Series Summary

Summary for the series part of the Maths 2 course.
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Sigma Notation

Basic Series

$\sum_{i=n}^{m} C$	$C \cdot (m-n+1)$
$\sum_{i=n}^{m} i$	$\begin{cases} f_1(m) & \text{if } n \in \{0,1\} \\ f_1(m) - f_1(n-1) & \text{else} \end{cases}$
$\sum_{i=n}^{m} i^2$	$\begin{cases} f_2(m) & \text{if } n \in \{0,1\} \\ f_2(m) - f_2(n-1) & \text{else} \end{cases}$
$\sum_{i=n}^{m} i^3$	$\begin{cases} f_3(m) & \text{if } n \in \{0,1\} \\ f_3(m) - f_3(n-1) & \text{else} \end{cases}$
$\sum_{i=n}^{m} a^i$	$\begin{cases} g(a, m+1) & \text{if } n = 0 \\ m+1 & \text{if } n = 0, a = 1 \\ g(a, m+1) - g(a, n) & \text{else} \end{cases}$

Index Manipulation

$\sum_{\substack{i=0 \\ i:\text{even}}}$	$i = 2k$
$\sum_{\substack{i=1 \\ i:\text{odd}}}$	$i = 2k \pm 1$
$\sum_{i=3}^{10} i^2$	$\sum_{i=1}^{8} (i+2)^2 = \sum_{i=1}^{8} i^2 + 4i + 4$
$\sum_{i=n}^{m} a^i$	$\xrightarrow{i=k+n} \sum_{k+n=n}^{k+n=m} a^{k+n} = \sum_{k=0}^{k=m-n} a^n \cdot a^k$ $= a^n \cdot g(a, m-n+1)$

Infinite Series

Famous Infinite Series

$\sum_{n=0}^{\infty} a^n$	Geometric Series $\begin{cases} \lvert a \rvert < 1 & \text{converges} \\ \lvert a \rvert \geq 1 & \text{diverges} \end{cases}$
$\sum_{n=1}^{\infty} \frac{1}{n}$	Harmonic Series diverges
$\sum_{n=1}^{\infty} \frac{1}{n^p}$	P Series $\begin{cases} P > 1 & \text{converges} \\ P \leq 1 & \text{diverges} \end{cases}$
$C \pm C$	$= C$
$C \pm D$	$= D$
$D + D$	$= D$
$D - D$	$= ?$

Strength of terms

Most	$f^n(n) : n^n, \ln(n)^n$
	$n!$
	a^n
	n^a
	$\log_a(n), \ln(n) \;:\; \log_n > \log_m$ if $m > n$
	a
Least	$\begin{cases} a^n & a < 1 \\ a^{-n} & a > 1 \end{cases}$

Series Tests

General Term	$\lim_{n \to \infty} U_n \begin{cases} \neq 0 & \text{diverges} \\ = 0 & \text{test fails} \end{cases}$
Integral	$\int_i^{\infty} U_n \begin{cases} \text{number} & \text{converges} \\ \pm\infty & \text{diverges} \end{cases}$
Alternating Series for $\sum^{n=\infty} (-1)^n b_n$	$\lim_{n \to \infty} b_n \begin{cases} = 0 & \text{converges} \\ \neq 0 & \text{diverges} \end{cases}$
Root for $\sum^{n=\infty} (f(n))^n$	$\lim_{n \to \infty} f(n) \begin{cases} < 1 & \text{converges} \\ > 1 & \text{diverges} \\ = 1 & \text{test fails} \end{cases}$
Ratio $2^n, n, n^2, n!$ in U_n	$\lim_{n \to \infty} \left\lvert \frac{U_{n+1}}{U_n} \right\rvert \begin{cases} < 1 & \text{converges} \\ > 1 & \text{diverges} \\ = 1 & \text{test fails} \end{cases}$
Comparison	Choose known V_n to compare $V_n \begin{cases} \text{convergent} & \begin{cases} U_n < V_n & \text{conv.} \\ U_n > V_n & \text{fail} \end{cases} \\ \text{divergent} & \begin{cases} U_n < V_n & \text{fail} \\ U_n > V_n & \text{div.} \end{cases} \end{cases}$
Limit Comparison	$\lim_{n \to \infty} \frac{U_n}{V_n}$ or $\frac{V_n}{U_n}$ $\begin{cases} = 0 & \text{fail} \\ = \infty & \text{fail} \\ \text{else} & U_n \text{ as } V_n \end{cases}$
Partial Sum $S_n = \sum_{i=1}^{n} U_i$	$\lim_{n \to \infty} S_n \begin{cases} \text{number} & \text{conv.} \\ \infty & \text{div.} \\ \text{oscillating} & \text{div.} \end{cases}$

Power Series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 \ldots$$

Series is convergent for all $x \in \text{I.C}$

$\lim_{n \to \infty} \left\lvert \frac{U_{n+1}}{U_n} \right\rvert < 1$ all values of x satisfying the inequality define the I.C, but the boundaries must be tested separately with other tests.

Taylor & Maclaurin Series

Maclaurin Expansion

$\text{I.C} = \mathbb{R}$ in elementary functions

about $x = 0$

$f(x)$	$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$
e^x	$\sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$
$\sinh(x)$	$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$
$\cosh(x)$	$\sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

Taylor Expansion

$f(a), (x-a)$ (about $x = a$)

$|x - a| < 1$

I.C determined using ratio test

$f(x)$	$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...$

Binomial Series

Combinatorics

${}^nC_r = \frac{n!}{r!(n-r)!}$	$n \geq r, {}^nC_a = {}^nC_{n-a}$

Binomial Expansion: Positive Integer n

\# of terms $= n + 1$

$\text{I.C} : x \in \mathbb{R}$

$(1+x)^n$	$1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ... + x^n$
General Term	${}^nC_rx^r$

Binomial Expansion: Negative or Fraction n

\# of terms $= \infty$

$\text{I.C} : |x| < 1$

$(1+x)^n$	$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$
$(1-x)^{-1}$	$x^r = 1 + x + x^2 + x^3 + ...$
$(1+x)^{-1}$	$(-1)^rx^r = 1 - x + x^2 - x^3 + ...$
$(1-x)^{-2}$	$(r+1)x^r = 1 + 2x + 3x^2 + 4x^3 + ...$
$(1+x)^{-2}$	$(-1)^r(r+1)x^r = 1 - 2x + 3x^2 - ...$

Fourier Series

$f(x)$	$f(x+p)$ is a periodic function with period p
$f(x)$	$A + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)$
f_{DC}	DC Component : $\begin{cases} A = \frac{\text{area of } f(x)}{\text{length of period}} \\ \text{avg of } f(x) \end{cases}$
f_{AC}	AC Component : $\begin{cases} \sum_{n=1}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x) \\ f(x) - A \end{cases}$
ω_0	$\frac{2\pi}{p} = 2\pi f$

a_0	$\frac{1}{p/2}\int_0^p f(x)\,\mathrm{d}x$
A	$\frac{a_0}{2}$
a_n	$\frac{1}{p/2}\int_0^p f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$
b_n	$\frac{1}{p/2}\int_0^p f(x) \cdot \sin(n\omega_0 x)\,\mathrm{d}x$

Special Cases and Extra Notes

$$A = \frac{\text{area of one sub-repetition}}{\text{length of one sub-repetition}}$$

$\mu = \text{length of smallest sub-repetition}$

$$A = \frac{1}{\mu} \cdot \int_0^{\mu} f(x)\,\mathrm{d}x$$

$$a_n = \frac{2}{\mu}\int_0^{\mu} f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$$

$$b_n = \frac{2}{\mu}\int_0^{\mu} f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$$

If x is not defined at the boundaries, for example: $-3 < x < 0$ and $0 < x < 3$, if $f(x_0)$ is needed (here x_0 can be $-3, 0, 3$)

$$f(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}$$

$$\sin(n\pi) = 0, \cos(n\pi) = (-1)^n$$

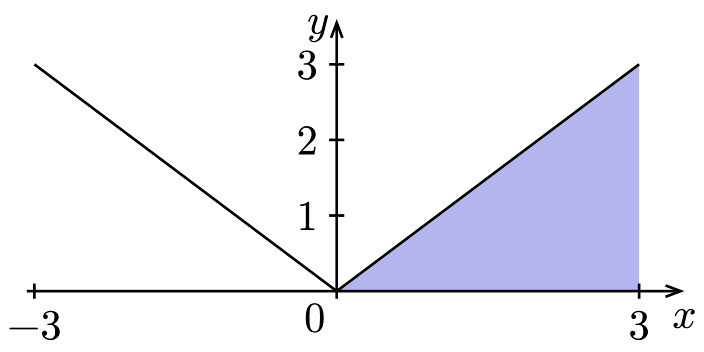
$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n : \text{even} \\ (-1)^{\frac{n-1}{2}} & n : \text{odd} \end{cases}$$

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n : \text{odd} \\ (-1)^{\frac{n}{2}} & n : \text{even} \end{cases}$$

Cosines Only (Even Function)

The first special case. The function is mirrored on the y-axis within one period. Example:

$$f(x) = \begin{cases} x, & 0 < x < 3 \\ -x, & -3 < x < 0 \end{cases}$$



Properties

Only even components. $b_n = 0$

$$f(x) = f(-x) = f(x+p)$$
$$= A + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 x)\,\mathrm{d}x$$

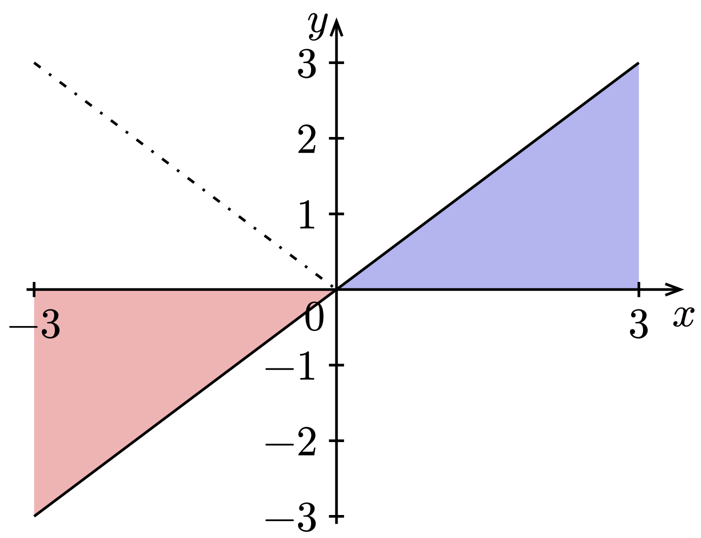
$$a_n = \tfrac{2}{p/2} \int_0^{p/2} f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$$

$$A = \frac{\text{highlighted area (half period)}}{\text{half period length}}$$
$$= \frac{a_0}{2} = \frac{1}{2} \cdot \frac{2}{p/2} \int_0^{p/2} f(x)\,\mathrm{d}x$$

Sines Only (Odd Function)

The function is mirrored on the y-axis then flipped on the x-axis, within one period. Example:

$$f(x) = x, -3 < x < 3$$



Properties

Truly sines only.

Because A is an even component, $A = 0$ here. It can also be proven from the fact that the area over the full period will be 0.

$$f(x) = -f(-x) = f(x+p)$$
$$= \sum_{n=1}^{\infty} b_n \sin(n\omega_0 x)$$

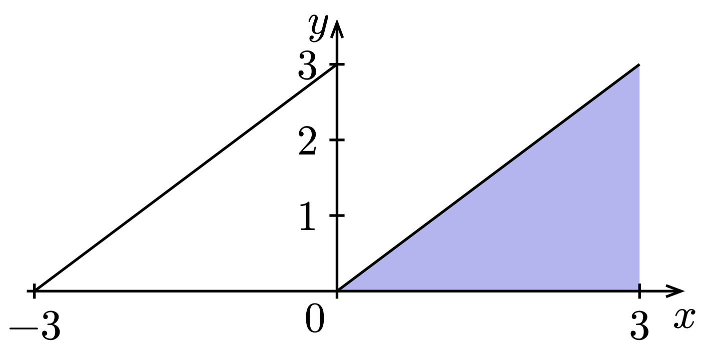
$$A = 0, a_0 = 0, a_n = 0$$

$$b_n = \tfrac{2}{p/2} \int_0^{p/2} f(x) \cdot \sin(n\omega_0 x)\,\mathrm{d}x$$

Even Harmonic

The function is repeated twice in one period. n is even. Example:

$$f(x) = \begin{cases} x, & 0 < x < 3 \\ x+3, & -3 < x < 0 \end{cases}$$



Properties

$$f(x) = f(x+p) = f\left(x + \frac{p}{2}\right)$$
$$= A + \sum_{n:\text{even}}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)$$

$$A = \frac{\text{highlighted area (half period)}}{\text{half period length}}$$
$$= \frac{a_0}{2} = \frac{1}{2} \cdot \frac{2}{p/2} \int_0^{p/2} f(x)\,\mathrm{d}x$$

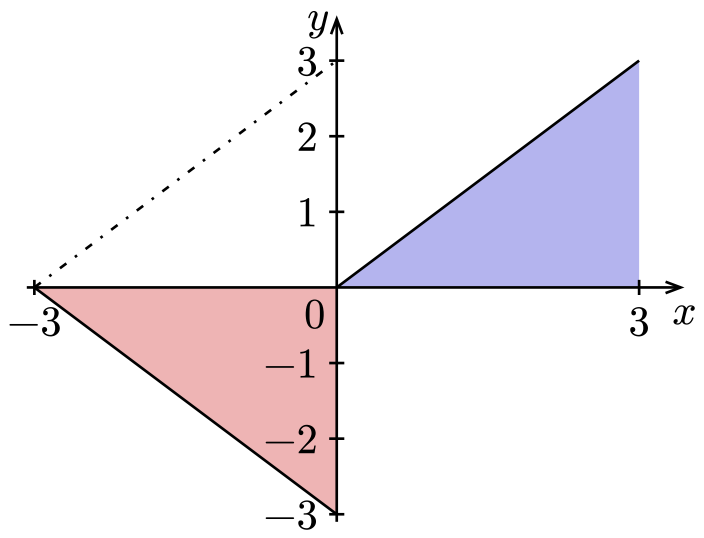
$$a_n = \frac{2}{p/2} \int_0^{p/2} f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$$

$$b_n = \frac{2}{p/2} \int_0^{p/2} f(x) \cdot \sin(n\omega_0 x)\,\mathrm{d}x$$

Odd Harmonic

The function is repeated twice in one period, but will be flipped about the x-axis every other repetition. n is odd. Example:

$$f(x) = \begin{cases} x, & 0 < x < 3 \\ -3 - x, & -3 < x < 0 \end{cases}$$



Properties

$$f(x) = f(x+p) = -f\left(x + \frac{p}{2}\right)$$

$$= \sum_{n:\text{odd}}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)$$

$$A = 0 \text{ (from graph)}, a_0 = 0$$

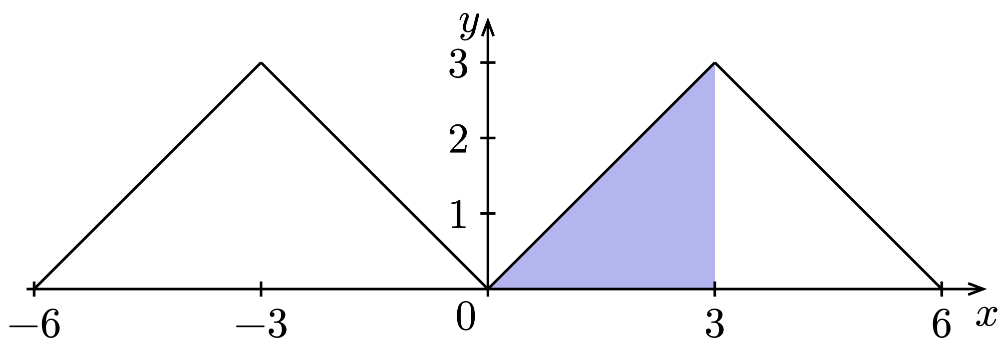
$$a_n = \frac{2}{p/2} \int_0^{p/2} f(x) \cdot \cos(n\omega_0 x)\, \mathrm{d}x$$

$$b_n = \frac{2}{p/2} \int_0^{p/2} f(x) \cdot \sin(n\omega_0 x)\, \mathrm{d}x$$

Even Cosines

Mirror about y-axis, then repeat after half-period. Example:

$$f(x) = \begin{cases} 6 + x, & -6 < x < -3 \\ -x, & -3 < x < 0 \\ x, & 0 < x < 3 \\ 6 - x, & 3 < x < 6 \end{cases}$$



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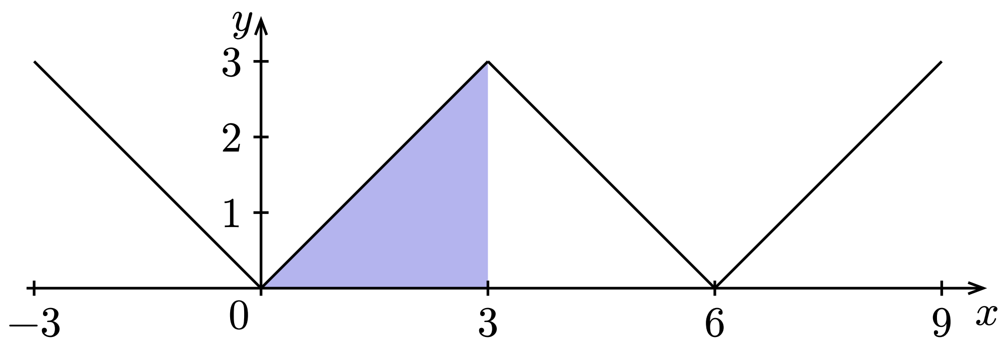


Properties

$$f(x) = f(-x) = f(x+p) = f\left(x + \frac{p}{2}\right)$$

$$= A + \sum_{n:\text{even}}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)$$

$$A = \frac{a_0}{2} = \frac{1}{2} \cdot \frac{4}{p/2} \int_0^{p/4} f(x)\, \mathrm{d}x$$

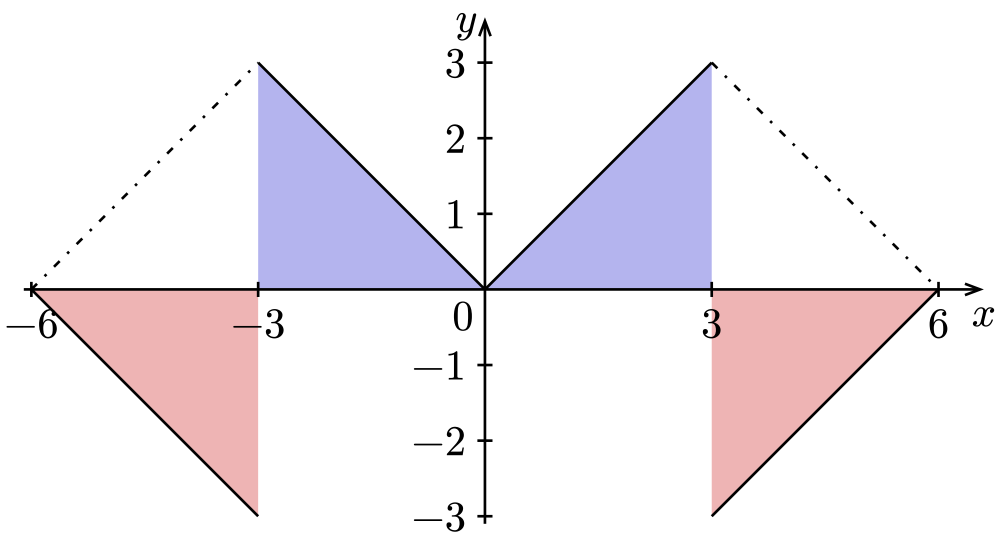
$$a_n = \frac{4}{p/2} \int_0^{p/4} f(x) \cos(n\omega_0 x)\, \mathrm{d}x$$

$$b_n = \frac{2}{p/2} \int_0^{p/4} f(x) \sin(n\omega_0 x)\, \mathrm{d}x$$

Odd Cosines

Mirror about y-axis, then repeat flipped about x-axis after half-period. Example:

$$f(x) = \begin{cases} -x - 6, & -6 < x < -3 \\ -x, & -3 < x < 0 \\ x, & 0 < x < 3 \\ x - 6, & 3 < x < 6 \end{cases}$$



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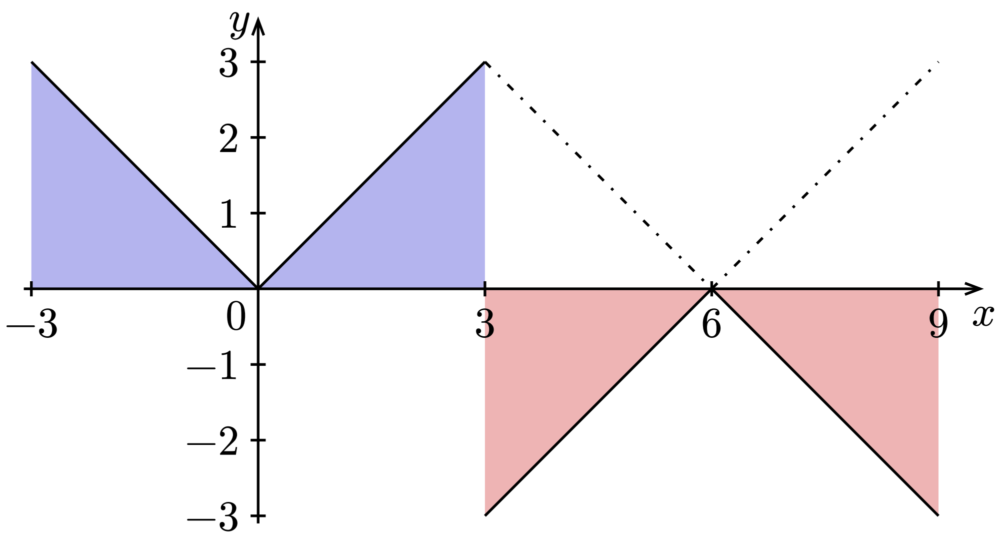


$$f(x) = f(-x) = f(x+p) = -f\left(x + \frac{p}{2}\right)$$

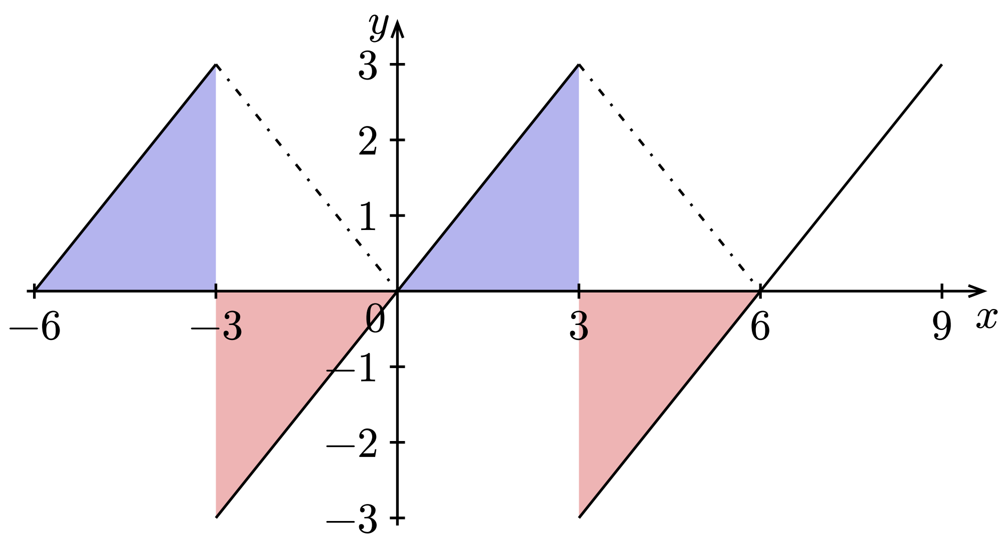
$$= \sum_{n:\text{odd}}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)$$

a_n and b_n are the same as those in even cosines

Even Sines

Mirror about y-axis then flip about x-axis. Then repeat after half-period.

$$f(x) = \begin{cases} x, & 0 < x < 3 \\ x - 6, & 3 < x < 9 \end{cases}$$



$$f(x) = -f(-x) = f(x+p) = f\left(x + \frac{p}{2}\right)$$

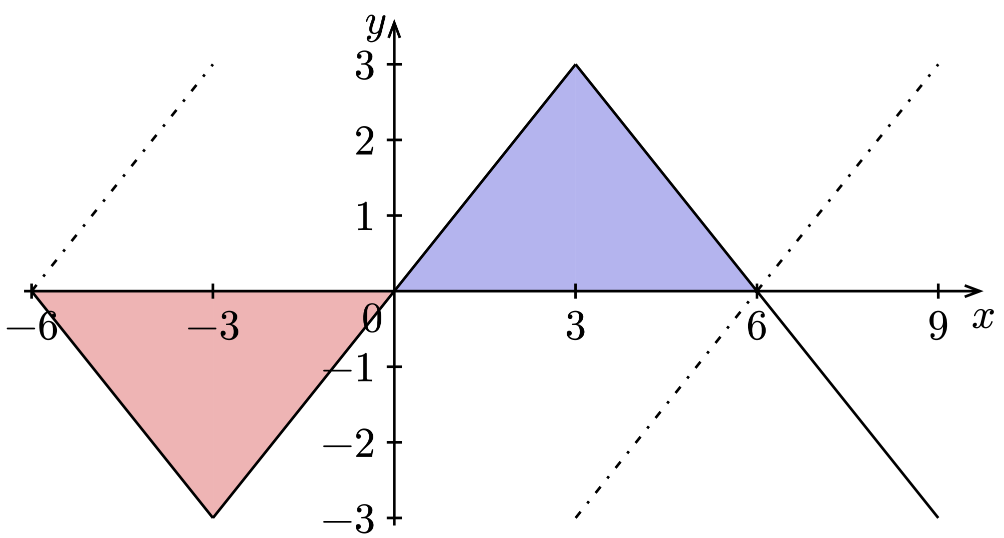
$$= \sum_{n:\mathrm{even}}^{\infty} b_n \sin(n\omega_0 x)$$

$$b_n = \frac{4}{p/2} \int_0^{p/4} f(x) \cdot \sin(n\omega_0 x)\,\mathrm{d}x$$

Odd Sines

Mirror about y-axis, then flip about x-axis. Then repeat flipped after half-period.

$$f(x) = \begin{cases} x, & 0 < x < 3 \\ 6 - x, & 3 < x < 9 \end{cases}$$



$$f(x) = -f(-x) = f(x+p) = -f\left(x + \frac{p}{2}\right)$$

$$= \sum_{n:\mathrm{odd}}^{\infty} b_n \sin(n\omega_0 x)$$

$$b_n = \frac{4}{p/2} \int_0^{p/4} f(x) \cdot \sin(n\omega_0 x)\,\mathrm{d}x$$