Dirichlet Series

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Suppose $s \in \mathbb{C}$. The Dirichlet series is defined as

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where $a_n \in \mathbb{C}$ is some complex sequence. We shall prove the following statements:

1. *Suppose $s_1, s_2 \in \mathbb{C}$ and $\Re(s_2) > \Re(s_1)$. If $D(s_1)$ is convergent, then $D(s_2)$ is also convergent.*

2. *Suppose $D(s)$ is convergent for some s and divergent for some other s. Then, there exists some $b \in \mathbb{R}$ such that for any s with $\Re(s) > b$, $D(s)$ is convergent; also, for any s with $\Re(s) < b$, $D(s)$ is divergent.*

3. *(1). The series*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is convergent if $\Re(s) > 1$, and is divergent if $\Re(s) < 1$. (2). If $m > 1$ and χ is a non-principal character $\bmod m$. Then, the series (the Dirichlet L-function)

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

is convergent if $\Re(s) > 0$, and is divergent if $\Re(s) < 0$.

4. *Suppose $s_1, s_2 \in \mathbb{C}$ and $\Re(s_2) > \Re(s_1) + 1$. Then, if $D(s_1)$ is convergent, then $D(s_2)$ is absolutely convergent.*

5. *Suppose that t is real and $t \neq 0$. Then, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+it}}$$

is divergent.

1. *Suppose $s_1, s_2 \in \mathbb{C}$ and $\Re(s_2) > \Re(s_1)$. If $D(s_1)$ is convergent, then $D(s_2)$ is also convergent.*

Proof. Define the "small" part of N to be

$$S(N) = \sum_{n=M}^{N} \frac{a_n}{n^{s_1}}.$$

This is well-defined as long as $N \geq M$. If $N = M - 1$, set $S(M-1) = 0$. Since $D(s_1)$ is convergent, given any $\varepsilon > 0$, there exits $X \in \mathbb{N}$ such that for any N, M with $N \geq M \geq X$, we always have

$$\left| \sum_{n=M}^{N} \frac{a_n}{n^{s_1}} \right| = |S(N)| < \varepsilon.$$

On the other hand, for any $n \geq M$, we always have

$$\frac{a_n}{n^{s_1}} = S(n) - S(n-1).$$

So,

$$\frac{a_n}{n^{s_2}} = \frac{a_n}{n^{s_1}} n^{s_1 - s_2} = \left(S(n) - S(n-1)\right) n^{s_1 - s_2}.$$

Let $S = \sum_{n=M}^{M-1} \frac{a_n}{n^{s_2}}$. From this we have

$$S = \sum_{n=M}^{N} S(n) n^{s_1 - s_2} - \sum_{n=M}^{N} S(n-1) n^{s_1 - s_2}.$$

According to our convention, the first summand in the second sum of the above expression is $S(M-1) = 0$, so the sum may take from $n = M+1$. Then, making a change of variable from n to $n-1$, we have

$$\sum_{n=M}^{N} S(n-1) n^{s_1 - s_2} = \sum_{n=M}^{N-1} S(n)(n+1)^{s_1 - s_2}.$$

Then,

$$\begin{aligned} S = &\sum_{n=M}^{N} S(n) n^{s_1 - s_2} - \sum_{n=M}^{N-1} S(n)(n+1)^{s_1 - s_2} \\ = &S(N) N^{s_1 - s_2} + \sum_{n=M}^{N-1} S(n) \left(n^{s_1 - s_2} - (n+1)^{s_1 - s_2} \right) \end{aligned}$$

We know that $|S(n)| < \varepsilon$ for any $M \leq n \leq N$. Also, since $\Re(s_1) < \Re(s_2)$, then let $\Re(s_1 - s_2) = -\sigma$ with $\sigma \in \mathbb{R}$ and $\sigma > 0$. Then, $|N^{s_1 - s_2}| = N^{\Re(s_1 - s_2)} < 1$. So,

$$|S| < \varepsilon + \varepsilon \sum_{n=M}^{N-1} \left| n^{s_1 - s_2} - (n+1)^{s_1 - s_2} \right|. \tag{1}$$

Note that

$$n^{s_1-s_2} - (n+1)^{s_1-s_2} = -(s_1 - s_2) \int_n^{n+1} y^{s_1-s_2-1} dy.$$

Therefore,

$$|n^{s_1-s_2} - (n+1)^{s_1-s_2}| = |s_1 - s_2| \int_n^{n+1} y^{-\sigma-1} dy.$$

Then,

$$\begin{aligned}
\sum_{n=M}^{N-1} |n^{s_1-s_2} - (n+1)^{s_1-s_2}| &= |s_1 - s_2| \sum_{n=M}^{N-1} \int_n^{n+1} y^{-\sigma-1} dy \\
&\leq |s_1 - s_2| \sum_{n=1}^{\infty} \int_n^{n+1} y^{-\sigma-1} dy \\
&= |s_1 - s_2| \int_1^{\infty} y^{-\sigma-1} dy \\
&= \frac{|s_1 - s_2|}{\sigma}.
\end{aligned}$$

Inserting this back to equation 1, we have

$$|S| < \varepsilon \left(1 + \frac{|s_1 - s_2|}{\sigma}\right),$$

and therefore $D(s_2)$ is convergent. This completes the proof. $\square$

2. *Suppose $D(s)$ is convergent for some s and divergent for some other s. Then, there exists some $b \in \mathbb{R}$ such that for any s with $\Re(s) > b$, $D(s)$ is convergent; also, for any s with $\Re(s) < b$, $D(s)$ is divergent.*

Proof. Let $S = \{s \in \mathbb{C} : D(s) \text{ is convergent}\}$. Then, let $b = \inf\{\Re(s) : s \in S\}$. Clearly b is well-defined, and claim that b satisfies the property. Given any s with $\Re(s) = \sigma > b$, let $\delta = \sigma - b$, then there exists some s' such that $\Re(s') < b + \delta/2$. Since $\Re(s') > b$, then $s' \in S$, so $D(s')$ is convergent. On the other hand, according to the result of 1, since $\Re(s) = \sigma = b + \delta > \Re(s')$, $D(s)$ is also convergent. This finishes the proof. $\square$

3. *(1). The series*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is convergent if $\Re(s) > 1$, and is divergent if $\Re(s) < 1$.

Proof. We know that if $s \in \mathbb{R}$, then $\zeta(s)$ is convergent if $s > 1$. This is, for any $s > 1$, given any $\varepsilon > 0$, there exits some X such that whenever $N \geq M \geq X$, we always have

$$\left| \sum_{n=M}^{N} \frac{1}{n^s} \right| < \varepsilon.$$

Then, given any $s \in \mathbb{C}$ with $\Re(s) = \sigma > 1$, we can also find X with $N \geq M \geq X$ so that

$$\sum_{n=M}^{N} \frac{1}{n^\sigma} = \left| \sum_{n=M}^{N} \frac{1}{n^s} \right| < \varepsilon.$$

This completes the proof. $\square$

(2). If $m > 1$ and χ is a non-principal character $\bmod m$. *Then, the series (the Dirichlet L-function)*

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

is convergent if $\Re(s) > 0$, and is divergent if $\Re(s) < 0$.

Proof. Let $N > 0$ and $S(N) = \sum_{n=1}^{N} \chi(n)$. Then, by division, $N = qm + r$ for some $0 \leq r < m$. Then,

$$|S(N)| = \left| \sum_{n=1}^{qm} \chi(n) + \sum_{n=qm+1}^{qm+r} \chi(n) \right|.$$

The first sum in the absolute value is 0, and the second sum, according to triangle inequality, can be bounded by

$$\sum_{n=qm+1}^{qm+r} |\chi(n)|.$$

Each $\chi(n)$ is less than 1 within this range, so therefore

$$|S(N)| \leq r < m.$$

So this proves that $|S(N)|$ is always bounded. Now, since

$$\chi(n) = S(n) - S(n-1),$$

we have

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{S(n) - S(n-1)}{n^s} = \sum_{n=1}^{\infty} \frac{S(n)}{n^s} - \sum_{n=1}^{\infty} \frac{S(n-1)}{n^s}.$$

The second sum, by the same change of variable method used in 1, can be transformed into a sum from 1 to ∞. Thus,

$$L(s, \chi) = \sum_{n=1}^{\infty} S(n) \left(n^{-s} - (n+1)^{-s}\right).$$

Similarly,

$$n^{-s} - (n+1)^{-s} = s \int_{n}^{n+1} u^{-s-1} du.$$

So

$$\left|(n^{-s} - (n+1)^{-s}\right| \leq |s| \int_{n}^{n+1} u^{-\sigma-1} du$$

with $\sigma = \Re(s)$. Then,

$$L(s, \chi) \leq m|s| \int_{1}^{\infty} u^{-\sigma-1} du.$$

The improper integral converges when $\sigma = \Re(s) > 0$ and diverges when $\sigma = \Re(s) < 0$, and so does $L(s, \chi)$. This completes the proof. $\square$

4. *Suppose* $s_1, s_2 \in \mathbb{C}$ *and* $\Re(s_2) > \Re(s_1) + 1$. *Then, if* $D(s_1)$ *is convergent, then* $D(s_2)$ *is absolutely convergent.*

Proof. Let $\Re(s_2) - \Re(s_1) = \sigma > 1$. Since $D(s_1)$ is convergent, the sequence $\{\left|\frac{a_n}{n^{s_1}}\right|\}_{n=1}^{\infty}$ is bounded. That is, there exists $M > 0$ such that for any n,

$$\left|\frac{a_n}{n^{s_1}}\right| < M.$$

Then,

$$\left|\frac{a_n}{n^{s_2}}\right| = \left|\frac{a_n}{n^{s_1}} n^{s_1 - s_2}\right| < M n^{-\sigma}.$$

Since $\sigma > 1$, $D(s_2)$ converges absolutely. $\square$

5. *Suppose that t is real and $t \neq 0$. Then, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+it}}$$

is divergent.

Proof. Let $N > M$ be positive integels and write

$$S = \sum_{n=M}^{N} \frac{1}{n^{1+it}}.$$

Further, define

$$T = \sum_{n=M}^{N} \int_{n}^{n+1} \frac{1}{x^{1+it}} dx.$$

We can do the integers inside T explicitly:

$$T = \int_{M}^{N+1} x^{-1-it} dx = \frac{1}{it} \left[M^{-it} - (N+1)^{-it} \right].$$

On the other hand, we have

$$S - T = \sum_{n=M}^{N} \int_{n}^{n+1} \left(n^{-1-it} - x^{-1-it} \right) dx$$

Note that

$$\begin{aligned} \left| n^{-1-it} - x^{-1-it} \right| &= \left| (1+it) \int_{n}^{x} u^{-2-it} du \right| \\ &\leq |1+it| \int_{n}^{n+1} u^{-2} du. \end{aligned}$$

From this we have

$$|S - T| \leq |1+it| \sum_{n=M}^{N} \int_{n}^{n+1} u^{-2} du \leq |1+it| \int_{M}^{\infty} u^{-2} du = \frac{|1+it|}{M}.$$

Now going back to T, since $|M^{-it}| = 1$we have

$$|T| = \frac{1}{|t|} \left| 1 - \left(\frac{N+1}{M} \right)^{-it} \right|.$$

Note that

$$\left(\frac{N+1}{M} \right)^{-it} = e^{\log\left(\frac{N+1}{M}\right)(-it)} = e^{\pm i \log\left(\frac{M+1}{N}\right)|t|} =: e^{-i\theta}.$$

Now we can choose $N > M$ such that

$$\frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

so that

$$\left|1 - \left(\frac{N+1}{M}\right)^{-it}\right| = |1 - \cos\theta \mp i\sin\theta| \geq |1 - \cos\theta| > 1.$$

In this case, we have

$$|S| \geq |T| - |S - T| > \frac{1}{|t|} - \frac{|1 - it|}{M}.$$

Now, let $\varepsilon = 1/2|t|$. Given any $X > 0$, we further bound M from below by

$$M > \max\{X, 2|t||1 + it|\}.$$

Then, by our construction of M, N, we have

$$|S| = \left|\sum_{n=M}^{N} \frac{1}{n^{1+it}}\right| > \frac{1}{2|t|} = \varepsilon.$$

Thus the series diverges, and this completes the proof.

$\square$