An Unreasonably Quick Introduction to String Theory, Conformal Field Theory and Geometry

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Abstract

A very quick introduction to the bosonic string, conformal field theory, the superstring and geometry. No background in quantum field theory is assumed and the omissions are vast. Based on four lectures at the 2024 Physical Mathematics of Quantum Field Theory Summer School.

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1 Day 1

Over the next 4 lectures we will rapidly whirl through the construction of string theory and conformal field theory assuming as little background as possible. Much of this is copied from the standard codices [1–4]. We reference these once and for all here and direct you to these sources for a far more complete derivation of what happens below.

1.1 The Harmonic Oscillator

We begin with the harmonic oscillator. This provides the foundational concepts in field theory and strings, but we begin with the humble single nonrelativistic particle.

1.1.1 Classical

The classical harmonic oscillator is viewed as a particle of mass m with a displacement x. The kinetic energy is given by $\frac{1}{2}m\dot{x}^2$, which we may also write as $p^2/2m$, where $p=m\dot{x}$ is the momentum.

The definition of the harmonic oscillator is that the potential energy is quadratic:

$$V = \frac{1}{2}m\omega^2 x^2,\tag{1.1}$$

and then the Hamiltonian H, i.e., total energy in this case, is the sum of the kinetic and potential energies.

The classical solution of this is that x(t) is a sine wave of (angular) frequency ω .

1.1.2 Quantum

In quantum mechanics we have a "space of states" given by a Hilbert space \mathcal{H} . The observables x and p are given by linear Hermitian, or self-adjoint, operators acting on \mathcal{H} . We then assert the commutation relation

$$[x, p] = i\hbar. \tag{1.2}$$

Dirac's trick involves introducing

$$u = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{i}{\sqrt{2m\omega\hbar}}p$$

$$u^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}x - \frac{i}{\sqrt{2m\omega\hbar}}p.$$
(1.3)

Then

$$[u, u^{\dagger}] = 1, \qquad H = \left(u^{\dagger}u + \frac{1}{2}\right)\hbar\omega.$$
 (1.4)

If $|E\rangle \in \mathcal{H}$ is an eigenvector of H, i.e.,

$$H|E\rangle = E|E\rangle,$$
 (1.5)

then

$$Hu |E\rangle = uH |E\rangle + [H, u] |E\rangle$$

$$= Eu |E\rangle + \hbar\omega [u^{\dagger}, u] u |E\rangle$$

$$= (E - \hbar\omega) u |E\rangle.$$
(1.6)

So u decreases E by $\hbar\omega$. Similarly u^{\dagger} increases E by $\hbar\omega$.

Next, observe that

$$\langle E | H | E \rangle = E ||E\rangle|^{2}$$

$$= \langle E | \left(u^{\dagger} u + \frac{1}{2} \right) |E\rangle \hbar \omega$$

$$= \left(||u|E\rangle|^{2} + \frac{1}{2} ||E\rangle|^{2} \right) \hbar \omega.$$
(1.7)

Since our inner product is positive definite, H has eigenvalue $E \geq \frac{1}{2}\hbar\omega$, with equality if and only if $u|E\rangle = 0$.

So we can construct the Hilbert space \mathscr{H} as follows. Let a basis element be the ground state $|g\rangle$ which is killed by u. Then the rest of the basis is defined by $(u\dagger)^n|g\rangle$ for n>0 (perhaps suitably normalized). This is the "Fock space" version of a Hilbert space.

Alternatively, the standard way to describe quantum mechanics is to say that \mathcal{H} is the Hilbert space of square-integrable functions of x. The operator x is then simply multiplication by x and the momentum operator is

$$p = -i\hbar \frac{d}{dx}. ag{1.8}$$

In this language one can show that the energy eigenstates of our harmonic oscillator are given by

$$\psi_n(x) = \langle x | (u^{\dagger})^n \rangle | g \rangle$$

$$= H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) \exp\left(-\frac{m\omega}{2\hbar} x^2 \right), \tag{1.9}$$

where H_n are polynomials of degree n in x known as Hermite polynomials. We'll need this in next lecture.

Let's set \hbar to 1 from here on.

1.2 A Quantum Violin String

Now extend this to a vibrating violin string.

Consider the string of figure 1. Here, σ is horizontal displacement and x is vertical displacement (which we assume to be small in some sense). We'll consider the string to be made out of catgut molecules of mass ν arranged with equal spacing d, and let x_n denote the displacement for the nth molecule.

Let the tension be T. We will take the limit of an infinite number of such molecules. The potential energy between sites is taken to be

$$U_i = \frac{T}{2d}(x_i - x_{i-1})^2. (1.10)$$

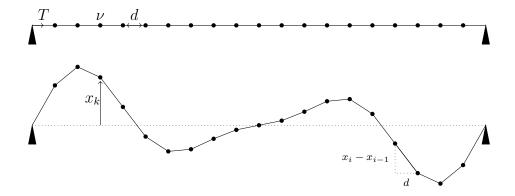


Figure 1: A violin string made up of particles: at rest, and then plucked

Let's Fourier transform $x \to \tilde{x}$. The Hamiltonian can then be shown to be

$$H = \sum_{n=1}^{\infty} \frac{1}{2\nu} \tilde{p}_n^2 + \frac{2T}{d} \sum_{n=1}^{\infty} n^2 \tilde{x}_n^2.$$
 (1.11)

So we immediately recognize this as an infinite set of decoupled harmonic oscillators, each with frequency proportional to n. Let us fix these various parameters so that the speed of sound along the string is 1 and the frequency is n.

Accordingly we introduce an infinite set of pairs u_n and u_n^{\dagger} satisfying

$$[u_m^{\dagger}, u_n] = \delta_{m.n}. \tag{1.12}$$

It is standard in string theory to define

$$\alpha_n = \sqrt{n}u_n \quad \text{for } n > 0$$

$$\alpha_{-n} = \sqrt{n}u_n^{\dagger} \quad \text{for } n > 0,$$
(1.13)

so that we have a commutation relationship

$$[\alpha_m, \alpha_n] = m\delta_{m+n}. (1.14)$$

Our notation is that a Kronecker delta with a single subscript is 1 precisely when that subscript is zero.

1.3 The Spectrum

Let's analyze the spectrum. Denote the lowest energy state, the ground state, as $|g\rangle$ again. So this is killed by α_n for n > 0:

$$\alpha_n |g\rangle = 0 \quad \text{if } n > 0.$$
 (1.15)

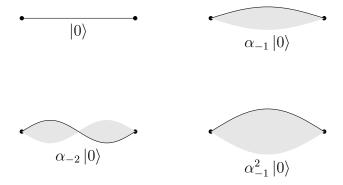


Figure 2: Various string excitations.

The Hamiltonian would appear to be

$$H \stackrel{?}{=} \sum_{n=1}^{\infty} \left(\alpha_{-n} \alpha_n + \frac{1}{2} n \right) \tag{1.16}$$

which implies

$$H|g\rangle \stackrel{?}{=} \frac{1}{2} \sum_{n=1}^{\infty} n|g\rangle, \qquad (1.17)$$

which is very unfortunate¹. Our so-called ground state has infinite energy!

We will take the attitude that we don't actually care about the absolute value of energy. What matters is differences in energies of our states. Of course, this might not be reasonable in a theory with gravity but let's ignore such issues. So let us reinterpret our Hilbert space. Relabel the ground state to be $|0\rangle$ and declare it to have zero energy. All other states, i.e., directions in the Hilbert space, are obtained by applying raising operators α_{-n} , n > 0, to $|0\rangle$. As such we build a Fock space again. Our new Hamiltonian is then

$$H = \sum_{n=1}^{\infty} \alpha_{-n} \alpha_n. \tag{1.18}$$

Our lowest level of excitation with energy 1 is then $\alpha_{-1}|0\rangle$. Next we have energy 2 states given by $\alpha_{-1}^2|0\rangle$ or $\alpha_{-2}|0\rangle$. These two states are different. See Figure 2.

Our violin string has fixed boundary conditions at each end. Boundary conditions in string theory lead to D-branes which are massively important in almost all aspects of string theory. Sadly we will completely ignore D-branes in these lectures.

Sometimes you might see claims by physicists at this point that $\sum n = -\frac{1}{12}$. We will not resort to such trickery here!

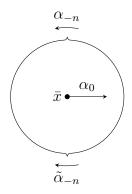


Figure 3: A closed string

Instead, let's focus on closed strings. See Figure 3. The closed string differs from our violin string in two very important respects

- 1. The string is free to float about. So it has a center of mass position, \bar{x} , and momentum which are new degrees of freedom.
- 2. The classical solutions are two decoupled sets of waves. One set moves counterclockwise around the string, while the other moves clockwise.

For the violin string, our sine waves were functions of σ . So, for the closed string we introduce

$$\sigma^{\pm} = t \pm \sigma, \tag{1.19}$$

and replace σ with σ^{\pm} for a doubled set of oscillators. Actually, let's complexify things and use -it instead of t, so that

$$\sigma^{\pm} = -it \pm \sigma. \tag{1.20}$$

We can now define the following new coordinates z and \bar{z} :

$$z = e^{i\sigma^+} \qquad \bar{z} = e^{i\sigma^+}. \tag{1.21}$$

As is common in complex analysis we will regard z and \bar{z} as independent. Holomorphic functions of z will represent the counterclockwise oscillations on our closed string, while the physically independent clockwise oscillations correspond to antiholomorphic functions.

Quantization of our closed string therefore has 2 sets of raising and lowering operators α_n and $\tilde{\alpha}_n$, as well as the center of mass position \bar{x} and momentum p. The commutation relations are

$$[\alpha_m, \alpha_n] = m\delta_{m+n}, \qquad [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{m+n}$$
$$[\tilde{\alpha}_m, \alpha_n] = 0$$
$$[\bar{x}, p] = i. \tag{1.22}$$

A little algebra, going back to our definition (1.3), then yields

$$x(z,\bar{z}) = \bar{x} + \frac{p}{4\pi i T} \log(z\bar{z}) + \frac{i}{2\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} (\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n}).$$
 (1.23)

Note we have yet to define α_0 . It is very convenient to set

$$\alpha_0 = \tilde{\alpha}_0 = p,\tag{1.24}$$

and $T = \frac{1}{4\pi}$. One then has simple relationships

$$i\partial x = \sum_{n \in \mathbb{Z}} \frac{\alpha_n}{z^{n+1}}$$

$$i\bar{\partial} x = \sum_{n \in \mathbb{Z}} \frac{\tilde{\alpha}_n}{\bar{z}^{n+1}},$$
(1.25)

where ∂ denotes $\partial/\partial z$ and $\bar{\partial}$ denotes $\partial/\partial \bar{z}$.

Let us define our ground state to satisfy $\alpha_0 |0\rangle = 0$ so that it corresponds to zero momentum as well as no oscillations. (For a full description we also need to add zero oscillator states $|0;p\rangle$ with nonzero momentum p.)

What we have here is a perfectly good (nonrelativistic) quantum mechanical description of a vibrating string. So what's all the fuss about string theory then if we can describe it with so little work? Well, this string is not really the one we think of when we say "string theory".

1.4 Fundamental Strings

A truly fundamental string is not made of cat molecules² or any other type of particle. It should be purely identified as itself as a string without any constituents. Quantum mechanically this makes a huge difference.

The way to state this is that σ (or σ^{\pm}) reparametrizations are a "gauge symmetry". A gauge symmetry is a symmetry of the description of our model, but one that nature is

²Perhaps we should emphasize that "catgut" is also not made out of cats.

unaware of. Our model makes explicit use of σ , indeed we don't know how to describe it without such a parametrization, but a fundamental string must have no kind of dependence on this parameter.

Reparametrizations of the circle are invertible maps $S^1 \to S^1$ which we will assume are differentiable. These form the group $Diff(S^1)$.

We write a reparametrization as

$$\sigma \to \sigma' = \sigma'(\sigma). \tag{1.26}$$

Classically, if we think of our universe as having a single spacial dimension with coordinate σ , then a "field" is simply a function of σ . If φ is a "scalar field" under a reparametrization we would replace it by a field φ' such that

$$\varphi'(\sigma') = \varphi(\sigma), \tag{1.27}$$

For the equivalent of vector fields, etc., we consider A such that

$$A'(\sigma') = \left(\frac{d\sigma}{d\sigma'}\right)^h A(\sigma) \tag{1.28}$$

where h is called the "weight". Such fields A are called "primary fields".

The Lie algebra of a group corresponds to elements infinitesimally close to the identity. So consider

$$\sigma \to \sigma + i\varepsilon e^{in\sigma}$$
. (1.29)

for small ε and only consider first order effects in ε . This induces

$$A(\sigma') = A(\sigma) + i\varepsilon e^{in\sigma} \frac{dA}{d\sigma}.$$
 (1.30)

Thus the generators of our Lie algebra can be written as

$$L_n = ie^{in\sigma} \frac{d}{d\sigma} \tag{1.31}$$

and it is easy to see these obey the commutation relation, i.e., Lie bracket:

$$[L_m, L_n] = (m-n)L_{m+n}. (1.32)$$

This Lie algebra is called the "Witt Algebra".

Now, for a change induced by L_m ,

$$\delta A = A'(\sigma) - A(\sigma)$$

$$= A'(\sigma' - i\epsilon e^{im\sigma}) - A(\sigma)$$

$$= A'(\sigma') - i\epsilon e^{im\sigma} \frac{dA}{d\sigma} - A(\sigma)$$

$$= \left(\frac{d\sigma}{d\sigma'}\right)^h A(\sigma) - i\epsilon e^{im\sigma} \frac{dA}{d\sigma} - A(\sigma)$$

$$= \left(1 - \epsilon m e^{im\sigma}\right)^{-h} A(\sigma) - i\epsilon e^{im\sigma} \frac{dA}{d\sigma} - A(\sigma)$$

$$= \epsilon e^{im\sigma} \left(-i\frac{d}{d\sigma} + mh\right) A.$$
(1.33)

So, if we take a Fourier decomposition of our primary field

$$A(\sigma) = \sum_{n} A_n e^{-in\sigma} \tag{1.34}$$

then

$$\epsilon^{-1}\delta A = \sum_{n} e^{im\sigma} \left(-i\frac{d}{d\sigma} + mh \right) A_{n} e^{-in\sigma}$$

$$= \sum_{n} (-n + mh) A_{n} e^{i(m-n)\sigma}$$

$$= \sum_{n} (m(h-1) - n) A_{m+n} e^{-in\sigma}$$
(1.35)

This implies, if we think of A as an operator,

$$[L_m, A_n] = (m(h-1) - n)A_{m+n}. (1.36)$$

In particular,

$$[L_0, A_n] = -nA_n. (1.37)$$

We view this as a generalization of "raising and lowering" operator statements. If L_0 is viewed as a kind of Hamiltonian, then A_n 's with positive subscripts are the lowering operators, while negative subscripts produce excitations.

Rather than a single σ , for a our closed string we have σ^{\pm} which we have written in terms of a complex coordinate z. So we really have two reparametrization groups and two Lie algebras with generators

$$L_n = -z^{n+1}\partial, \quad \tilde{L}_n = -\bar{z}^{n+1}\bar{\partial}.$$
 (1.38)

Written this way, the L_n 's form the Lie algebra of (meromorphic) complex reparametrizations. Such mappings of the complex plane are *conformal*, i.e., angle preserving.

So, by starting out trying to analyze string theory, we've ended up with a model on the complex plane where we are interested in invariance under conformal maps. This is how "conformal field theory" is part of string theory.

We want to think of how the L_n 's (and \tilde{L}_n 's) act on our quantum mechanical string. We've described our string in terms of a Fock space \mathscr{H} in which we rather delicately removed an infinity by suitably defining our ground state. It would be nice if we could describe the action of the Witt algebra purely in terms of this language so that we don't disturb the validity of this construction.

In other words, can we write the L_n 's purely in terms of the only operators we have, namely the α_n 's? First note that if A is a primary field of weight h then the change of coordinates from σ^+ to z means that the Fourier series becomes (up to factors we can drop)

$$A(z) = \sum_{n} \frac{A_n}{z^{n+h}}. ag{1.39}$$

Given how derivatives transform, we see from (1.28) that $i\partial x(z)$ has h=1 assuming it is a primary field, and that its Fourier modes are therefore α_n . Thus, (1.36) gives

$$[L_m, \alpha_n] = -n\alpha_{m+n}. (1.40)$$

So our goal is to build L_m from the α_n 's such that this is true. Now, a bracket is a derivation, i.e., it obeys Leibniz rule, and we have (1.22). This implies L_m must be quadratic in the α 's. Indeed the following attempt yields (1.40)

$$L_m \stackrel{?}{=} \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha_{m+k} \alpha_{-k} \tag{1.41}$$

But then we would have

$$L_0 |0\rangle \stackrel{?}{=} \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha_k \alpha_{-k} |0\rangle \stackrel{?}{=} \frac{1}{2} \sum_{k=1}^{\infty} k |0\rangle.$$
 (1.42)

Our old divergent nemesis has returned. But actually we are being a bit too quick here. Defining L_n purely by its commutator with something means we could always have added an arbitrary constant term. Furthermore, changing the order of the α_n 's in the definition of L_m would shift L_0 by a constant. So we are completely free to reorder and this can remove the divergence. Define "normal ordering":

$$:\alpha_m \alpha_n := \begin{cases} \alpha_m \alpha_n & \text{if } m \le n \\ \alpha_n \alpha_m & \text{if } m > n \end{cases}$$
 (1.43)

and then define

$$L_m = \frac{1}{2} \sum_{k \in \mathbb{Z}} : \alpha_{m+k} \alpha_{-k} : . \tag{1.44}$$

This does not mess up (1.40) but now

$$L_n |0\rangle = 0, \quad \forall n > -1, \tag{1.45}$$

and $L_n|0\rangle$ with n<-1 only has a finite number of excited oscillator modes, so our divergences have gone away. Also, L_0 coincides with the Hamiltonian (1.18).

Now, however, with sufficient stamina and care, one can show this normal ordering has changed the algebra to

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}m(m^2 - 1)\delta_{m+n}. \tag{1.46}$$

(We encourage this as a homework exercise.)

Note we still have freedom to add an arbitrary *finite* constant to L_0 . This comes back to haunt us shortly.

1.5 The Virasoro Algebra

The Virasoro Algebra is defined as having a basis L_n , $n \in \mathbb{Z}$ together with c and brackets

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n}, \qquad (1.47)$$

and any bracket with c equal to zero. That is, c is a *central* element. The Virasoro algebra is thus a *central extension* of the Witt algebra.

In any representation of this algebra, it is common to take the generator c to be a multiple of the identity operator. Then, reinterpreting notation a little, we think of c as this multiple. So c is then a number. This latter convention is standard in the physics literature and then c is called the *central charge* of the Virasoro algebra.

So we have shown that the reparametrization of our quantum mechanical string corresponds to a Virasoro algebra with central charge c = 1.

Now, to be more general we could think of a string vibrating in D dimensions by indexing our displacement as x^{μ} , $\mu = 1, ..., D$. We would then have oscillators α_n^{μ} obeying commutation relations

$$[\alpha_m^{\mu}, \alpha_n^{\nu}] = m \delta^{\mu,\nu} \delta_{m+n}. \tag{1.48}$$

It is then an easy matter to show that the resulting model has central charge

$$c = D. (1.49)$$

Indeed, the central charge is additive in the sense that if we take the combination of two (non-mutually-interacting) quantum mechanical systems with a Virasoro algebra then the combined system has a central charge equal to to sum of its parts. As such, the central charge is a measure of the "content" of a system.

So we think of our string theory having a D-dimensional "target space". But is this space or spacetime? We really have been cheating somewhat so far compared to a more careful analysis of string theory. We have been able to get away with a non-relativistic view of string oscillations and we have not really addressed time. The correct treatment is to view a two dimensional "worldsheet" Σ with local parameters σ and τ and map this into spacetime with coordinates x^{μ} , one of which is timelike. So we have a kind of time coordinate on the worldsheet and another time coordinate in spacetime. Furthermore we now have to worry about the bigger group of reparametrizations of the two-dimensional string worldsheet. Fortunately none of all this makes too much of a difference to our simplified presentation. We refer to a standard treatment such as [1] for the full story. What matters is that the interesting part of the reparametrization symmetry is still the (left and right-moving) Virasoro algebra.

Now we understand the reparametrization symmetry of the initial model, we need to treat it as a gauge symmetry and get rid of it. A naïve approach would be to look at the part of the Hilbert space invariant under the Virasoro algebra. Sadly this produces a completely empty theory. Instead we should play the game of living correctly in a Fock space picture of the Hilbert space. The nicest way of doing this is via BRST quantization which, sadly, we have nowhere near enough time to cover.

What BRST quantization does is to add more content to the theory in the form of "ghosts". These form their own representation of the Virasoro algebra but this time with c = -26. It is very satisfying that this central charge is negative as gauging a symmetry should be taking away degrees of freedom in some sense. Finally one shows that the content of the resulting theory is only nontrivial if the central charges of the ghosts and spacetime parts cancel and so we end up with the result that we require

$$D = 26. (1.50)$$

That is, this particular model of a fundamental string only works if spacetime has 26 dimensions.

Remember that finite constant we could add to L_0 ? It turns out that the BRST process fixes this constant too, but in such a way that the ground state $|0\rangle$ has negative energy squared. This is physically really bad and can be interpreted as predicting "tachyons". Fortunately the superstring that we see on day 3 fixes this.

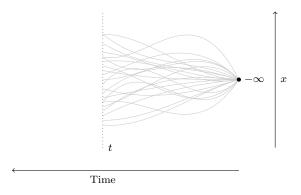


Figure 4: A Wavefunction in a Path Integral.

2 Day 2

2.1 Path Integrals

So far we have a non-interacting string. To get something more interesting we need some kind of field theory picture. In particular, we will describe two-dimensional conformal field theory, which is, in many ways, the simplest of all quantum field theories to analyze.

We begin with Feynman's interpretation of quantum mechanics via path integrals [5]. For a quantum mechanical particle living on a real line \mathbb{R} parametrized by x we have a wavefunction $\psi(x)$. $\psi(x)$ represents an amplitude while $|\psi(x)|^2$ represents the probability density of a particle being at x.

In the path integral approach we assume we have some kind of boundary condition lurking in the infinite past. Fixing a time t, we can then compute the wave function at this time by an integral

$$\psi_t(x') = \int_{-\infty}^{x',t} \mathcal{D}x \, e^{iS},\tag{2.1}$$

where we integrate over all possible paths starting in the infinite past and ending at our point of interest as shown in figure 4. Note we have time flowing from right to left in this picture, which we need to be consistent with the usual bra-ket notation. The weighting factor is associated to the *action* S of a given path. Analytically this path integral is awkward to define and we use the vague notation $\mathcal{D}x$ to denote the measure in this crazy integral.

We want to replace the notion of a particle with a closed string. So paths are replaced by cylinders to give something like figure 5. The string wavefunction is now a

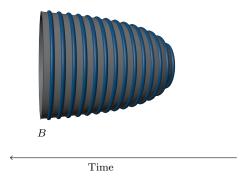


Figure 5: A String Worldsheet.

map

$$\psi: L(\mathbb{R}) \to \mathbb{C},\tag{2.2}$$

where $L(\mathbb{R})$ is the loopspace of \mathbb{R} . That is, a point in $L(\mathbb{R})$ is a map $S^1 \to \mathbb{R}$. Let's say we're interested in strings starting out at $t = -\infty$ and then we look at them at time t = 0. So we want to impose some boundary condition, B, at t = 0. We decompose this boundary conditions into Fourier modes. For t = 0 we put

$$x(e^{i\sigma}) = \sum_{n \in \mathbb{Z}} x_n e^{in\sigma}, \tag{2.3}$$

where $x_{-n} = \bar{x}_n$ as x is real. So, the path integral gives

$$\psi(x_0, x_1, \dots, x_n, \dots) = \int_{\mathcal{P}} \mathscr{D}x \, e^{iS}, \tag{2.4}$$

where we impose the boundary condition B given by (2.3) at time t = 0.

This is easier to picture if we use our complex coordinate $z = e^{i\sigma^+}$. Now $t = -\infty$ is mapped to the origin z = 0 while t = 0 is mapped to the circle |z| = 1. So our path integral is now over all *disks* centered at the origin with some fixed condition on the boundary of the disk.

The action S is given by the integral of the Lagrangian which is the kinetic energy minus the potential energy. This turns out to be given quite nicely as

$$S = \frac{1}{4\pi} \int \partial x \bar{\partial} x \, d^2 z. \tag{2.5}$$

Consider a classical solution x_{cl} which minimizes the action for a given boundary condition at t = 0. We can then write

$$x = x_{\rm cl} + x', \tag{2.6}$$

where x' represents a quantum fluctuation around this classical value.

The Euler–Lagrange equations for this action amount to $\partial \bar{\partial} x = 0$ and so we can write

$$x_{\rm cl} = x_0 + \sum_{n>0} (x_n z^n + x_{-n} \bar{z}^n), \tag{2.7}$$

where the constants x_n are fixed by the boundary condition. We then have

$$S = \frac{1}{4\pi} \int \partial x \bar{\partial} x \, dz d\bar{z}$$

$$= \frac{1}{4\pi} \int \partial x_{\text{cl}} \bar{\partial} x_{\text{cl}} \, dz d\bar{z} + \frac{1}{4\pi} \int \partial x' \bar{\partial} x' \, dz d\bar{z},$$
(2.8)

since minimizing the action removes cross terms. Integrating over |z| < 1 we explicitly obtain the classical contribution to the action

$$\frac{1}{4\pi} \int \partial x_{\text{cl}} \bar{\partial} x_{\text{cl}} dz d\bar{z} = \frac{1}{4\pi} \sum_{n>0, m>0} nm x_n x_{-m} \int z^{n-1} \bar{z}^{m-1} dz d\bar{z}
= \frac{1}{4\pi} \sum_{n>0, m>0} nm x_n x_{-m} \int_0^{2\pi} \int_0^1 r^{n+m-2} e^{i(n-m)\theta} 2r dr d\theta
= \frac{1}{2} \sum_{n>0} n|x_n|^2,$$
(2.9)

where we used $z = re^{i\theta}$.

This implies that

$$\psi(x_0, x_1, \ldots) = C_0 \exp\left(-\frac{1}{2} \sum_{n>0} n|x_n|^2\right),$$
 (2.10)

where

$$C_0 = \int_{B_0} \mathcal{D}x' \exp\left(-\frac{1}{4\pi} \int \partial x' \bar{\partial}x' \, dz d\bar{z}\right)$$
 (2.11)

is the integral over all worldsheets which look like a disk $|z| \leq 1$ with boundary condition B_0 given by x' = 0 at |z| = 1. We have no idea how to compute C_0 but it does not depend on the x_n 's so just gives some constant factor in the wave function.

This is great! We've managed to get the interesting part of the path integral, that is the part that depends on the boundary condition at t = 0, while hiding the nasty part in some normalization we don't really care about.

So the path integral with boundary modes x_n yields

$$\psi(x_0, x_1, \dots) = C_0 \exp\left(-\frac{1}{2} \sum_{n>0} n|x_n|^2\right).$$
 (2.12)

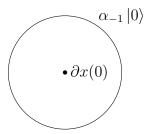


Figure 6: A vertex operator creates a state at |z| = 1.

But now recall that the wavefunction of a harmonic oscillator is generated by the basis of Hermite polynomials and is given by

$$\psi(x) = H_n(x)e^{-\omega x^2}. (2.13)$$

So we have magically discovered the string wavefunction (2.12) at t = 0 where every oscillator mode is in its ground state. This is kind of weird as we never explicitly needed to put any boundary condition at $t = -\infty$. Somehow the path integral picture "wants" to do the right thing and produce the state $|0\rangle$ at t = 0 in the absence of any other information.

For a bit more magic, let's now stick $\partial x(0)$ into the path integral:

$$\int_{B} \mathscr{D}x \, e^{-S} \, \partial x(0). \tag{2.14}$$

But

$$\partial x_{cl}(0) = x_1, \tag{2.15}$$

and the x' contribution is unaffected. So the path integral evaluates to

$$\psi(x_0, x_1, \dots) = C_0 x_1 \exp\left(-\frac{1}{2} \sum_n |x_n|^2\right).$$
 (2.16)

But x_1 corresponds to the first Hermite polynomial and so the lowest mode of the lowest frequency oscillator is excited. This is α_{-1} excitation. So somehow the function $\partial x(0)$ produced this excitation back at $t = -\infty$. The function $\partial x(0)$ creates a state as shown in figure 6.

Similarly we may insert

$$\partial^2 x(0), \tag{2.17}$$

to give the α_{-2} excitation, etc.

2.2 Vertex Operator Picture

Notice that we are inserting nothing more than simple *functions* into the path integral to produce new states. We get a very similar picture by thinking of *operators* on our Hilbert space. Let's temporarily use a hat to emphasize operators over functions. So we have

$$i\partial \hat{x}(z) = \sum_{n \in \mathbb{Z}} \frac{\alpha_n}{z^{n+1}},\tag{2.18}$$

which, when applied to the ground state yields

$$i\partial\hat{x}(0)|0\rangle = \alpha_{-1}|0\rangle. \tag{2.19}$$

So the operator $i\partial \hat{x}(0)$ produced an excitation just like the function in the path integral did. We will therefore follow the usual physics convention, confuse these two notions and freely call functions operators, and, indeed also use the moniker "fields" too!

What happens if we instead insert the following operator into the path integral

$$i\partial x(z)$$
? (2.20)

Since $i\partial x(0)$ produced a state at z=0, we will assume the above "creates" this state at some arbitrary point z in the worldsheet. We thus produce a kind of two-dimensional quantum field theory. We can create new things by "inserting" functions into the path integral. Things like $i\partial x(z)$ used this way are often called "vertex operators".

So far we have looked at the past up to some time. To get a complete story we should continue forward to $t = +\infty$. We thus consider the future vacuum $\langle 0|$ to be the adjoint of our past $|0\rangle$. So we propagate from the past to some boundary condition B at t = 0 and then forward to the infinite future. This produces an inner product

$$\langle 0|0\rangle = \int_{\text{all possible } B's} |\psi_1|^2 dx_1 dx_2 \dots = \int_{\text{all spheres}} \mathscr{D}x \, e^{-S}. \tag{2.21}$$

We have no idea how to actually compute this but, we can choose to normalize it by asserting $\langle 0|0\rangle = 1$.

Next we may put some vertex operator into this and compute, for example,

$$\langle 0 | i\partial x(z) | 0 \rangle = \left\langle 0 \left| \sum_{n \in \mathbb{Z}} \frac{\alpha_n}{z^{n+1}} \right| 0 \right\rangle = 0,$$
 (2.22)

since every α_n for non-negative n kills $|0\rangle$ and, for non-positive n, it kills $\langle 0|$ to the left. Often one takes the vacuum states as implicit and writes this "one-point" function as

$$\langle \partial x(z) \rangle = 0. \tag{2.23}$$

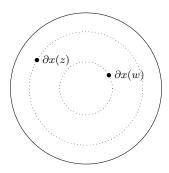


Figure 7: A Product of Vertex Operators

More interestingly, next compute two vertex operators as in figure 7.

$$\langle \partial x(z)\partial x(w)\rangle = -\sum_{m>0,n<0} \frac{\langle 0 | \alpha_m \alpha_n | 0 \rangle}{z^{m+1}w^{n+1}}$$

$$= -\sum_{m>0,n<0} \frac{m\delta_{m+n}}{z^{m+1}w^{n+1}} = -\sum_{m>0} \frac{m}{zw} \left(\frac{w}{z}\right)^m$$

$$= -\frac{1}{(z-w)^2},$$
(2.24)

assuming |w| < |z|. This assumption is correct as w appears to the right of z in our path integral and so w happens "first". In complex plane language this time ordering is exactly |w| < |z| as shown in the figure.

2.3 The Operator Product Expansion

Note if we write the above product of vertex operators naked outside a path integral we get

$$\partial \hat{x}(z)\partial \hat{x}(w) = -\sum_{m \in \mathbb{Z}} \frac{m}{zw} \left(\frac{w}{z}\right)^m, \qquad (2.25)$$

which *nowhere* converges. So this product of operators really does not have literal validity.

So we can either be careful to implicitly "dress" our products carefully inside $\langle \ldots \rangle$ or we can deal with noncovergent power series by treating them formally. This latter point of view leads to the mathematical formulation of "Vertex Operator Algebras". We refer to [6], for example, for a full treatment. We will not pursue this formal language here. Instead we assume operator products have some kind of intrinsic existence manifested by what happens when they are put in a path integral.

That said, the above is an example of an "operator product expansion", which might look typically like

$$\mathcal{O}_i(z)\mathcal{O}_j(w) = \sum_{n,k} \frac{c_{ij}^{k,n} \mathcal{O}_k(w)}{(z-w)^k},$$
(2.26)

for constants $c_{ij}^{k,n}$. Note that in particular cases there may additionally be terms like $\log(z-w)$ and/or fractional powers.

A key fact is that there are relationships between OPE's and commutation relations. This relates the "canonical commutator" picture of quantum mechanics (1.2) to the path integral. The order in which we write operators such as α_n becomes time ordering in the path integral, which, in turn, becomes radial ordering in the complex plane. For example, given the above operator product we can reproduce $[\alpha_m, \alpha_n]$ using complex analysis as follows. The expansion (1.25) gives

$$\alpha_n = \frac{1}{2\pi} \oint_C z^n \partial x(z) dz, \qquad (2.27)$$

where C encloses the origin once, and thus,

$$[\alpha_m, \alpha_n] = \frac{1}{4\pi^2} \oint \oint_{|w| < |z|} z^m w^n \partial x(z) \partial x(w) \, dz \, dw - \frac{1}{4\pi^2} \oint \oint_{|w| > |z|} z^m w^n \partial x(z) \partial x(w) \, dz \, dw,$$

$$(2.28)$$

where each integral contour encloses the original once under the inequalities shown. If we assume the only singularities of the operator product occur at z = w, we may change the z-contour for a fixed value of w as in figure 8. Using (2.24) we recover

$$[\alpha_{m}, \alpha_{n}] = -\frac{1}{4\pi^{2}} \oint_{w=0} dw \oint_{z=w} \frac{z^{m} w^{n}}{(z-w)^{2}} dz$$

$$= \frac{1}{2\pi i} \oint_{w=0} m w^{m+n-1} dw$$

$$= m \delta_{m+n}.$$
(2.29)

Note that anything nonsingular in the operator product will not contribute to the contour integrals and so does not affect the commutator. Therefore, the above calculation only determines the operator product to leading order:

$$\partial x(z)\partial x(w) = -\frac{1}{(z-w)^2} + \dots, \qquad (2.30)$$

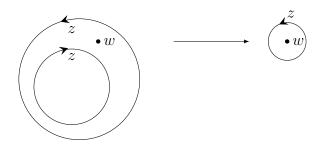


Figure 8: Change of contours.

where the ellipses denote something finite as $z \to w$. Such use of ellipses is standard in operator product notation. Note, however, that these finite terms can still be important. If we compute a correlation function with more than two vertex operators, finite terms can lead to singular ones by the further operator products. This all adds to the complications of the vertex operator picture.

We can do the analogue of normal ordering for operator products:

$$:\mathcal{O}(z)\mathcal{O}(w):=\mathcal{O}(z)\mathcal{O}(w)-(\text{singular poles as }z\to w), \tag{2.31}$$

and define

$$:\mathcal{O}(z)\mathcal{O}(z):=\lim_{z\to w}:\mathcal{O}(z)\mathcal{O}(w):. \tag{2.32}$$

Notice we haven't used the unadorned field x(z) itself as a vertex operator. While it can be, the logarithm in (1.23) produces cuts that make it awkward. Differentiating rids us of this, but we can also exponentiate. This implies it is interesting to consider

$$V(z) = e^{ikx(z)} (2.33)$$

Indeed, this is the vertex operator that creates a string state with center of mass momentum p = k. We'll implicitly assume normal ordering in this operator from now on. One can derive the following products:

$$e^{ikx(z)} e^{ilx(z)} = (z - w)^{kl} e^{ikx(z) + ilx(w)}$$

$$i\partial x(z) e^{ikx(w)} = \frac{k}{z - w} e^{ikx(z)} + \dots,$$
(2.34)

which we will need later.

Another operator of note is T(z) whose Fourier modes are built out of the L_m 's.

$$T(z) = \sum_{m \in \mathbb{Z}} \frac{L_m}{z^{m+2}}.$$
(2.35)

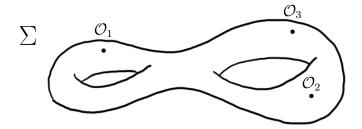


Figure 9: A Riemann surface with vertex operators at marked points.

This is known as the "stress (energy) tensor".

The Virasoro algebra then becomes

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$
 (2.36)

2.4 Conformal Field Theory

We can now sort of give a very rough outline of a "definition" of conformal field theory. The data is a collection of vertex operators $\mathcal{O}_1, \mathcal{O}_2, \ldots$, together with *correlation functions* computed as follows. Given a fixed Riemann Surface Σ we can insert any vertex operators at points on this surface and compute the path integral

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\dots\mathcal{O}_j(z_j)\dots\rangle\rangle_{\Sigma}\in\mathbb{C}.$$
 (2.37)

One of the vertex operators in our collection is T(z). Everything must then be invariant under the action of the resulting conformal symmetry. For example, L_{-1} must correctly generate translations leading to "Ward" identities. Also, if Σ is a sphere then all correlation functions are invariant under the Möbius group. An immediate consequence, for example, is that a suitably normalized self-adjoint vertex operator will always have a product

$$\mathcal{O}(z)\mathcal{O}(w) = \frac{1}{(z-w)^{2h}} + \dots, \tag{2.38}$$

where h is the weight of \mathcal{O} . All of this is carefully explained in [4] and we need not reproduce it here.

Another consideration is that Riemann surfaces can degenerate into connected sums of other Riemann surfaces. Thus one can make statements about how the correlation functions vary as we move in the moduli space of (marked) Riemann surface. Again, we need not pursue this here.

Finally, of course, we have only talked about the *holomorphic* dependence of z and have thus completely ignored the right-moving string modes. To get to final answers one needs to assemble the holomorphic and antiholomorphic parts together.

A correlation function of particular interest is that of a genus one Riemann surface with no operators inserted. This is a function of the complex structure of the Riemann surface alone and is thus a relatively simple object. It is called the "partition function". Again, a sensible course on conformal field theory would have much to say about this but we need to move on.

3 Day 3

3.1 Fermions

Everything we need to know about the "bosonic" oscillator modes of our string we have considered so far can be derived from the commutator

$$[\alpha_m, \alpha_n] = m\delta_{m+n}. (3.1)$$

We'd now like to consider some new kind of degree of freedom our string may exhibit. We'll denote its modes by a_r . So let's try the simplest thing one might think of:

$$[a_r, a_s] = \delta_{r+s}. (3.2)$$

Now, if we exchange r and s the right-hand-side of this equation is invariant. So this cannot be a commutator. Instead, the obvious thing to do is to declare it to be an anticommutator:

$$[a_r, a_s] = a_r a_s + a_s a_r. (3.3)$$

We thus say that these a_r excitations correspond to a *fermion*. Some people like using braces for anticommutators but one can write general equations more easily if you use the convention that [A, B] is a commutator unless both A and B are fermions in which case it is an anticommutator.

If we assume a_r comes from a primary field we can find its weight by computing the Jacobi identity (with the correct signs allowing for fermions):

$$[L_m, [a_r, a_s]] - [a_s, [L_m, a_r]] + [a_r, [a_s, L_m]] = 0.$$
(3.4)

From (3.2) and (1.36) this yields

$$m(2h-1)\delta_{m+r+s} = 0, (3.5)$$

which fixes $h=\frac{1}{2}$. So we define our new fermion vertex operator as

$$\psi(z) = \sum_{r} \frac{a_r}{z^{r+1/2}},\tag{3.6}$$

and we get an operator product

$$\psi(z)\psi(w) = \frac{1}{z - w} + \text{finite as } z \to w.$$
 (3.7)

It's fun to note we have a very simple picture of a two-dimensional spin-statistics theorem at work here. We see from (2.38) that exchanging vertex operators changes sign or not whether h is in \mathbb{Z} or $\mathbb{Z} + \frac{1}{2}$. Consistent with this, the bosons $i\partial x(z)$ have h = 1 and these fermions have $h = \frac{1}{2}$.

Now, we see from (3.6) that either $r \in \mathbb{Z}$ and $\psi(z)$ has a branch cut at the origin, or $\psi(z)$ is single-valued and $r \in \mathbb{Z} + \frac{1}{2}$. So, which is more natural? It turns out that we need to consider both possibilities, leading to two *sectors* of fermions. They are named as

$$r \in \begin{cases} \mathbb{Z} + \frac{1}{2} & \text{Neveu Schwarz (NS)} \\ \mathbb{Z} & \text{Ramond (R)}. \end{cases}$$
 (3.8)

Following the process we did earlier for the boson, we find Virasoro generators giving the correct commutation relations with a_r from

$$L_n = \frac{1}{2} \sum_r r : a_{-r} a_{n+r};, \tag{3.9}$$

which is equivalent to

$$T(z) = -\frac{1}{2} : \psi(z)\partial\psi(z):. \tag{3.10}$$

Again, a careful computation shows that this then obeys the Virasoro algebra, but this time the central charge is $c = \frac{1}{2}$, in either sector. (This works perfectly for the NS sector but for the Ramond sector we also see a shift in L_0 by c/24.) So these fermions are somehow "worth" one half of a boson!

To construct the spectrum, we first need a vacuum. There is a stark difference between the NS vacuum and the R vacuum which will be absolutely key for us.

The NS vacuum is easy enough to define since r is not integral:

$$a_r |0\rangle_{NS} = 0 \text{ for } r > 0.$$
 (3.11)

We can declare this to be the unique NS vacuum and the lowest excited state is $a_{-\frac{1}{2}}|0\rangle_{\rm NS}$, etc.

For the Ramond sector we need to work harder because a_0 does not change the energy. Consider D dimensions by using operators a_r^{μ} , for $\mu = 1, \ldots, D$. We have

$$[a_0^{\mu}, a_0^{\nu}] = \delta^{\mu,\nu}. \tag{3.12}$$

This is well-known in mathematics as a Clifford Algebra on \mathbb{R}^D (with standard inner product).

We cannot say a_0^{μ} kills the vacuum as this is inconsistent with $(a_0^{\mu})^2 = \frac{1}{2}$, and we cannot say a_0^{μ} leaves the vacuum invariant (or rescales it) as that is inconsistent with the Clifford algebra for $\mu \neq \nu$. So the Ramond vacuum must be multidimensional (assuming D > 1).

We can build things like

$$\dots a_0^{\mu_3} a_0^{\mu_2} a_0^{\mu_1} |0\rangle_R, \qquad (3.13)$$

choosing to include any a_0^{μ} or not and these vectors in the Hilbert space all have the same energy. If all such possibilities are considered linearly independent we would construct a 2^D -dimensional vacuum, but this turns out to be rather redundant. To be precise, it is *reducible*. There is only one irreducible representation of the Clifford algebra (assuming D is even) and that corresponds to a Dirac spinor of dimension $2^{\frac{D}{2}}$ (see, for example, [7]). As such, the Ramond vacuum corresponds to a spacetime *spinor*. In contrast the NS vacuum is a spacetime *scalar* whilst the NS states

$$a_{-\frac{1}{2}}^{\mu} |0\rangle_{\rm NS} ,$$
 (3.14)

form a spacetime vector.

It is essential here to keep track of whether we are talking about the worldsheet or spacetime when it comes to bosons and fermions. We have bosons $\partial x(z)$ and fermions $\psi(z)$ on the worldsheet, whereas spacetime bosons are in the NS sector and spacetime fermions (i.e., spinors) are in the R sector.

3.2 Supersymmetry

Let's have both α_n^{μ} and a_r^{μ} degrees of freedom for our *D*-dimensional string. So the Virasoro generators are the sum of (1.44) and (3.9). Define the fermionic object

$$G_r = \sum_{n,\mu} \alpha^{\mu}_{-n} a^{\mu}_{n+r},\tag{3.15}$$

(which may be in the NS or R sector). Then (grading the brackets correctly)

$$[G_r, \alpha_m^{\mu}] = -ma_{r+m}^{\mu}, \qquad [G_r, a_s^{\mu}] = \alpha_{r+s}^{\mu}.$$
 (3.16)

This is therefore a "worldsheet supersymmetry". It exchanges worldsheet fermions with worldsheet bosons.

One can then show (in the NS sector) that this obeys the "N = 1 superconformal algebra" defined by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n}$$

$$[L_m, G_r] = (\frac{1}{2}m - r)G_{m+r}$$

$$[G_r, G_s] = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s},$$
(3.17)

with $c = \frac{3}{2}D$. (Again, that last term in the third equation is a good exercise in treating normal ordering correctly.) The N = 1 refers to the fact we only have one G. We'll get a second one shortly. Alternatively, introducing the "supercurrent"

$$G(z) = \sum_{r} \frac{G_r}{z^{r+3/2}},$$
(3.18)

this algebra can be written as an operator product statement as (2.36) together with

$$T(z)G(w) = \frac{\frac{1}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \dots$$

$$G(z)G(w) = \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2T(w)}{z-w} + \dots$$
(3.19)

If we merely added these fermionic degrees of freedom to our reparametrization-invariant string theory we'd need $\frac{3}{2}D = 26$, which we can't solve for integral D. Instead we define the *superstring* as the model having the N=1 superconformal algebra as a *gauge symmetry*. So, we have some kind of super-reparametrization invariance. This requires adding in "superghosts" to the BRST process which raises a whole new collection of subtleties.

The superghosts themselves have a central charge of 11 so now we require

$$\frac{3}{2}D - 26 + 11 = 0, (3.20)$$

and we obtain the well-known result that the superstring likes to live in 10 dimensions. Modular invariance and a peculiar fact about the ground state of the superghosts force a thing called the GSO projection to be taken onto odd fermion numbers and that very nicely removes the offending tachyon we noted earlier. The full story for this doesn't appear until volume 2 of Polchinski [2] and so we certainly do not have enough time to dig deeper into this here.

3.3 Compactification

If we wanted to claim that the universe is described by superstring theory we need to explain away this issue of 10 dimensions. If D were equal to 4 our net central charge would be -9. So what we need to do is to introduce more "stuff" into our conformal field theory to add 9 to c. Assuming everything factors nicely, we can treat this stuff as its own c = 9 conformal field theory.

So what we need to describe a "realistic" superstring theory is some c=9 conformal field theory. One way to interpret this theory is as providing the 6 "missing dimensions". In other words, ten-dimensional spacetime is viewed as four-dimensional spacetime times some six-dimensional space X that is "compact" enough that we wouldn't notice it. As such this process is called "compactification".

It should be emphasized that at no time did string theory demand that our c=9 conformal field theory have any kind of geometric interpretation. That said, all the fruitful applications of string theory to geometry are based on this assumption.

From now on we define d as follows. Let us say that the conformal field theory we are using to compactify has central charge c = 3d. This means we are compactifying down to 10 - 2d observable non-compactified spacetime and any putative compactification space X is of (real) dimensional 2d.

3.4 Spacetime Supersymmetry

As we've described things so far, we have one vacuum $|0\rangle_{NS}$ for the NS sector and and a whole subspace of vacua $|0\rangle_{R}$ for the R sector. We can describe this more succinctly using the vertex operator language. Let the vacuum $|0\rangle$ be the NS vacuum. This is, after all, unique. Then suppose we have a vertex operator $\Sigma^{+}(z)$ that creates a direction in the R-vacuum from this.

This means that $\Sigma^+(z)$ must create a branch cut in the complex plane, as in figure 10, so that fermions pick up a sign as we cross the cut and bosons do not. Our operator product with fermions must therefore contain square roots, something like

$$\psi^{\mu}(z)\Sigma^{+}(w) = \frac{\Xi(w)}{\sqrt{z-w}} + \dots,$$
 (3.21)

whereas the operator product between Σ^+ and bosons should be single-valued.

This operator Σ^+ transforms the NS sector to the R sector. As such, it takes spacetime bosons to fermions and so is a *spacetime supersymmetry*.

Let Σ^- be the adjoint of Σ^+ . We then claim

$$\Sigma^{+}(z)\Sigma^{-}(w) = \frac{1}{(z-w)^{d/4}} + \frac{J(w)}{(z-w)^{d/4-1}} + \text{less singular},$$
(3.22)

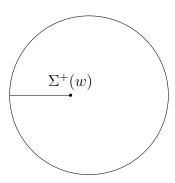


Figure 10: A vertex operator creates a branch cut.

where J(w) is an h=1 vertex operator. This is proven as follows. That shift in the L_n commutation relations we noted earlier for the R vacuum means Σ^{\pm} has conformal weight d/8. That accounts for the first term above. We then argue that the branch cuts in the product must be consistent so that we get single-valued correlation functions at the end of the day. This implies all powers of z-w must differ by an integer. This yields the second term so long as we can show that term in nonzero. That can be shown by looking at 4-point functions of Σ^{\pm} [8]. Furthermore, this 4-point function yields

$$J(z)J(w) = \frac{d}{(z-w)^2} + \dots$$
 (3.23)

So, to recap here, the existence of spacetime supersymmetry has required us to have a new field with h = 1:

$$J(z) = \sum_{n} \frac{J_n}{z^{n+1}}. (3.24)$$

Any h = 1 field is a "current" in the sense that it has a conserved change

$$J_0 = \frac{1}{2\pi i} \oint_C J(z)dz, \tag{3.25}$$

so that if a field A(z) obeys

$$J(z)A(w) = \frac{q(A)A(w)}{z - w} + \dots,$$
 (3.26)

then A(z) is said to have charge q(A), and interactions will conserve charges by a simple contour argument as in figure 11. (Our weight one field $i\partial x(z)$ has a conserved charge given by center of mass momentum.)

This new charge, measured by J_0 , is called "R-charge".

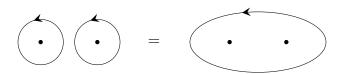


Figure 11: Conservation of Charge.

Now we have a new field J(z) we should add it to our super-Virasoro algebra. The only ingredient we are missing is the charge of G(z) with respect to our new current. One can argue

$$G(z) = \frac{1}{\sqrt{2}} \left(G^{+}(z) + G^{-}(z) \right), \tag{3.27}$$

where G^{\pm} has R-charge ± 1 .

All in all, this gives a new algebra with generators T(z), $G^{\pm}(z)$ and J(z) and operator products

$$T(z)T(w) = \frac{\frac{3}{2}d}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

$$T(z)G^{\pm}(w) = \frac{\frac{3}{2}G^{\pm}(w)}{(z-w)^2} + \frac{\partial G^{\pm}(w)}{z-w} + \dots$$

$$T(z)J(w) = \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} + \dots$$

$$J(z)G^{\pm}(w) = \pm \frac{G^{\pm}(w)}{z-w} + \dots$$

$$J(z)J(w) = \frac{d}{(z-w)^2} + \dots$$

$$G^{+}(z)G^{+}(w) = G^{-}(z)G^{-}(w) = 0$$

$$G^{+}(z)G^{-}(w) = \frac{2d}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{z-w} + \dots$$
(3.28)

This is called the "N=2 superconformal algebra". So spacetime supersymmetry implies this N=2 algebra. If we demand a similar condition in the right-moving sector we get an antiholomorphic copy of this generated by $\tilde{T}(z)$, $\tilde{G}^{\pm}(z)$ and $\tilde{J}(z)$ to give an N=(2,2) superconformal field theory.

It is important to note that we do *not* gauge this N=2 superconformal algebra. No further ghosts or superghosts are added to the N=1 system and so no central charge conditions are modified.

We can play a useful trick with vertex operator algebras to understand the R charges more. Let's assume we can integrate J(z) to form a new boson $\phi(z)$. If this

boson behaves just like x we need to set

$$J(z) = i\sqrt{d}\partial\phi(z),\tag{3.29}$$

to get normalizations right. It follows from (2.34) that any field of the form

$$A(z) = e^{\frac{ij}{\sqrt{d}}\phi(z)}A'(z), \tag{3.30}$$

where A'(z) has no ϕ -dependence, has R-charge equal to j. One may choose

$$\Sigma^{\pm}(z) = e^{\pm \frac{i}{2}\sqrt{d}\phi(z)},\tag{3.31}$$

and so, again from (2.34), it follows that

$$\Sigma^{+}(z)A(w) = (z - w)^{\frac{j}{2}}....$$
(3.32)

So, the rule for how the operator Σ^+ makes branch cuts with respect to other operators becomes

$$j = \begin{cases} \text{even for bosons} \\ \text{odd for fermions} \end{cases}$$
 (3.33)

In other words, the \mathbb{Z}_2 -grading of fermion number is extended to a \mathbb{Z} -grading for the R-charge. So spacetime supersymmetry requires both an N=2 superconformal algebra, together with the above R-charge quantization condition.

The fact that R-charge gives this \mathbb{Z} -grading is a profound statement in many modern applications of conformal field theory and string theory to geometry. If there is some kind of \mathbb{Z} -grading (coming from, say, a differential graded algebra) it ultimately comes from something like the spacetime supersymmetry condition. Without it, you probably only have a \mathbb{Z}_2 -grading.

4 Day 4

4.1 Chiral Primary Fields

Consider an N = (2, 2) superconformal field theory producing spacetime supersymmetry as we discussed in the previous lecture. The last of the products in (2.34) can be written

$$[G_r^+, G_s^-] = 2L_{r+s} + (r-s)J_{r+s} + d(r^2 - \frac{1}{4})\delta_{r+s}. \tag{4.1}$$

Suppose $|\psi\rangle$ has eigenvalues h and j for L_0 and J_0 respectively. Then

$$\left\langle \psi \left| [G_{1/2}^{-}, G_{-1/2}^{+}] \right| \psi \right\rangle = \left\| G_{-1/2}^{+} |\psi\rangle \right\|^{2} + \left\| G_{1/2}^{-} |\psi\rangle \right\|^{2}$$

$$= (2h - j) \||\psi\rangle\|^{2}.$$
(4.2)

So $j \leq 2h$ with equality if and only if

$$G_{1/2}^{-} |\psi\rangle = G_{-1/2}^{+} |\psi\rangle = 0.$$
 (4.3)

This is called a "chiral primary field" [9]. We may similarly prove $-j \leq h$ with equality when

$$G_{1/2}^{+} |\psi\rangle = G_{-1/2}^{-} |\psi\rangle = 0.$$
 (4.4)

defining an "antichiral primary field". Furthermore, by considering the case $r = \frac{3}{2}$ it is easy to show

$$-d \le j \le d. \tag{4.5}$$

The same conditions hold in the right-moving sector with the same statements for \bar{h} and $\bar{\jmath}$. Fields which are chiral primary in both sectors are called (c,c)-fields, etc.

Let $k^{j,\bar{j}}$ be the dimension of the space of (c,c) states with eigenvalues of J_0 , \tilde{J}_0 equal to j,\bar{j} respectively. Note we have already shown that j and \bar{j} are integers, so we have a finite set of numbers. It is traditional to arrange them in a diamond:

$$k^{0,0}$$

$$k^{1,0} k^{0,1}$$

$$k^{2,0} k^{1,1} k^{0,2}$$

$$\vdots$$

$$k^{d,0} k^{d-1,1} \cdots k^{0,d}$$

$$\vdots$$

$$k^{d,d-1} k^{d-1,d}$$

$$k^{d,d}$$

$$k^{d,d}$$

$$(4.6)$$

The product of $\Sigma^+(z)$ with itself produces a useful field

$$\Omega(z) = e^{i\sqrt{d}\phi(z)},\tag{4.7}$$

which itself is chiral primary of charge d. Furthermore it is the only such chiral primary field. It follows that $d \in \mathbb{Z}$, and we determine the corners of our diamond:

$$k^{0,0} = k^{d,0} = k^{0,d} = k^{d,d} = 1.$$
 (4.8)

The product of $\Omega(z)$ with an antichiral primary field of charge -j yields a chiral primary field of charge d-j. The adjoint of a (c,c)-field of charge (j,\bar{j}) is an (a,a)-field of charge $(-j,-\bar{j})$ and we can then use $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ to map this to a (c,c)-field of charge $(d-j,d-\bar{j})$. This is an invertible process. It follows that

$$k^{j,\bar{\jmath}} = k^{d-j,d-\bar{\jmath}}. (4.9)$$

If we have some kind of holomorphic-antiholomorphic symmetry on the worldsheet, then it might follow that $k^{j,\bar{j}} = k^{\bar{j},j}$. Then our diamond has left-right and up-down symmetry. This is almost (but not quite) always the case in any example in the literature.

4.2 The Torus

The simplest case is d = 1 and then our diamond is completely determined:

We can construct this conformal field theory as follows. Take two bosons $x^{1,2}$ and two fermions $\psi^{1,2}$ (plus $\tilde{\psi}^{1,2}$ for right movers).

Define

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^1 \pm ix^2), \qquad \psi^{\pm} = \frac{1}{\sqrt{2}}(\psi^1 \pm i\psi^2).$$
 (4.11)

That is, we use complex coordinates on the *target space* now as well as the worldsheet. If we impose periodic conditions, we can thus view x^+ as the coordinate on a complex torus of complex dimension 1.

Now we obtain our desired N = (2, 2) superconformal field theory with

$$T(z) = -\partial x^{+}(z)\partial x^{-}(z) - \frac{1}{2}\psi^{+}(z)\partial\psi^{-}(z) - \frac{1}{2}\psi^{-}(z)\partial\psi^{+}(z)$$

$$G^{\pm}(z) = i\sqrt{2}\partial x^{\mp}(z)\psi^{\pm}(z)$$

$$J(z) = \psi^{+}(z)\psi^{-}(z),$$
(4.12)

and similarly for the antiholomorphic sector.

Note that $\psi^+(z)$ gives $k^{1,0}=1$ and $\tilde{\psi}^+(z)$ gives $k^{0,1}=1$. In this case $\Omega(z)=\psi^+(z)$.

To get a higher value of d, we can simply take d copies of this. The cleanest notation for this is to use x^i , i = 1, ..., d for the holomorphic coordinates and $x^{\bar{\imath}}$, $\bar{\imath} = 1, ..., d$ for the antiholomorphic coordinates. Again, we need to emphasize the distinction between worldsheet and target space. Complex conjugation on the worldsheet would take ∂x^i to $\bar{\partial} x^i$, whereas complex conjugation in the target space takes ∂x^i to $\partial x^{\bar{\imath}}$!

We can now build (c,c) fields by taking combinations of ψ^i 's and $\tilde{\psi}^i$'s (which anticommute but otherwise do not interact with each other). For example, taking 3 copies of our complex torus we get d = 3 and a (c,c)-diamond

4.3 The Hodge Diamond

Anyone familiar with complex geometry will instantly recognize (4.13) as the "Hodge Diamond" of the complex 3-torus. Let us quickly review what this is. A complex manifold X has local homolorphic coordinates x^i and antiholomorphic coordinates $x^{\bar{\imath}}$. A differential form of type (p,q) is of the form

$$\omega = f(x^{i}, x^{\bar{i}}) dx^{i_{1}} dx^{i_{2}} \dots dx^{i_{p}} dx^{\bar{i}_{1}} dx^{\bar{i}_{1}} \dots dx^{\bar{i}_{q}}, \tag{4.14}$$

where the differentials dx anticommute. Let $\Omega^{p,q}$ be the space of all such forms. We can then form a double complex

with differentials

$$\partial \omega = \sum_{j} \frac{\partial f(x^{i}, x^{\bar{\imath}})}{\partial x^{j}} dx^{j} dx^{i_{1}} dx^{i_{2}} \dots dx^{i_{p}} dx^{\bar{\imath}_{1}} dx^{\bar{\imath}_{1}} \dots dx^{\bar{\imath}_{q}}$$

$$\bar{\partial} \omega = \sum_{\bar{\jmath}} \frac{\partial f(x^{i}, x^{\bar{\imath}})}{\partial x^{\bar{\jmath}}} dx^{\bar{\jmath}} dx^{i_{1}} dx^{i_{2}} \dots dx^{i_{p}} dx^{\bar{\imath}_{1}} dx^{\bar{\imath}_{1}} \dots dx^{\bar{\imath}_{q}}.$$

$$(4.16)$$

By a construction called the Frölicher spectral sequence, this computes the de Rham cohomology of X with respect to a total differential

$$d = \mathbf{d} + \bar{\mathbf{d}},\tag{4.17}$$

with a filtration that determines the *Hodge Numbers* $h^{p,q}$ as the dimension of the spaces in this grid after applying cohomology with respect to $\bar{\mathbf{d}}$ and $\bar{\mathbf{d}}$ (i.e., the E_2 stage of the spectral sequence, where it degenerates). We refer to [10] for a nice account of this.

So the question is, can we relate our (c,c)-dimensions $k^{j,\bar{j}}$ to the Hodge numbers $h^{p,q}$? The first thing to note is that the two superscripts in $k^{j,\bar{j}}$ are swapped by world-sheet complex conjugation whilst the two superscripts in $h^{p,q}$ are swapped by target space complex conjugation, and we have emphasized these are not the same thing! So we need to combine them somehow. The trick is to consider (c,a) fields of the form

$$A(z,\bar{z}) = f(x^i, x^{\bar{\imath}})\psi^{i_1}\psi^{i_2}\dots\psi^{i_p}\tilde{\psi}^{\bar{\imath}_1}\tilde{\psi}^{\bar{\imath}_1}\dots\tilde{\psi}^{\bar{\imath}_q}, \tag{4.18}$$

Now, if we define the worldsheet supersymmetry action

$$Q^{+}\mathcal{O}(w) = \frac{1}{2\sqrt{2}\pi} \oint_{z=w} G^{+}(z)\mathcal{O}(w) dz.$$
 (4.19)

Then

$$Q^{+}x^{i} = \psi^{i}, Q^{+}x^{\bar{\imath}} = 0, Q^{+}\psi^{i} = 0, Q^{+}\psi^{\bar{\imath}} = \partial x^{\bar{\imath}}, Q^{+}\tilde{\psi}^{i} = 0, Q^{+}\tilde{\psi}^{\bar{\imath}} = 0, (4.20)$$

with the obvious generalizations to Q^- and \widetilde{Q}^{\pm} . Then

$$Q^{+}A(z,\bar{z}) = \sum_{j} \frac{\partial f(x^{i},x^{\bar{\imath}})}{\partial x^{j}} \psi^{j} \psi^{i_{1}} \psi^{i_{2}} \dots \psi^{i_{p}} \tilde{\psi}^{\bar{\imath}_{1}} \tilde{\psi}^{\bar{\imath}_{1}} \dots \tilde{\psi}^{\bar{\imath}_{q}}$$

$$\tilde{Q}^{-}A(z,\bar{z}) = \sum_{\bar{\imath}} \frac{\partial f(x^{i},x^{\bar{\imath}})}{\partial x^{\bar{\jmath}}} \tilde{\psi}^{\bar{\imath}_{1}} \psi^{i_{2}} \dots \psi^{i_{p}} \tilde{\psi}^{\bar{\imath}_{1}} \tilde{\psi}^{\bar{\imath}_{1}} \dots \tilde{\psi}^{\bar{\imath}_{q}}.$$

$$(4.21)$$

So the obvious similarity between (4.16) and (4.21) shows we have a correspondence:

$$\partial = Q^+, \quad \bar{\partial} = \widetilde{Q}^-,$$
(4.22)

and the (c,a) fields thus correspond to (p,q)-forms. Given that $k^{j,\bar{j}}$ was defined using (c,c) fields, we have

$$h^{p,q} = k^{p,d-q}. (4.23)$$

Everything we have done today was done in a single flat coordinate patch for torus. One can try to extend this to general Riemannian manifolds but this is where all the difficulty of the geometry of string theory comes into play. This involves nonlinear σ -models, gauged linear σ -models, etc., which is well beyond what we can cover here.

That said, we might generally hope that, given an N = (2, 2) superconformal field theory with the correct charge quantization condition we might find some complex manifold X whose Hodge numbers agree with (4.23). In particular, this would imply that

$$h^{d,0} = 1, (4.24)$$

which is the Calabi–Yau condition. Thus we expect a correspondence between N = (2,2) theories and Calabi–Yau manifolds. But, then again, we have no right to expect there to be a geometric interpretation of a particular superconformal field theory and so this correspondence is far from perfect.

This identification between differential forms of type (p, q) with the (c,a) fields is known as the A-model and naturally yields de Rham cohomology. There is also a B-model story which goes via the (c,c) fields. They are related by mirror symmetry, which, of course, is a long story in itself!

4.4 Topological Field Theory

The chiral primary fields have a nice ring structure thanks to operator products and this contains a good deal of information about the N=(2,2) theory. In fact, it is surprising how much interesting information can be extracted purely from just these chiral fields. We can modify an N=(2,2) theory in such a way as to leave only this (c,c) information so we can hone in on this part of the bigger N=(2,2) picture. This leads to "topological field theory". Much of the modern analysis of string theory in the context of geometry is presented as a topological field theory so we will present it here, albeit very briefly.

The last term in (3.28) suggests a new field

$$F(z) = T(z) - \frac{1}{2}\partial J(z), \tag{4.25}$$

might be an interesting object. In terms of modes we write $F_n = L_n - \frac{1}{2}(n+1)J_n$. Let us use this to eliminate T(z) in favor of F(z) in the N=2 algebra. Furthermore, let's rename $G^{\pm}(z)$ to Q(z) and G(z) respectively. We then obtain

$$F(z)F(w) = \frac{2F(w)}{(z-w)^2} + \frac{\partial F(w)}{z-w} + \dots$$

$$F(z)Q(w) = \frac{Q(w)}{(z-w)^2} + \frac{\partial Q(w)}{z-w} + \dots$$

$$F(z)G(w) = \frac{2G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \dots$$

$$F(z)J(w) = \frac{d}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} + \dots$$

$$Q(z)G(w) = \frac{2d}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2F(w)}{z-w} + \dots$$

$$J(z)Q(w) = \frac{Q(w)}{z-w} + \dots$$

$$J(z)G(w) = -\frac{G(w)}{z-w} + \dots$$

$$J(z)J(w) = \frac{d}{(z-w)^2} + \dots$$

Now let's try to reinterpret this as a conformal field theory with stress tensor F(z). Note that by doing this, we really are considering an inequivalent conformal field theory. The first equation tells us that the new central charge is 0. But this is supposed to be a measure of the complexity of the conformal field theory, so clearly our new theory is very simple!

Indeed, if c=0 then a positive inner product on the Hilbert space forces the condition that $F_n |\psi\rangle = 0$ for any n [4]. So any field is completely invariant under reparametrization. The fact that $F_0 |\psi\rangle = 0$ is precisely that $h - \frac{1}{2}j = 0$ and so the only fields in our new theory are chiral primary fields from the old theory. The vast majority of fields from the original N=2 theory have been lost.

When we simply "twist" the N=2 algebra to the topological version above, the adjoint of Q(z) is G(z). One can also choose to alter the inner product structure such that Q(z) becomes self-adjoint. The positive definiteness of the inner product is now lost unless we restrict attention to Q-cohomology. In this case the setup looks similar to BRST quantization and, because of this, people often refer to the R-charge as "ghost number" instead.

The condition $F_{-1} | \psi \rangle = 0$ is the condition that translations in z are invariant. This means that any correlation function will not depend on the location of vertex operators (at least so long as they don't collide or hit singularities in the Riemann surface). This is why the theory is called "topological".

The chiral ring structure of the original N=2 theory is preserved in this topological field theory and this gives a context in which it is much easier to analyze. Sadly, we have no time to pursue this further.

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