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1. $(a_n) \in \mathbb{C}$ $T: \ell^2 \to \ell^2$

$$(x_1, \cdots) \xmapsto{T} (a_1 x_1, a_2 x_2, \cdots)$$

Show that T is compact iff $|a_n| \to 0$

"$\Rightarrow$": Consider $T_n: \ell^2 \to \ell^2$

$$(x_1, x_2, \cdots x_n, \cdots) \xmapsto{T_n} (a_1 x_1, a_2 x_2, \cdots a_n x_n, 0, \cdots 0, \cdots)$$

It is easy to see $T_n \in B(\ell^2)$ and $\dim \mathrm{Range}(T_n) < \infty$

Thus $T_n \in C(\ell^2)$.

$$\|T - T_n\|^2 = \sup_{\|x\|_2 = 1} \|(a_{n+1} x_{n+1}, a_{n+2} x_{n+2}, \cdots)\|_2^2$$

$\forall \varepsilon > 0$ $\exists N$. $\forall n > N$, $|a_n| \le \varepsilon$

For $n > N$, $\|T - T_n\|^2 \le \sup_{\|x\|_2 = 1} \varepsilon^2 \sum_{k=n+1}^{\infty} |x_k|^2 \le \varepsilon^2$

Thus $\|T - T_n\| \to 0$ $(n \to \infty)$. Since $C(\ell^2)$ is closed in $B(\ell^2)$

we have $T \in C(\ell^2)$.

"$\Leftarrow$" Since T is compact. $\sigma(T) \setminus \{0\} \subseteq \sigma_p(T)$

$Tx = \lambda x$, $x \neq 0$ $\Leftrightarrow$ $\lambda = a_i$, $i = 1, 2, \cdots$, $x = k e_i$, $k \in \mathbb{C} \setminus \{0\}$.

Thus $a_n \neq 0 \Rightarrow a_n \in \sigma(T) \setminus \{0\}$. We know from the properties of compact operator $\sigma(T) \setminus \{0\}$ is bounded (say by M) and does not have a limit point at $z \neq 0$.

If $|a_n| \not\to 0$ $\exists \varepsilon_0 > 0$, and subsequence $\{n_k\}$. s.t. $|a_{n_k}| \geq \varepsilon_0$, $\forall k = 1, 2, \cdots$

Thus $a_{n_k} \in \{z \in \mathbb{C} \mid \varepsilon_0 \le |z| \le M\}$

$\{a_{n_k}\}$ must have a limit point in $\{z \in \mathbb{C} \mid \varepsilon_0 \le |z| \le M\}$.

which is not 0. A contradiction!

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2. Right-shift operator $S_R: \ell^2 \to \ell^2$ defined as

$$(a_1, a_2, \cdots) \xmapsto{S_R} (0, a_1, a_2, \cdots)$$

(a) Show $\sigma(S_R) = \{\lambda \in \mathbb{C} \mid |\lambda| \le 1\}$

It is easy to see for $a \in \ell^2$, $\|S_R a\|_2 = \|a\|_2$. Thus $\|S_R\| = 1$

For $\lambda \in \sigma(S_R)$, $|\lambda| \le \|S_R\| = 1$. $\sigma(S_R) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \le 1\}$

On the other hand,

For $\lambda \in \mathbb{C}$, $|\lambda| \le 1$, we show $\mathrm{Range}(S_R - \lambda I) \neq \ell^2$.

If $y = (-1, 0, 0, \cdots) \in \mathrm{Range}(S_R - \lambda I)$

there exists $x = (x_1, x_2, \cdots) \in \ell^2$, s.t. $(S_R - \lambda I)x = y$

i.e. $-1 = -\lambda x_1$, $0 = x_1 - \lambda x_2$, $0 = x_2 - \lambda x_3$, $\cdots$

We have $\lambda \neq 0$ and

$x_1 = \frac{1}{\lambda}$ $x_2 = \frac{1}{\lambda^2}$, $\cdots$ $x_n = \frac{1}{\lambda^n}$, $\cdots$

$\|x\|_2^2 = \sum_{n=1}^{\infty} \frac{1}{|\lambda|^{2n}} = \infty$. A contradiction.

Thus $\{\lambda \in \mathbb{C} : |\lambda| \le 1\} \subseteq \sigma(S_R)$. $\{\lambda \in \mathbb{C}, |\lambda| \le 1\} = \sigma(S_R)$. #

(b) $\sigma_p(S_R) = \phi$

If not, let $\lambda \in \sigma_p(S_R)$, $\exists\, a = (a_n) \neq 0 \in \ell^2$, $S_R a = \lambda a$.

$\Rightarrow$ $(0, a_1, a_2, \cdots) = (\lambda a_1, \lambda a_2, \lambda a_3, \cdots)$

If $\lambda = 0$, $a_n = 0$, $n = 1, 2, \cdots$ contradiction!

If $\lambda \neq 0$, $a_1 = 0$, $a_2 = 0$, $\cdots$ $a_n = 0$, $\cdots$ contradiction!

Thus $\sigma_p(S_R) = \phi$

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(c) $\{\lambda \in \mathbb{C} \mid |\lambda|<1\} \subset \sigma_r(S_R)$

Consider the left shift operator S_L in question 3.

For $x=(x_1, x_2, \dots) \in \ell^2$, $y=(y_1, y_2, \dots) \in \ell^2$,

$$\langle (S_R - \lambda I)x, y \rangle = \sum_{n=1}^{\infty} y_{n+1} x_n - \lambda \sum_{n=1}^{\infty} x_n y_n = \langle x, (S_L - \bar{\lambda} I) y \rangle$$

$\Rightarrow$ $(S_R - \lambda I)^* = S_L - \bar{\lambda} I$

$\Rightarrow$ $\text{Range}(S_R - \lambda I)^{\perp} = \ker(S_L - \bar{\lambda} I)$

Let $a=(a_1, a_2, \dots) \in \ell^2$, $a \neq 0$, $a \in \ker(S_L - \bar{\lambda} I) \iff a_n = a_1 (\bar{\lambda})^{n-1}$, $n=2, \dots$

$a \in \ell^2 \iff \sum_{n=1}^{\infty} |a_n|^2 < \infty \iff |\bar{\lambda}| < 1$

Thus $\ker(S_L - \bar{\lambda} I) \neq \{0\} \iff |\lambda| < 1$, $\lambda \in \mathbb{C}$. $(\triangle)$

This means $\lambda \in \mathbb{C}$, $|\lambda|<1 \Rightarrow \overline{\text{Range}(S_R - \lambda I)} = \ker(S_L - \bar{\lambda} I)^{\perp} \neq \ell^2$

$\Rightarrow \{\lambda \in \mathbb{C} \mid |\lambda|<1\} \subseteq \sigma_r(S_R)$

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(d) $\{\lambda \mid |\lambda|=1\} = \sigma_c(S_R)$

From $(\triangle)$ in (c) we see that

$|\lambda|=1$, $\lambda \in \mathbb{C} \Rightarrow \ker(S_L - \bar{\lambda} I) = \{0\}$

$\overline{\text{Range}(S_R - \lambda I)} = \ker(S_L - \bar{\lambda} I)^{\perp} = \ell^2$

From (b), $\{\lambda \mid |\lambda|=1\} \subseteq \sigma(S_R)$, i.e. $\text{Range}(S_R - \lambda I) \neq \ell^2$

Thus $\{\lambda \mid |\lambda|=1\} \subseteq \sigma_c(S_R)$

It is easy to see from (b), (c)

$\sigma_c(S_R) \subseteq \{\lambda \mid |\lambda|=1\}$

We are done. #

3. Left-shift operator $S_L : \ell^2 \to \ell^2$

$$(a_1, a_2, \cdots) \xmapsto{S_L} (a_2, a_3, a_4, \cdots)$$

Find $\sigma_p(S_L)$, $\sigma_r(S_L)$, $\sigma_c(S_L)$.

For $a \in \ell^2$, $\|S_L a\|_2^2 = \sum_{n=2}^{\infty} |a_n|^2 \le \|a\|_2^2$ $\quad \|S_L\| \le 1$

$\Rightarrow \sigma(S_L) \subseteq \{z \in \mathbb{C}, \ |z| \le 1\}$.

The argument in dotted box in question 2(c) shows

$\ker(S_L - \lambda I) \ne 0 \iff |\lambda| < 1$. Thus $\sigma_p(S_L) = \{z \in \mathbb{C} \mid |z| < 1\}$.

We know $\sigma_c(S_L) \cup S_r(S_L) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}$.

In fact, $\{z \in \mathbb{C} \mid |z| = 1\} = \sigma_c(S_L) \cup Sr(S_L)$ since $\sigma(S_L)$ is compact.

$$(S_L - \lambda I)^* = S_R - \bar{\lambda} I$$

$$\Rightarrow \quad \text{Range}(S_L - \lambda I)^{\perp} = \ker(S_R - \bar{\lambda} I) \overset{\text{question 2(b)}}{=\!=\!=} \{0\}$$

i.e. $\overline{\text{Range}(S_L - \lambda I)} = \ell^2$

Thus, $\{z \in \mathbb{C} \mid |z| = 1\} \subseteq \sigma_c(S_L)$. We are done.

In summary, $\sigma_p(S_L) = \{z \in \mathbb{C} \mid |z| < 1\}$

$\sigma_c(S_L) = \{z \in \mathbb{C} \mid |z| = 1\}$

$\sigma_r(S_L) = \phi$.

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