1. $T \in B(\mathcal{H})$, $\mathcal{H}$ Hilbert. T is positive operator if

$\langle Tx, x\rangle \geq 0, \ \forall x \in \mathcal{H}$

(a) T positive $\Rightarrow$ T self-adjoint? Depends on if $\mathcal{H}$ is real or complex.

No, if $\mathcal{H}$ can be real Hilbert space. $T = \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$, $\mathcal{H} = \mathbb{R}^2$

$\langle Te_1, e_2\rangle = 0 \quad \langle e_1, Te_2\rangle = 2 \quad T$ not self-adjoint.

$\langle T(ae_1+be_2), ae_1+be_2\rangle = a(a+2b) + b^2 = (a+b)^2 \geq 0 \quad T$ positive

Yes, if $\mathcal{H}$ is complex Hilbert space:

$\langle Tx, y\rangle = \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle)$

$\langle x, Ty\rangle = \overline{\langle Ty, x\rangle} = \frac{1}{4}(\langle T(y+x), y+x\rangle - \langle T(y-x), y-x\rangle$

$-i\langle T(y+ix), y+ix\rangle + i\langle T(y-ix), y-ix\rangle)$ (using $\langle Tx, x\rangle \in \mathbb{R}$)

However, $\langle T(x+iy), x+iy\rangle - \langle T(x-iy), x-iy\rangle = \langle Tx, x\rangle + i\langle Ty, x\rangle - i\langle Tx, y\rangle$

$+ \langle Ty, y\rangle - (\langle Tx, x\rangle - i\langle Ty, x\rangle + i\langle Tx, y\rangle + \langle Ty, y\rangle)$

$= 2i\langle Ty, x\rangle - 2i\langle Tx, y\rangle$

changes sign if we exchange x and y.

We've proven $\langle Tx, y\rangle = \langle x, Ty\rangle$. #

(b) $T: l^2 \to l^2 \quad (a_n) \xmapsto{T} (a_n/n)$ is positive

For $\{a_n\} \in l^2$, $\|T\{a_n\}\|^2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^2} \leq \sum_{n=1}^{\infty} |a_n|^2$, $\|T\| \leq 1$

Also T is linear $\Rightarrow T \in B(\mathcal{H})$.

$\langle T\{a_n\}, \{a_n\}\rangle = \sum_{n=1}^{\infty} \frac{a_n}{n} \overline{a_n} = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n} \geq 0$

$\Rightarrow T$ is positive operator. #

(c) T^*T is positive and self-adjoint for $T \in B(H)$

for $x, y \in \mathcal{H}$, $\langle T^*Tx, y\rangle = \langle Tx, Ty\rangle = \langle x, T^*Ty\rangle$

(as $(T^*)^* = T$)

$\Rightarrow T^*T$ is self-adjoint

$\|T^*T\| \le \|T\|^2 < \infty \Rightarrow T^*T \in B(\mathcal{H})$

$\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \geq 0$, $\forall x \in \mathcal{H}$.

$\Rightarrow T^*T$ is positive operator. #

(d) T is positive, compact, self-adjoint. There exists positive compact operator S, $T = S^2$

The non-zero eigenvectors of T form a basis of Hilbert space $\mathcal{H}$.

WLOG let T has infinite nonzero spectrum. Let $\{e_n\}_{n=1}^{\infty} \oplus \mathcal{H}_0$ be eigenbasis, $T|_{\mathcal{H}_0} = 0$

with eigenvalue $\{\lambda_n\}_{n=1}^{\infty}$, $|\lambda_1| \geq |\lambda_2| \geq \cdots$ $|\lambda_i| \to 0$

$\langle Te_n, e_n\rangle = \langle \lambda_n e_n, e_n\rangle = \lambda_n \|e_n\|^2 \geq 0 \Rightarrow \lambda_n \geq 0$

For $v \in \mathcal{H}$ we have $Tv = T \sum_{n=1}^{\infty} \langle v, e_n\rangle e_n = \sum_{n=1}^{\infty} \lambda_n \langle v, e_n\rangle e_n$

Define $S: \mathcal{H} \to \mathcal{H}$

$v \mapsto \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle v, e_n\rangle e_n$

S is clearly linear. $\|Sv\|^2 = \sum_{n=1}^{\infty} \lambda_n |\langle v, e_n\rangle|^2 \le \lambda_1 \sum_{n=1}^{\infty} |\langle v, e_n\rangle|^2 \le \lambda_1 \|v\|^2$

$\|S\| \le \sqrt{\lambda_1}$, $S \in B(\mathcal{H})$.

$S^2 v = S\left(\sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle v, e_n\rangle e_n\right) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle v, e_n\rangle S e_n$

$= \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle v, e_n\rangle \sqrt{\lambda_n} e_n = \sum_{n=1}^{\infty} \lambda_n \langle v, e_n\rangle e_n = Tv$, $v \in \mathcal{H} \Rightarrow S^2 = T$

Denote $S_N: \mathcal{H} \to \mathcal{H}$ clearly $\|S - S_N\| \to 0 \Rightarrow S$ compact.

$v \mapsto \sum_{n=1}^{N} \sqrt{\lambda_n} \langle v, e_n\rangle e_n$

$\langle Sv, v\rangle = \langle \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle v, e_n\rangle e_n, \sum_{n=1}^{\infty} \langle v, e_n\rangle e_n\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n} |\langle v, e_n\rangle|^2 \geq 0$

$\Rightarrow S$ positive. We are done. #

2. $T: L^2(0,1) \to L^2(0,1)$

$$Tf(x) = \int_0^x f(t)\,dt$$

(a) Show T is compact

Let $K(t,x) = 1_{t \le x}$

$$Tf(x) = \int_0^x f(t)\,dt = \int_0^1 1_{t\le x} f(t)\,dt = \int_0^1 K(t,x) f(t)\,dt$$

$$\|Tf(x)\|_2^2 = \int_0^1 \left|\int_0^1 K(t,x) f(t)\,dt\right|^2 dx$$

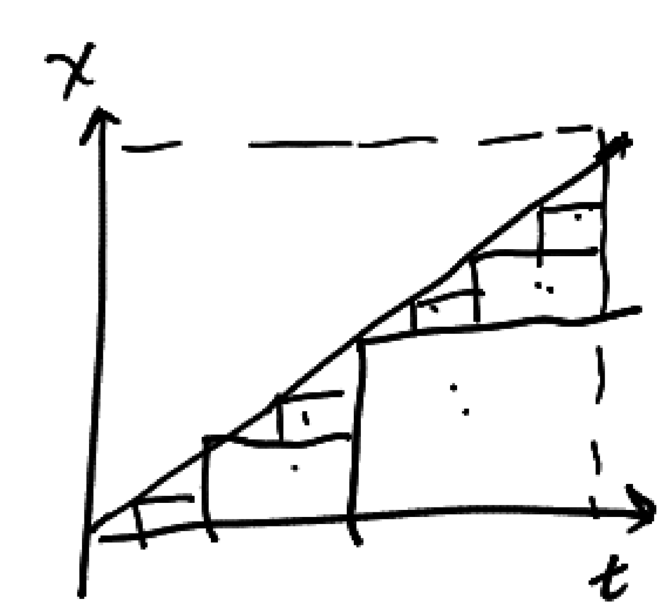
$$\le \int_0^1 \int_0^1 K^2(t,x)\,dt \int_0^1 f^2(t)\,dt\,dx$$

$$= \|f\|_2^2 \int_0^1\int_0^1 K^2(t,x)\,dt = \frac{\|f\|^2}{2} \qquad T \in B(L^2(0,1))$$

Let $K_n = \sum_{m=1}^{2^n-1} 1_{A_m}(x)\, 1_{B_m}(t)$

s.t. $\|K_n - K\|_2^2 = \int_0^1\int_0^1 |(K_n - K)(t,x)|^2\,dt\,dx \to 0, \; n\to\infty$

(A construction is illustrated in the figue)



Let $T_n f(x) = \int_0^1 K_n(t,x) f(t)\,dt$

We note that T_n are of finite rank

$$\|(T-T_n)f\|_2^2 = \int_0^1 \left|\int_0^1 (K-K_n)(t,x) f(t)\,dt\right|^2 dx$$

$$\le \|K-K_n\|_2^2\, \|f\|_2^2 \qquad \text{(Cauchy-Schwarz)}$$

$$\|T-T_n\|_2 \le \|K-K_n\|_2 \to 0 \quad (n\to\infty)$$

$\Rightarrow T$ is compact.

#

(b) Compute $\sigma(T^*T)$.

We know T^*T is compact since T is compact.

Thus $0 \in \sigma(T^*T)$.

For $K(t,x) = 1_{t\le x}$

Let $Sf(x) = \int_0^1 K(x,t) f(t)\, dt$

$$\langle f, Sg \rangle = \int_0^1 f(x) \overline{\int_0^1 K(x,t) g(t)\, dt}\, dx$$
$$= \int_0^1\int_0^1 K(x,t) f(x) \overline{g(t)}\, dx\, dt \quad (\text{Fubini})$$
$$= \int_0^1\int_0^1 K(t,x) f(t) \overline{g(x)}\, dt\, dx = \int_0^1 \int_0^1 K(t,x) f(t)\, dt\, \overline{g(x)}\, dx$$
$$= \langle Sf, g \rangle$$

$\Rightarrow S = T^*$

$$T^*T f(x) = \int_0^1 K(x,t)\, Tf(t)\, dt$$
$$= \int_0^1 K(x,t) \int_0^1 K(s,t) f(s)\, ds\, dt$$
$$= \int_0^1\int_0^1 K(x,t) K(s,t) f(s)\, dt\, ds \quad (\text{Fubini})$$
$$= \int_0^1 \left(\int_0^1 K(x,t) K(s,t)\, dt\right) f(s)\, ds$$

Let $G(x,s) = \int_0^1 K(x,t) K(s,t)\, dt = 1 - \max(x,s)$

$$T^*Tf(x) = \int_0^1 G(x,s) f(s)\, ds$$

Compute eigenvalues: $T^*T\, v(x) = \int_0^1 (1-\max(x,s))\, v(s)\, ds = \lambda v(x)$

$(1-x)\int_0^x v(s)\, ds + \int_x^1 (1-s) v(s)\, ds = \lambda v(x) \quad \Rightarrow v \in C^1$

$\lambda v'(x) = v(x) - \int_0^x v(s)\, ds - x v(x) + (x-1) v(x) \quad \Rightarrow v' \in C^1,\ v \in C^2$

$\lambda v''(x) = -v(x), \quad v(1) = 0,\ v'(0) = 0$ This is 2nd order ODE system.

Solution is $\lambda_n = \dfrac{1}{\left(\frac{\pi}{2} + n\pi\right)^2}$, $n \in \mathbb{Z}$ $\quad v_n(x) = \cos\left(n+\frac{1}{2}\right)\pi x$

Thus $\sigma(T^*T) = \{0\} \cup \left\{\dfrac{4}{(2n+1)^2\pi^2},\ n \in \mathbb{Z}\right\}$ #