parts greater than 0. That means that the system is unstable.
The double poles in zero also implies that the system is unstable.

Bode diagrams

To plot the three Bode diagrams we need the three transfer functions GS1(s), GS2(2), GS3(s) , defined as:
Y1(s) = GS1(s) • U(s) //All function are Laplace Transformed
Y2(s) = GS2(s) • U(s) //All function are Laplace Transformed
Y3(s) = GS3(s) • U(s) //All function are Laplace Transformed

The transfer functions:
Matlab command **tf** (A,B,C,D) gives the three transfer functions:

Transfer function from input u to output y1

$$GS1(s): \frac{0.01\ s^4 - 4.441e\text{-}017\ s^3 - 0.225\ s^2 + 6.395e\text{-}016\ s + 1}{s^6 + 1.332e\text{-}015\ s^5 - 23.75\ s^4 - 1.421e\text{-}014\ s^3 + 120\ s^2}$$

Transfer function from input u to output y2

$$GS2(s): \frac{-0.075\ s^2 + 6.661e\text{-}017\ s + 1}{s^6 + 1.332e\text{-}015\ s^5 - 23.75\ s^4 - 1.421e\text{-}014\ s^3 + 120\ s^2}$$

Transfer function from input u to output y3

$$GS3(s): \frac{-0.005\ s^4 - 4.441e\text{-}018\ s^3 + 0.025\ s^2 + 2.842e\text{-}016\ s + 1}{s^6 + 1.332e\text{-}015\ s^5 - 23.75\ s^4 - 1.421e\text{-}014\ s^3 + 120\ s^2}$$

The Plot of Bode diagrams

Matlab command **bode**(GS1(s)) , **bode**(GS2(s)) and **bode**(GS3(s)) gives the Bode diagrams:

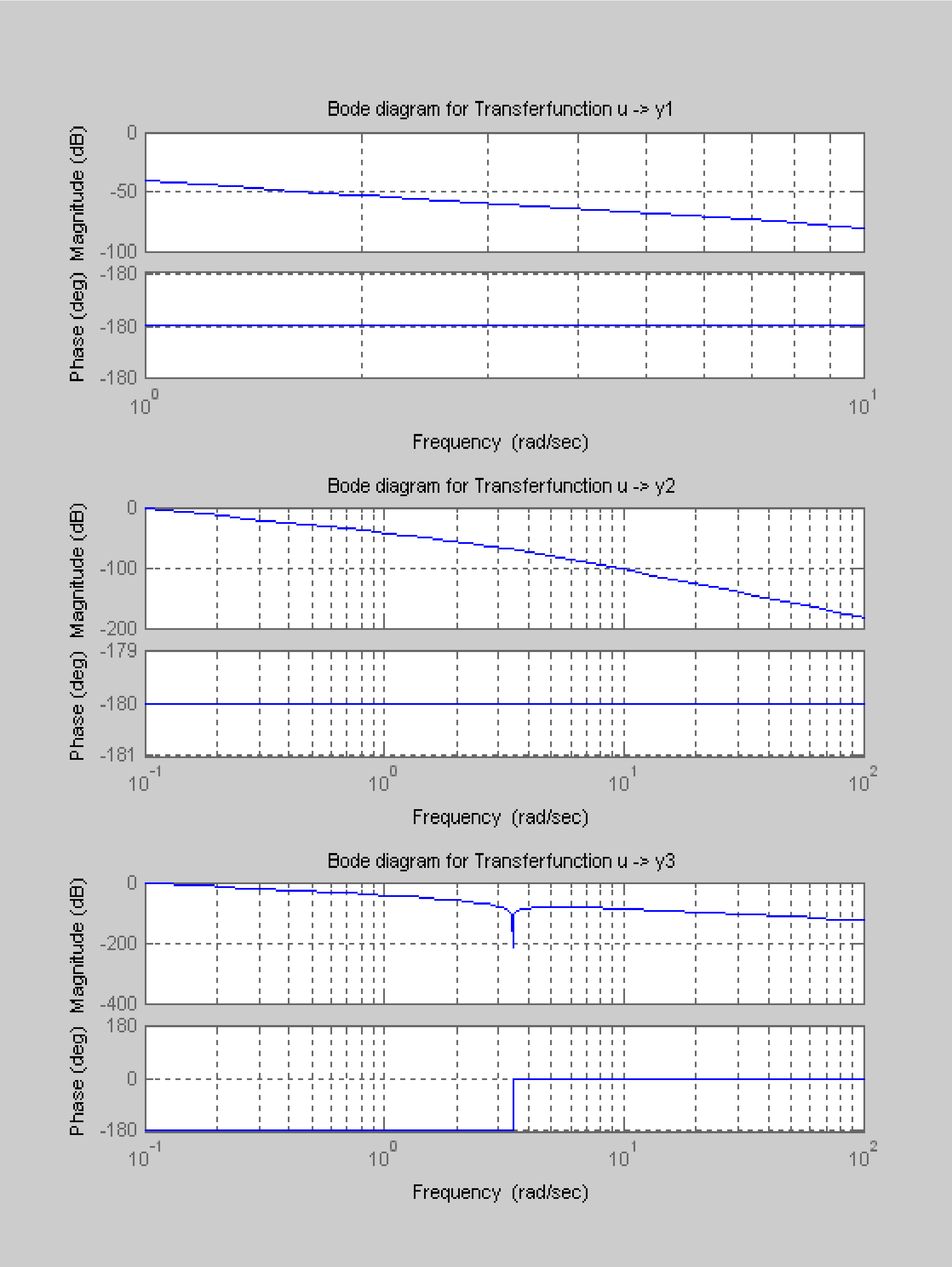

Figure 3 shows the bode diagrams of the open Pendulum system

Short analysis of the Bode diagrams

Bode diagrams shows how the Gain (Margin) and Phase from input to output changes according to input frequency.

The Bode diagrams of the transfer functions GS1(s) and GS2(s) are good looking for control as the Magnitude function is less than one at any Phase angle.

For GS3(s) we see an interesting resonance phenomena. At an angle frequency ω of about 3.4 for the input u there are sudden big dip to -200dB in Amplitude of y3 and sudden raise of 180^0 in the Phase. That means with an input of 3.4 Hz, the top of the upper pendulum hardly moves as much as the cart does, an even worse the pendulum moves in the opposite direction as the cart for frequency below 3.4 Hz and in the same direction for higher frequencies than 3.4 Hz. This fact makes it impossible to control the system at this frequency.

5 Simulation of linearized closed system

Stabilization

The linearized system A B C is unstable and to control it we first must make system stable. This is done by a constant feedback from the states x to the input u. After feedback we have a closed system which we can represent in the block diagram in figure 4.

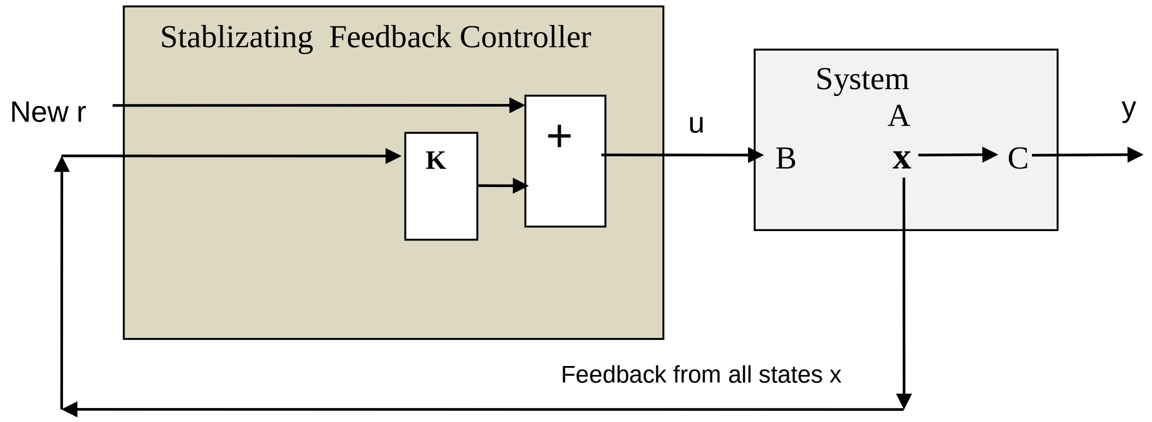

*Figure 4 shows the block diagram of the System with a static stabilization feedback Controller **u= Kx** using the state variables **x,** which we assume available.*

Pole placement

The theory says that if the system is controllable it is always possible to assign its poles in an arbitrary position in the complex plane with a linear static feedback **u=Kx** from the state **x**.
We choose some places, all in the stable left half plane of the complex plane. The closed loop system should now be stablilized. The new chosen λ's are [-1.0, -1.1, -1.2, -1.3. -1.4, -1.5]

The gain matrix K for the pole placement is found in Matlab with the command
K= **- place**(A,B,C,D,[-1.0, -1.1, -1.2, -1.3. -1.4, -1.5])

As a check we compute the eigenvalues of the matrix A+BK of the new system to see if eigenvalues are those we have chosen. Matlab gives **eig**(A+BK) = [-1.0, -1.1, -1.2, -1.3. -1.4, -1.5] , so yes they are the ones we have chosen.

Simulation of the closed loop system

The new system A+BK, B, C should now be stable. To see how it behaves we simulate the zero-input response with Initial Condition (I.C.), which is enough to describe the system's behavior. The chosen I.C. for the state variables are [-5,0,0,0,0,0] which means that the system at start stays -5m from origin with the pendulum both pointing upright in the vertical position.
The simulation is done with the Matlab lsim-command.
[y, x] = lsim((A+BK), B, C, D, [-5,0,0,0,0,0])

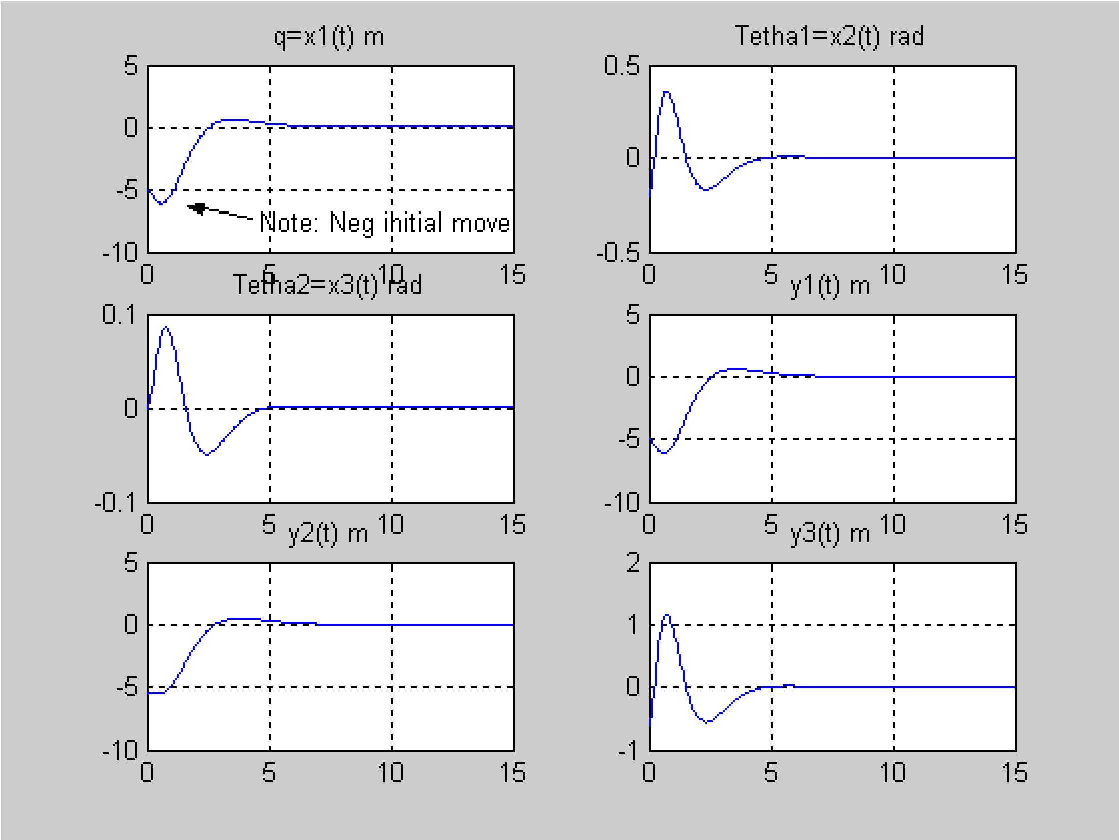

Figure 5 shows the stabilized system response to initial condition. As seen by the figures the system is stable and the final state is the equilibrium point 0.

6 Obtain a dynamic state observer

When an observer is included in the Controller we get the block diagram in figure 6. The original System is not changed

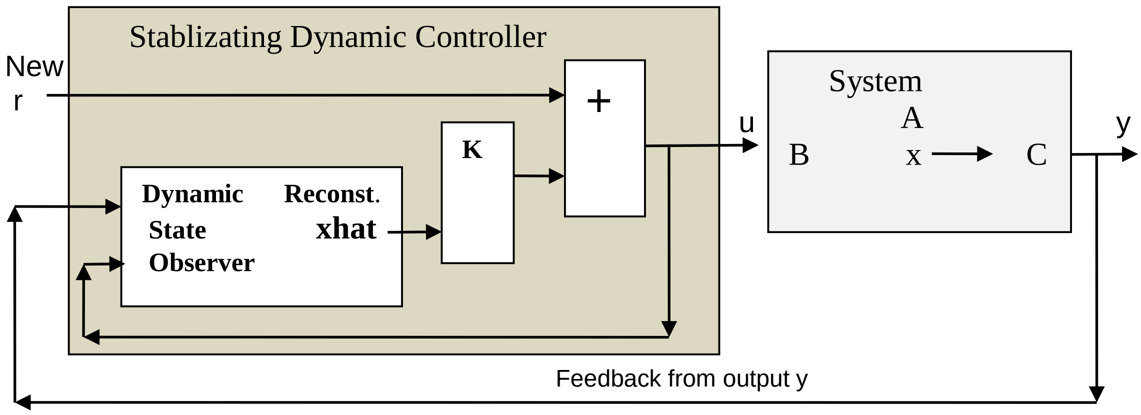

Figure 6 shows block diagram of a System and a Controller with a Dynamic State Observer which reconstructs
*the state variables **x** from system output **y.** The reconstructed values of **x** are called **xhat** which is used in the static stabilization feedback **u= Kxhat.** (before u=K x)*

The dynamic Observer

According to theory of observers, the observer can be implemented as a dynamic system with almost the same dynamics as the original system driven by both **u** and **y**. The real part of the eigenvalues of the observer should however be placed more negative than the system (A+BK, B, C). To succeed in designing the observer, then the observed system should be observable, and this was discussed above in the section observable issues.

The observer has the equation:

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

According to the separation principle the Observer and the dynamical feedback can be designed and also used independent of each other so we now work with the observer.

The observer system matrices

We implement the observer as a time invariant dynamic system which shall observe the already stabilized system (A+BK, B, C).

The observer itself is according theory a system which is pole placed version of (A+BK, B, C) with help of L.

Choosing L in the observer

We have placed the eigenvalues of that system to [-1.0, -1.1, -1.2, -1.3. -1.4, -1.5].
We choose the eigenvalues more negative for the observer: [-4.1,-4.2,-4.3,-4.4,-4.5,-4.6];
The pair (A, C) is observable, which according to duality principle means that the pair (A', C') is controllable.
We can use Matlab command **place** to find L as follows.
L = place(A',B,C',D, [-4.1,-4.2,-4.3,-4.4,-4.5,-4.6])'
K= **-** L'

The Observer gain L becomes:
L =

-66.2094	198.8223	-126.1258
32.8966	-94.4117	62.6755
4.1484	-12.4472	8.3100
-86.3663	239.8836	-125.6542
37.2668	-100.1332	58.6206
18.5712	-50.7230	32.200

Simulation of the observer

To test the observer we use output data from a simulation of system (A+BK, B, C). Note that there is no external input, when we simulate the system (A+BK, B, C). The input of the observer is the output from a simulation of the initial condition response of system (A+BK, B, C).
The observers reconstructed values of x is Xhat. The corresponding output for Xhat is Yhat.
The errors **e** in the reconstructed values of **x** is difference of the known **x** and the reconstructed **Xhat**.
We show two simulation of the observer, done in Matlab with the command **lsim.**
The plots of two simulations are shown below. Our comment on the simulation in the text under the plots.

Simulation 1 of the Observer when the initial conditions of the observer is the same as the System A, B, C, D.

The initial conditions for the system (A+BK, B, C) is [-5,0,0,0,0,0] and observer's initial condition is [-5, 0, 0, 0, 0, 0].

Plot of simulation:

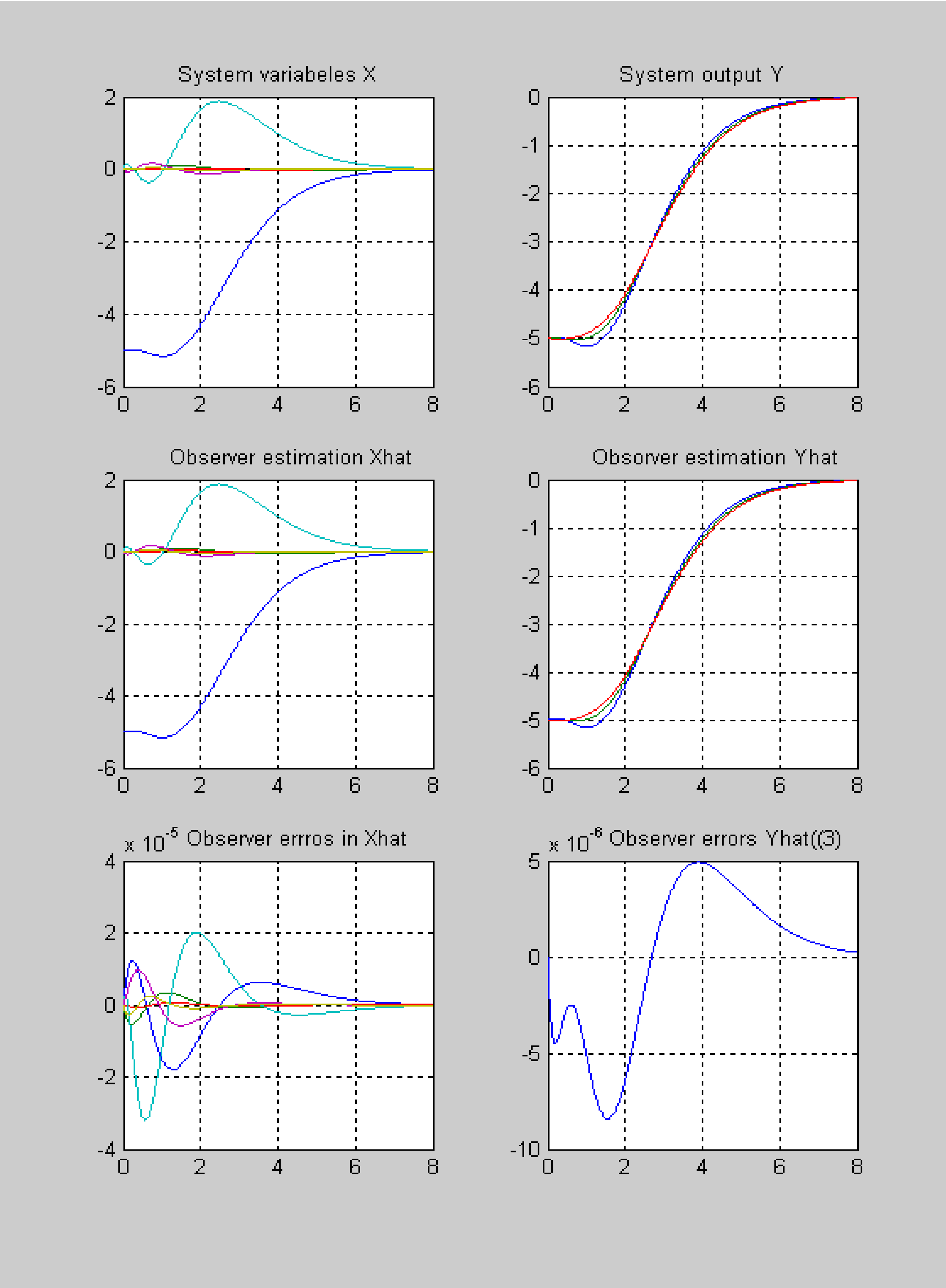

Figure 7 shows the observer output when the initial conditions of the observer are the same as the observed System.We can see that the observer output is almost exactly as the system.

Simulation 2 of the Observer when the initial conditions of the observer differs from the System A, B, C, D.

We now choose the initial condition of the observer to be [-4, 0.05, -0.05, 0 , 0, 0] which differs from the initial conditions of the observed System's value [-5, 0 ,0 ,0, 0, 0]

Simulation plot:

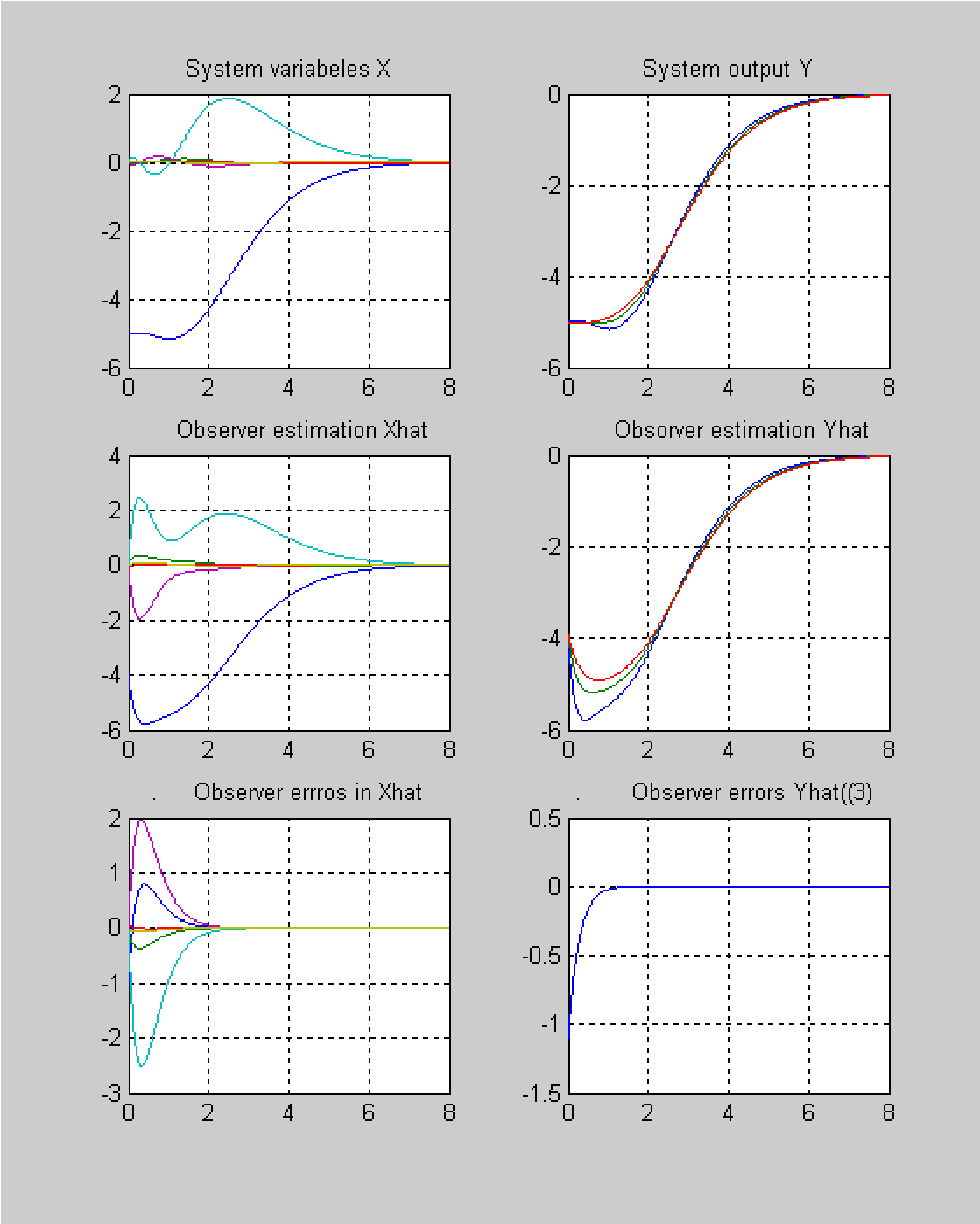

Figure 8 shows the observer output when the initial conditions of the observer differs from the observed System. We can see that the observer output has transients that converge to zero after a few seconds. This is according to the chosen negative eigenvalues of the observer, which are about -4.

7 Combine the state feedback and the dynamic observer

The combined system when we include the observer in the feedback chain becomes a "double sized system" of dimension 12, where the new system contains a system of dimension 6 and an observer of dimension 6.

The combined system has the same block diagram shown in figure 9.
Note that the Pendulum System in the figure 9 still represents the original unstable open system.

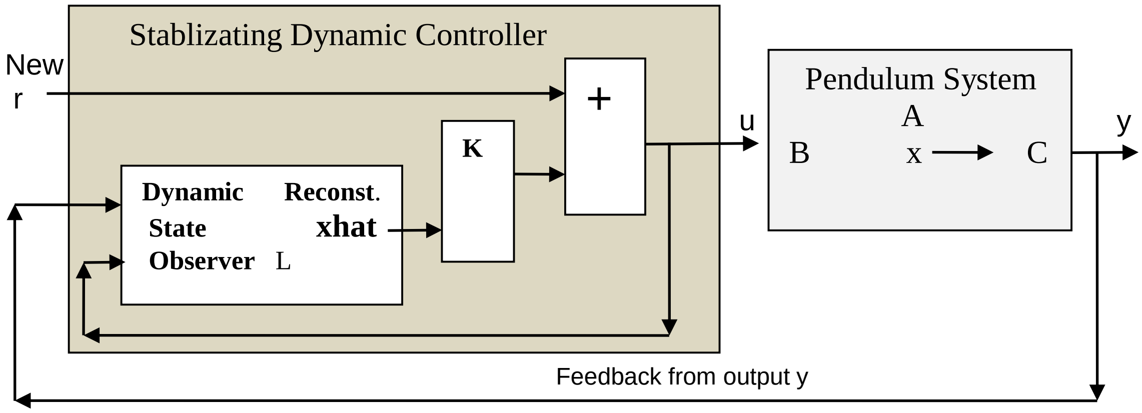

Figure 9 shows block diagram of the combined System when the Dynamic State Observer is included in the feedback loop.

The combined system equations with no external inputs is:

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Simulation of combined system
The simulation of the above equations is done in Matlab with the command **lsim.**
We show the plot of two simulations and comment the simulation result in the text under the figures.

Simulation 1. The initial condition of pendulum system is [-5,0,0,0,0,0], the observer initial conditions is [0,0,0,0,0,0]

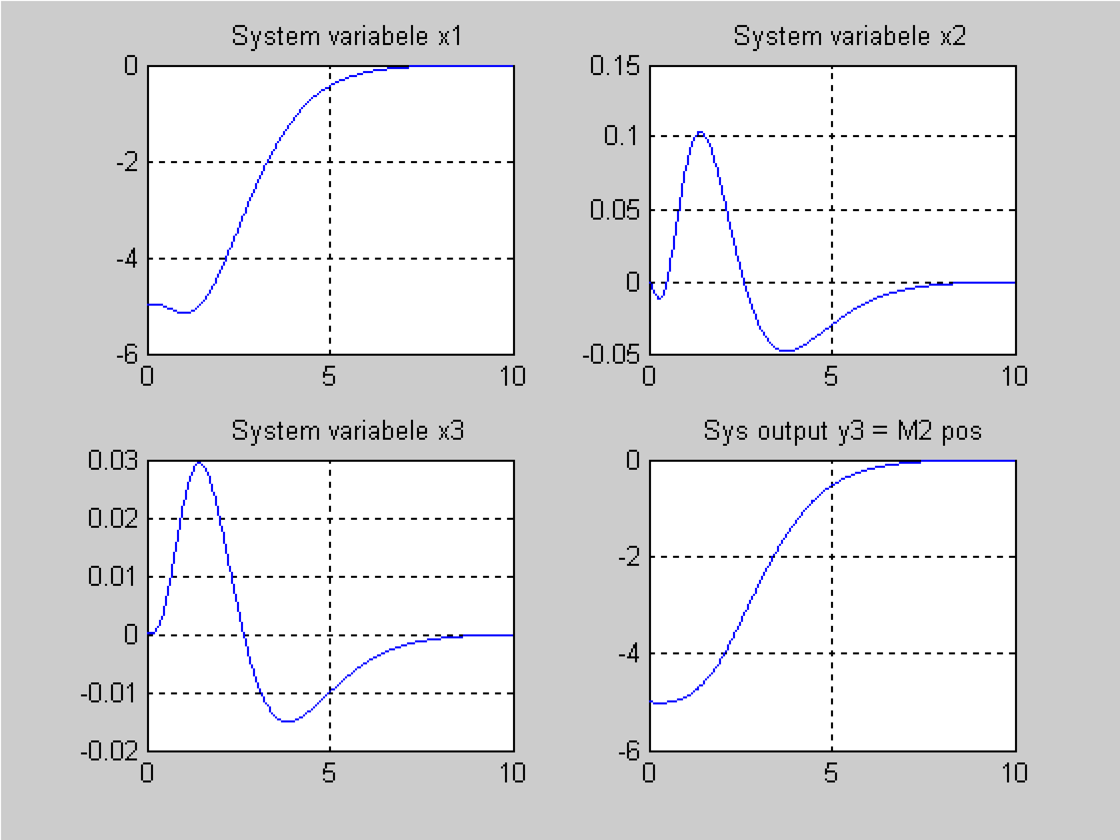

Figure 10 shows the simulation plots for the combined system with observer output when the initial conditions of the system is [-5,0,0,0,0,0] and the observer initial conditions is [0,0,0,0,0,0].

We can see that the Pendulums (y3) nicely move to the equilibrium point 0, though the observer has a different initial conditions then the Pendulum System. The simulation shows that the combined system is stable.

Simulation 2. The initial condition of pendulum system is [-5, 0, 0, 0, 0, 0], the observer initial condition is [-5, 0, 0, 0, 0, 0]

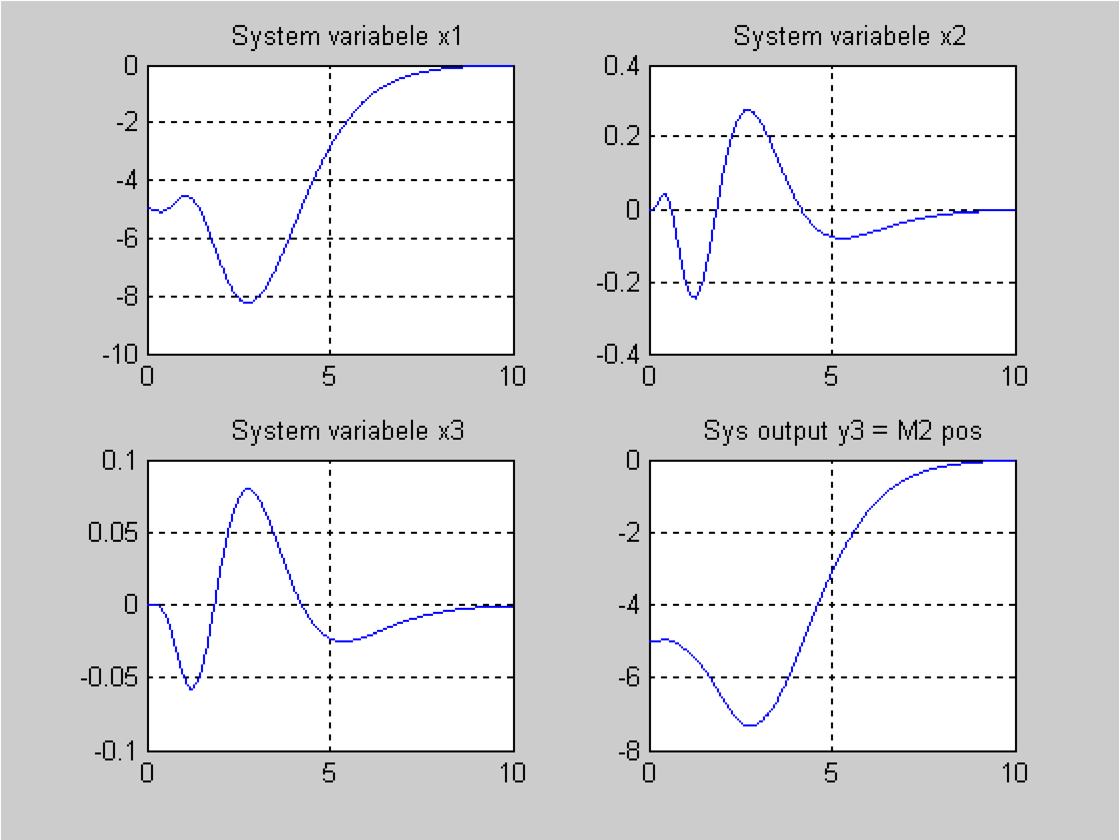

Figure 11 shows the simulation plots for the combined system with observer output when the initial conditions of the system is [-5,0,0,0,0,0] and the observer initial condition is the same [-5,0,0,0,0,0].
We can see that the Pendulums (y3) nicely move to the equilibrium point 0 when the observer has the same initial condition as the Pendulum System. However the movement is more oscillatory. The simulation shows that the combined system is stable.

8 Appendix 1

```
Clear[eqa1, eqa2, eqa3, leqa1, leqa2, leqa3, M, M1, M2, L1, L2, g, x1, x2, x3, x4, x5, x6, u, dx4, dx5, dx6]
eqa1 = -L1 M1 Sin[x2] x5^2 - L1 M2 Sin[x2] x5^2 - L2 M2 Sin[x2 + x3] x5 x6 - L2 M2 Sin[x2 + x3] x6^2 + (M + M1 + M2) dx4 + L1 (M1 + M2) Cos[x2] dx5 + L2 M2 Cos[x2 + x3] dx6 == u
eqa2 = -g L1 (M1 + M2) Sin[x2] - g L2 M2 Sin[x2 + x3] + L2 M2 Sin[x2 + x3] x4 x6 - L1 L2 M2 Sin[x3] x6^2 + L1 (M1 + M2) Cos[x2] dx4 + L1^2 (M1 + M2) dx5 + L1 L2 M2 Cos[x3] dx6 == 0
eqa3 = L2 M2 (Cos[x2 + x3] dx4 + L1 Cos[x3] dx5 + L2 dx6 - g Sin[x2 + x3] - Sin[x2 + x3] x4 x5) == 0

dx4 (M + M1 + M2) + dx5 L1 (M1 + M2) Cos[x2] + dx6 L2 M2 Cos[x2 + x3] - L1 M1 x5^2 Sin[x2] - L1 M2 x5^2 Sin[x2] - L2 M2 x5 x6 Sin[x2 + x3] - L2 M2 x6^2 Sin[x2 + x3] == u

dx5 L1^2 (M1 + M2) + dx4 L1 (M1 + M2) Cos[x2] + dx6 L1 L2 M2 Cos[x3] - g L1 (M1 + M2) Sin[x2] - L1 L2 M2 x6^2 Sin[x3] - g L2 M2 Sin[x2 + x3] + L2 M2 x4 x6 Sin[x2 + x3] == 0

L2 M2 (dx6 L2 + dx5 L1 Cos[x3] + dx4 Cos[x2 + x3] - g Sin[x2 + x3] - x4 x5 Sin[x2 + x3]) == 0

dxEqa = Solve[{eqa1, eqa2, eqa3}, {dx4, dx5, dx6}] // Simplify;
dxEqa // Transpose

{{dx4 -> -((L1 (-2 M1 - M2 + M2 Cos[2 x3]) (2 u + 2 L1 (M1 + M2) x5^2 Sin[x2] + 2 L2 M2 x6 (x5 + x6) Sin[x2 + x3] - g M2 Sin[2 (x2 + x3)] - M2 x4 x5 Sin[2 (x2 + x3)]) +
      (-(2 M1 + M2) Cos[x2] + M2 Cos[x2 + 2 x3]) (-L1 (g (2 M1 + M2) - M2 x4 x5) Sin[x2] + M2
         (-2 L1 L2 x6^2 Sin[x3] - 2 L2 (g - x4 x6) Sin[x2 + x3] + L1 (g + x4 x5) Sin[x2 + 2 x3]))) / (2 L1 (2 M M1 + M1^2 + M M2 + M1 M2 - M1 (M1 + M2) Cos[2 x2] - M M2 Cos[2 x3])))},
 {dx5 -> -((2 L1 (2 M1 + M2) u Cos[x2] - 2 L1 M2 u Cos[x2 + 2 x3] - 4 g L1 M M1 Sin[x2] - 4 g L1 M1^2 Sin[x2] - 2 g L1 M M2 Sin[x2] - 4 g L1 M1 M2 Sin[x2] +
        2 L1 M M2 x4 x5 Sin[x2] + 2 L1 M1 M2 x4 x5 Sin[x2] + 2 L1 M2^2 x4 x5 Sin[x2] + 2 L1^2 M1^2 x5^2 Sin[2 x2] + 3 L1^2 M1 M2 x5^2 Sin[2 x2] + L1^2 M2^2 x5^2 Sin[2 x2] +
        2 L1 L2 M1 M2 x5 x6 Sin[x3] + 2 L1 L2 M2^2 x5 x6 Sin[x3] - 4 L1 L2 M M2 x6^2 Sin[x3] - 2 L1 L2 M1 M2 x6^2 Sin[x3] + L1^2 M1 M2 x5^2 Sin[2 x3] + L1^2 M2^2 x5^2 Sin[2 x3] -
        4 g L2 M M2 Sin[x2 + x3] - 4 g L2 M1 M2 Sin[x2 + x3] - 3 g L2 M2^2 Sin[x2 + x3] + 4 L2 M M2 x4 x6 Sin[x2 + x3] + 4 L2 M1 M2 x4 x6 Sin[x2 + x3] +
        3 L2 M2^2 x4 x6 Sin[x2 + x3] - L1^2 M1 M2 x5^2 Sin[2 (x2 + x3)] - L1^2 M2^2 x5^2 Sin[2 (x2 + x3)] + g L2 M2^2 Sin[3 (x2 + x3)] - L2 M2^2 x4 x6 Sin[3 (x2 + x3)] +
        2 L1 L2 M1 M2 x5 x6 Sin[2 x2 + x3] + L1 L2 M2^2 x5 x6 Sin[2 x2 + x3] + 2 L1 L2 M1 M2 x6^2 Sin[2 x2 + x3] + 2 g L1 M M2 Sin[x2 + 2 x3] + 2 L1 M M2 x4 x5 Sin[x2 + 2 x3] +
        L1 M1 M2 x4 x5 Sin[x2 + 2 x3] + L1 M2^2 x4 x5 Sin[x2 + 2 x3] - L1 M1 M2 x4 x5 Sin[3 x2 + 2 x3] - L1 M2^2 x4 x5 Sin[3 x2 + 2 x3] - L1 L2 M2^2 x5 x6 Sin[2 x2 + 3 x3]) /
      (2 L1^2 (2 M M1 + M1^2 + M M2 + M1 M2 - M1 (M1 + M2) Cos[2 x2] - M M2 Cos[2 x3])))},
 {dx6 -> (2 L1 (M1 + M2) u Cos[x2 - x3] - 2 L1 (M1 + M2) u Cos[x2 + x3] - 2 g L2 M M2 Sin[x2] - 2 g L2 M1 M2 Sin[x2] - 2 g L2 M2^2 Sin[x2] + 2 L2 M M2 x4 x6 Sin[x2] +
      2 L2 M1 M2 x4 x6 Sin[x2] + 2 L2 M2^2 x4 x6 Sin[x2] + L1 L2 M1 M2 x5 x6 Sin[2 x2] + L1 L2 M2^2 x5 x6 Sin[2 x2] - 2 g L1 M M1 Sin[x2 - x3] -
      2 g L1 M M2 Sin[x2 - x3] + L1 M1^2 x4 x5 Sin[x2 - x3] + 2 L1 M1 M2 x4 x5 Sin[x2 - x3] + L1 M2^2 x4 x5 Sin[x2 - x3] + L1^2 M1^2 x5^2 Sin[2 x2 - x3] +
      2 L1^2 M1 M2 x5^2 Sin[2 x2 - x3] + L1^2 M2^2 x5^2 Sin[2 x2 - x3] + 2 L1^2 M1^2 x5^2 Sin[x3] + 4 L1^2 M1 M2 x5^2 Sin[x3] + 2 L1^2 M2^2 x5^2 Sin[x3] +
      L1 L2 M1 M2 x5 x6 Sin[2 x3] + L1 L2 M2^2 x5 x6 Sin[2 x3] - 2 L1 L2 M M2 x6^2 Sin[2 x3] + 2 g L1 M M1 Sin[x2 + x3] + 2 g L1 M M2 Sin[x2 + x3] +
      4 L1 M M1 x4 x5 Sin[x2 + x3] + 2 L1 M1^2 x4 x5 Sin[x2 + x3] + 4 L1 M M2 x4 x5 Sin[x2 + x3] + 4 L1 M1 M2 x4 x5 Sin[x2 + x3] + 2 L1 M2^2 x4 x5 Sin[x2 + x3] -
      L1 L2 M1 M2 x5 x6 Sin[2 (x2 + x3)] - L1 L2 M2^2 x5 x6 Sin[2 (x2 + x3)] - L1^2 M1^2 x5^2 Sin[2 x2 + x3] - 2 L1^2 M1 M2 x5^2 Sin[2 x2 + x3] -
      L1^2 M2^2 x5^2 Sin[2 x2 + x3] - L1 M1^2 x4 x5 Sin[3 x2 + x3] - 2 L1 M1 M2 x4 x5 Sin[3 x2 + x3] - L1 M2^2 x4 x5 Sin[3 x2 + x3] - 2 g L2 M M2 Sin[x2 + 2 x3] -
      g L2 M1 M2 Sin[x2 + 2 x3] - g L2 M2^2 Sin[x2 + 2 x3] + 2 L2 M M2 x4 x6 Sin[x2 + 2 x3] + L2 M1 M2 x4 x6 Sin[x2 + 2 x3] + L2 M2^2 x4 x6 Sin[x2 + 2 x3] +
      g L2 M1 M2 Sin[3 x2 + 2 x3] + g L2 M2^2 Sin[3 x2 + 2 x3] - L2 M1 M2 x4 x6 Sin[3 x2 + 2 x3] - L2 M2^2 x4 x6 Sin[3 x2 + 2 x3]) /
    (2 L1 L2 (2 M M1 + M1^2 + M M2 + M1 M2 - M1 (M1 + M2) Cos[2 x2] - M M2 Cos[2 x3]))}}
```

9 Appendix 2

```
In[12]:= Eigenvalues[J]

Out[12]= {0, 0, -1/(L1^2 L2 M M1)
   (Sqrt[(1/2 g L1^4 L2 M^2 M1^2 + 1/2 g L1^3 L2^2 M^2 M1^2 + 1/2 g L1^3 L2^2 M M1^3 + 1/2 g L1^4 L2 M^2 M1 M2 - 1/2 g L1^3 L2^2 M^2 M1 M2 + 1/2 g L1^2 L2^3 M^2 M1 M2 + 1/2 g L1^3 L2^2 M M1^2 M2 + 1/2 g L1^2 L2^3 M M1^2 M2 -
      1/2 Sqrt[((-g L1^4 L2 M^2 M1^2 - g L1^3 L2^2 M^2 M1^2 - g L1^3 L2^2 M M1^3 - g L1^4 L2 M^2 M1 M2 + g L1^3 L2^2 M^2 M1 M2 - g L1^2 L2^3 M^2 M1 M2 - g L1^3 L2^2 M M1^2 M2 - g L1^2 L2^3 M M1^2 M2)^2 -
         4 (g^2 L1^7 L2^3 M^4 M1^4 + g^2 L1^7 L2^3 M^3 M1^5 + g^2 L1^7 L2^3 M^4 M1^3 M2 + 2 g^2 L1^7 L2^3 M^3 M1^4 M2 + g^2 L1^7 L2^3 M^3 M1^3 M2^2))])]), 1/(L1^2 L2 M M1)
  (Sqrt[(1/2 g L1^4 L2 M^2 M1^2 + 1/2 g L1^3 L2^2 M^2 M1^2 + 1/2 g L1^3 L2^2 M M1^3 + 1/2 g L1^4 L2 M^2 M1 M2 - 1/2 g L1^3 L2^2 M^2 M1 M2 + 1/2 g L1^2 L2^3 M^2 M1 M2 + 1/2 g L1^3 L2^2 M M1^2 M2 + 1/2 g L1^2 L2^3 M M1^2 M2 -
      1/2 Sqrt[((-g L1^4 L2 M^2 M1^2 - g L1^3 L2^2 M^2 M1^2 - g L1^3 L2^2 M M1^3 - g L1^4 L2 M^2 M1 M2 + g L1^3 L2^2 M^2 M1 M2 - g L1^2 L2^3 M^2 M1 M2 - g L1^3 L2^2 M M1^2 M2 - g L1^2 L2^3 M M1^2 M2)^2 -
         4 (g^2 L1^7 L2^3 M^4 M1^4 + g^2 L1^7 L2^3 M^3 M1^5 + g^2 L1^7 L2^3 M^4 M1^3 M2 + 2 g^2 L1^7 L2^3 M^3 M1^4 M2 + g^2 L1^7 L2^3 M^3 M1^3 M2^2))])]), -1/(L1^2 L2 M M1)
   (Sqrt[(1/2 g L1^4 L2 M^2 M1^2 + 1/2 g L1^3 L2^2 M^2 M1^2 + 1/2 g L1^3 L2^2 M M1^3 + 1/2 g L1^4 L2 M^2 M1 M2 - 1/2 g L1^3 L2^2 M^2 M1 M2 + 1/2 g L1^2 L2^3 M^2 M1 M2 + 1/2 g L1^3 L2^2 M M1^2 M2 + 1/2 g L1^2 L2^3 M M1^2 M2 +
      1/2 Sqrt[((-g L1^4 L2 M^2 M1^2 - g L1^3 L2^2 M^2 M1^2 - g L1^3 L2^2 M M1^3 - g L1^4 L2 M^2 M1 M2 + g L1^3 L2^2 M^2 M1 M2 - g L1^2 L2^3 M^2 M1 M2 - g L1^3 L2^2 M M1^2 M2 - g L1^2 L2^3 M M1^2 M2)^2 -
         4 (g^2 L1^7 L2^3 M^4 M1^4 + g^2 L1^7 L2^3 M^3 M1^5 + g^2 L1^7 L2^3 M^4 M1^3 M2 + 2 g^2 L1^7 L2^3 M^3 M1^4 M2 + g^2 L1^7 L2^3 M^3 M1^3 M2^2))])]), 1/(L1^2 L2 M M1)
  (Sqrt[(1/2 g L1^4 L2 M^2 M1^2 + 1/2 g L1^3 L2^2 M^2 M1^2 + 1/2 g L1^3 L2^2 M M1^3 + 1/2 g L1^4 L2 M^2 M1 M2 - 1/2 g L1^3 L2^2 M^2 M1 M2 + 1/2 g L1^2 L2^3 M^2 M1 M2 + 1/2 g L1^3 L2^2 M M1^2 M2 + 1/2 g L1^2 L2^3 M M1^2 M2 +
      1/2 Sqrt[((-g L1^4 L2 M^2 M1^2 - g L1^3 L2^2 M^2 M1^2 - g L1^3 L2^2 M M1^3 - g L1^4 L2 M^2 M1 M2 + g L1^3 L2^2 M^2 M1 M2 - g L1^2 L2^3 M^2 M1 M2 - g L1^3 L2^2 M M1^2 M2 - g L1^2 L2^3 M M1^2 M2)^2 -
         4 (g^2 L1^7 L2^3 M^4 M1^4 + g^2 L1^7 L2^3 M^3 M1^5 + g^2 L1^7 L2^3 M^4 M1^3 M2 + 2 g^2 L1^7 L2^3 M^3 M1^4 M2 + g^2 L1^7 L2^3 M^3 M1^3 M2^2))])])}
```

```
In[94]:= Simplify[eigEqa1 == -eigEqa2]
         Simplify[eigEqa3 == -eigEqa4]

Out[94]= True

Out[95]= True

In[231]:= Solve[eigEqa2 == 0]
          Solve[eigEqa4 == 0]
          Solve[{eigEqa2 == 0} /. {g -> 0}]
          Solve[{eigEqa2 == 0} /. {g -> 1}]
          Solve[{eigEqa2 == 0} /. {g -> 9.8}]

Out[231]= {{g -> 0}, {M2 -> -M - M1}, {M2 -> -M1}}

Out[232]= {{g -> 0}, {M2 -> -M - M1}, {M2 -> -M1}}

Out[233]= {{}}

Out[234]= {{M2 -> -M - M1}, {M2 -> -M1}}

Out[235]= {{M2 -> 0.5 (0. - 2. M1)}, {M2 -> 0.5 (-2. M - 2. M1)}}
```