Discrete Structures for Computing – SSP – Final Exam 2024
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Recreated and solved by a student

1. Prove by induction $\sum_{k=1}^{n}(k \cdot k!) = (n+1)! - 1$

   Note: $n \geqslant 1$

Let $P(n) : \sum_{k=1}^{n}(k \cdot k!) = (n+1)! - 1$

Prove $P(n)\ \forall n \geqslant 1$
1. **Base Case**, let $n = 1$ and prove that $P(1)$ is true.

$$\begin{aligned} \sum_{k=1}^{1}(k \cdot k!) &= (n+1)! - 1 && \text{(Base Case)} \\ 1 \times 1! &= (1+1)! - 1 && \text{(Substitute)} \\ 1 &= 2 - 1 = 1 && \text{(Proven)} \end{aligned}$$

2. **Inductive Step**, prove that $P(x) \to P(x+1) \quad \forall x \geqslant 1$
3. **Inductive Hypothesis**, $n = x$
Assume $P(x)$, prove $P(x+1)$

$$\begin{aligned} \sum_{k=1}^{x}(k \cdot k!) &= (x+1)! - 1 && \text{(Assume } P(x)\text{)} \\ P(x+1) = \sum_{k=1}^{x+1}(k \cdot k!) &= ((x+1)+1)! - 1 && \text{(R.T.P } P(x+1)\text{)} \\ \sum_{k=1}^{x+1}(k \cdot k!) &= (x+2)! - 1 && \text{(Simplify)} \end{aligned}$$

$$\begin{aligned} \text{L.H.S} &= \sum_{k=1}^{x}(k \cdot k!) + (x+1) \cdot (x+1)! && \text{(Expand Series)} \\ &= (x+1)! - 1 + (x+1) \cdot (x+1)! && \text{(Sub. from } P(x)\text{)} \\ &= (x+1)! \cdot (1 + (x+1)) - 1 && \text{(Simplify)} \\ &= (x+1)! \cdot (x+2) - 1 && \text{(Simplify)} \\ &= (x+2)! - 1 && \text{(Factorial Simplification)} \\ &= \text{R.H.S} && \text{(Proven)} \end{aligned}$$

2. Is the function $f : \mathbb{Z} \longrightarrow \mathbb{Z}, f(x) = 4 - 3x$, onto? prove your answer.

NO
(Counter-example)
Find $n \in \mathbb{Z}$ which has no pre-image under f (in range, but not in co-domain).
Let $n = 0$:

$$\begin{aligned} 0 &= 4 - 3x \\ -4 &= -3x \\ x &= \frac{4}{3} \end{aligned}$$

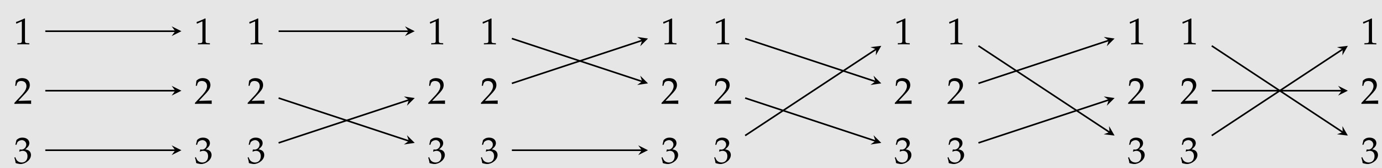
$x = \dfrac{4}{3} \notin \mathbb{Z}$, hence f is NOT onto.

3. Prove that the function $f : \mathbb{R} - \{2\} \longrightarrow \mathbb{R} - \{5\}, f(x) = \dfrac{5x+1}{x-2}$, is one-to-one.

Definition of a one-to-one function: $f(n) = f(m)$ if, and only if, $n = m$.
(Direct Proof)
$n = m \rightarrow f(n) = f(m)$ does not need proving.
R.T.P: $f(n) = f(m) \rightarrow n = m$

$$\begin{aligned} f(n) &= f(m) \\ \frac{5n+1}{n-2} &= \frac{5m+1}{m-2} && \text{(Substitute)} \\ (m-2)(5n+1) &= (n-2)(5m+1) \\ \underline{5mn} + m - 10n - \underline{2} &= \underline{5mn} + n - 10m - \underline{2} \\ m - 10n &= n - 10m \\ 11m &= 11n \\ m &= n && \text{(Proven)} \end{aligned}$$

4. Draw diagrams of all bijections $f : X \longrightarrow X, X = \{1, 2, 3\}$



5. How many ways to arrange the word **TENNESSEE**?

(9 Letters: 1 T, 2 N, 2 S, 4 E) Number of ways: ${}^{9}C_4 \times {}^{5}C_2 \times {}^{3}C_2 \times {}^{1}C_1$

6. How many ways to distribute 8 identical balls among 4 people?

$$\binom{8+4-1}{8} = {}^{11}C_8$$

7. There are 9 players, among them are 3 good front row players. How many teams of five players can you form if at least one good front row player must be included?

Let F = good front row players, N = normal (other) players.

$1F, 4N = {}^{3}C_1 \times {}^{6}C_4$

$2F, 3N = {}^{3}C_2 \times {}^{6}C_3$

$3F, 2N = {}^{3}C_3 \times {}^{6}C_2$

$$\text{Total} = {}^{3}C_3 \times {}^{6}C_2 + {}^{3}C_2 \times {}^{6}C_3 + {}^{3}C_1 \times {}^{6}C_4 = {}^{9}C_5 - {}^{6}C_5$$

8. You are going to use the 7 characters a, b, c, d, e, f, g to make an ordered list of characters.
   (a) How many lists can you make if no repetition is allowed?

   $7!$

   (b) How many lists can you make if repetitions are allowed?

   7^7

   (c) How many lists can you make if at least one character should be repeated?

   $7^7 - 7!$

9. You are going to build a binary string of length 8.
   (a) How many strings end with a 1?

   2^7

   (b) How many strings have exactly four ones?

   ${}^{8}C_4$

   (c) How many strings end with 1 or have exactly four ones?

   $2^7 + {}^{8}C_4 - {}^{7}C_3$

10. If 10 contestants are ranked without ties, how many ways are there for first, second, third places?

${}^{10}P_3$

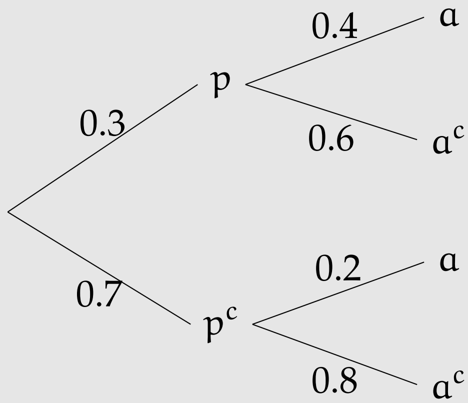
11. Find the coefficient of $x^8 \cdot y^5$ in $(4x^2 - 7y)^9$

Binomial theorem

$$
\begin{aligned}
(4x^2 - 7y)^9 &= \sum_{k=0}^{9} {}^{9}C_k (4x^2)^k \cdot (\text{-}7y)^{9-k} && \text{(General Term)} \\
2k &= 8 && \text{(Power of } x\text{)} \\
k &= 4 && \\
{}^{9}C_4 \cdot (4x^2)^4 \cdot (\text{-}7y)^{9-4} & && \text{(Substitute)} \\
{}^{9}C_4 \cdot 4^4 x^8 \cdot (\text{-}7^5) \cdot y^5 & && \text{(Simplify)} \\
{}^{9}C_4 \cdot 4^4 \cdot (\text{-}7^5) \cdot x^8 \cdot y^5 & && \text{(Re-order)} \\
{}^{9}C_4 \cdot 4^4 \cdot (\text{-}7^5) & && \text{(Isolate Coefficient)}
\end{aligned}
$$

12. In a certain population, insurance policy holders are either accident prone or non-accident prone. The probability for an accident prone to have an accident within this year is 0.4. The probability for a non-accident prone to have an accident within this year is 0.2. If 30% of the population is accident prone:

Let p = is accident prone, p^c = is non-accident prone, a = will have an accident this year, a^c = will not have an accident this year.



(a) What is the probability of a policy holder to get into an accident within this year?

$$P(a) = 0.3 \times 0.4 + 0.7 \times 0.2 = 0.26$$

(b) What is the probability of a policy holder to be accident prone if he had an accident within this year?

$$P(p|a) = \frac{P(p \cap a)}{P(a)} = \frac{0.3 \times 0.4}{0.26} = \frac{6}{13} \approx 0.46$$