Series Summary

Summary for the series part of the Maths 2 course.
Made with ❤ by Youssef Samy using Typst.

Sigma Notation

Basic Series

$$\sum_{i=n}^{m} C = C \cdot (m-n+1)$$

$$\sum_{i=n}^{m} i = \begin{cases} f_1(m) & \text{if } n \in \{0,1\} \\ f_1(m) - f_1(n-1) & \text{else} \end{cases}$$

$$\sum_{i=n}^{m} i^2 = \begin{cases} f_2(m) & \text{if } n \in \{0,1\} \\ f_2(m) - f_2(n-1) & \text{else} \end{cases}$$

$$\sum_{i=n}^{m} i^3 = \begin{cases} f_3(m) & \text{if } n \in \{0,1\} \\ f_3(m) - f_3(n-1) & \text{else} \end{cases}$$

$$\sum_{i=n}^{m} a^i = \begin{cases} g(a, m+1) & \text{if } n = 0 \\ m+1 & \text{if } n = 0, a = 1 \\ g(a, m+1) - g(a, n) & \text{else} \end{cases}$$

Index Manipulation

$$\sum_{\substack{i=0 \\ i:\text{even}}} \quad i = 2k$$

$$\sum_{\substack{i=1 \\ i:\text{odd}}} \quad i = 2k \pm 1$$

$$\sum_{i=3}^{10} i^2 = \sum_{i=1}^{8} (i+2)^2 = \sum_{i=1}^{8} i^2 + 4i + 4$$

$$\sum_{i=n}^{m} a^i \quad i=k+n \rightarrow \sum_{k+n=n}^{k+n=m} a^{k+n} = \sum_{k=0}^{k=m-n} a^n \cdot a^k$$

$$= a^n \cdot g(a, m-n+1)$$

Infinite Series

Famous Infinite Series

Geometric Series

$$\sum_{n=0}^{\infty} a^n \quad \begin{cases} |a| < 1 & \text{converges} \\ |a| \geq 1 & \text{diverges} \end{cases}$$

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}$$

P Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} P > 1 & \text{converges} \\ P \leq 1 & \text{diverges} \end{cases}$$

$C \pm C$	$= C$
$C \pm D$	$= D$
$D + D$	$= D$
$D - D$	$= ?$

Strength of terms

Most	$f^n(n) : n^n, \ln(n)^n$
	$n!$
	a^n
	n^a
	$\log_a(n), \ln(n) \;:\; \log_n > \log_m$ if $m > n$
	a
Least	$\begin{cases} a^n & a < 1 \\ a^{-n} & a > 1 \end{cases}$

Series Tests

General Term	$\lim_{n \to \infty} U_n \begin{cases} \neq 0 & \text{diverges} \\ = 0 & \text{test fails} \end{cases}$
Integral	$\int_i^{\infty} U_n \begin{cases} \text{number} & \text{converges} \\ \pm\infty & \text{diverges} \end{cases}$

Alternating Series for $\sum^{n=\infty} (-1)^n b_n$

$$\lim_{n \to \infty} b_n \begin{cases} = 0 & \text{converges} \\ \neq 0 & \text{diverges} \end{cases}$$

Root for $\sum^{n=\infty} (f(n))^n$

$$\lim_{n \to \infty} f(n) \begin{cases} < 1 & \text{converges} \\ > 1 & \text{diverges} \\ = 1 & \text{test fails} \end{cases}$$

Ratio $2^n, n, n^2, n!$ in U_n

$$\lim_{n \to \infty} \left| \frac{U_{n+1}}{U_n} \right| \begin{cases} < 1 & \text{converges} \\ > 1 & \text{diverges} \\ = 1 & \text{test fails} \end{cases}$$

Comparison

Choose known V_n to compare

$$V_n \begin{cases} \text{convergent} & \begin{cases} U_n < V_n & \text{conv.} \\ U_n > V_n & \text{fail} \end{cases} \\ \text{divergent} & \begin{cases} U_n < V_n & \text{fail} \\ U_n > V_n & \text{div.} \end{cases} \end{cases}$$

Limit Comparison

$$\lim_{n \to \infty} \frac{U_n}{V_n} \text{ or } \frac{V_n}{U_n}$$

$$\begin{cases} = 0 & \text{fail} \\ = \infty & \text{fail} \\ \text{else} & U_n \text{ as } V_n \end{cases}$$

Partial Sum

$$S_n = \sum_{i=1}^{n} U_i \qquad \lim_{n \to \infty} S_n \begin{cases} \text{number} & \text{conv.} \\ \infty & \text{div.} \\ \text{oscillating} & \text{div.} \end{cases}$$

Power Series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 \ldots$$

Series is convergent for all $x \in \text{I.C}$

$\lim_{n \to \infty} \left| \frac{U_{n+1}}{U_n} \right| < 1$ all values of x satisfying the inequality define the I.C, but the boundaries must be tested separately with other tests.

Taylor & Maclaurin Series

Maclaurin Expansion

I.C $= \mathbb{R}$ in elementary functions

about $x = 0$

$f(x)$	$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$
e^x	$= \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
$\sin(x)$	$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$
$\cos(x)$	$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
$\sinh(x)$	$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$
$\cosh(x)$	$= \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

Taylor Expansion

$f(a), (x-a)$ (about $x = a$)

$|x - a| < 1$

I.C determined using ratio test

$f(x)$	$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$

Binomial Series

Combinatorics

${}^nC_r = \frac{n!}{r!(n-r)!}$	$n \geq r, {}^nC_a = {}^nC_{n-a}$

Binomial Expansion: Positive Integer n

of terms $= n + 1$

I.C : $x \in \mathbb{R}$

$(1+x)^n$	$= 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots + x^n$
General Term	${}^nC_r x^r$

Binomial Expansion: Negative or Fraction n

of terms $= \infty$

I.C : $|x| < 1$

$(1+x)^n$	$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$
$(1-x)^{-1}$	$= x^r = 1 + x + x^2 + x^3 + \ldots$
$(1+x)^{-1}$	$= (-1)^r x^r = 1 - x + x^2 - x^3 + \ldots$
$(1-x)^{-2}$	$= (r+1)x^r = 1 + 2x + 3x^2 + 4x^3 + \ldots$
$(1+x)^{-2}$	$= (-1)^r(r+1)x^r = 1 - 2x + 3x^2 - \ldots$

Fourier Series

$f(x)$	$f(x)$ is a periodic function with period p $= f(x+p)$ $= A + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)$
f_{DC}	DC Component : $= \begin{cases} A = \frac{\text{area of } f(x)}{\text{length of period}} \\ \text{avg of } f(x) \end{cases}$
f_{AC}	AC Component : $= \begin{cases} \sum_{n=1}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x) \\ f(x) - A \end{cases}$

General Definitions

ω_0	$= \frac{2\pi}{p} = 2\pi f$
a_0	$= \frac{1}{\text{half period}} \int_{\text{full period}} f(x)\,\mathrm{d}x$
A	$= \frac{1}{\text{full period}} \int_{\text{full period}} f(x)\,\mathrm{d}x = \frac{a_0}{2}$
a_n	$= \frac{1}{\text{half period}} \int_{\text{full period}} f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$
b_n	$= \frac{1}{\text{half period}} \int_{\text{full period}} f(x) \cdot \sin(n\omega_0 x)\,\mathrm{d}x$

Extra Notes

- You can shift a function horizontally to be able to see its periodic nature, but it can't be shifted vertically.
- If x is not defined at the boundaries, for example:

$-3 < x < 0$ and $0 < x < 3$

if $f(x_0)$ is needed (here x_0 can be $-3, 0, 3$)

$$f(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}$$

- Sine and cosine properties for simplification:

$$\sin(n\pi) = 0$$

$$\cos(n\pi) = (-1)^n$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n : \text{even} \\ (-1)^{\frac{n-1}{2}} & n : \text{odd} \end{cases}$$

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n : \text{odd} \\ (-1)^{\frac{n}{2}} & n : \text{even} \end{cases}$$

Generalization for Special Cases

$$A = \frac{\text{area of one sub-repetition}}{\text{length of one sub-repetition}}$$

$$\mu = \text{length of smallest sub-repetition}$$
$$= \begin{cases} p & \text{generally} \\ p/2 & \text{sines/cosines only, even/odd harmonic} \\ p/4 & \text{combinations of two of the above} \end{cases}$$

$$A = \frac{1}{\mu} \cdot \int_0^\mu f(x)\,\mathrm{d}x$$

$$a_n = \frac{2}{\mu} \int_0^\mu f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$$

$$b_n = \frac{2}{\mu} \int_0^\mu f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$$

Special Cases

Special cases of the fourier series may include one or two specifically oriented repetitions within one period. For example, in a cosines-only function, one half of the period is mirrored in the other half.
In the following examples and drawings, the center of the period is assumed to be the origin. Whenever that is not the case, you should either shift the origin horizontally to be at the center of the period, or replace all the short forms like:

$$a_n = \frac{1}{p/2} \int_0^p f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$$

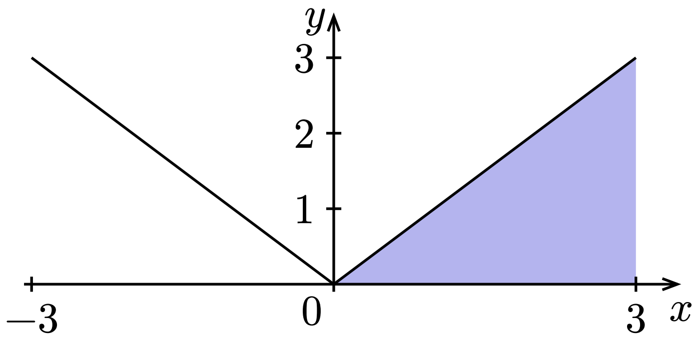
with their original form, as in:

$$a_n = \frac{1}{\text{half period}} \int_{\text{full period}} f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$$

Cosines Only (Even Function)

The first special case. The function is mirrored on the y-axis within one period. Example:

$$f(x) = \begin{cases} x, & 0 < x < 3 \\ -x, & -3 < x < 0 \end{cases}$$



Properties

Only even components. $b_n = 0$

$$f(x) = f(-x) = f(x+p)$$
$$= A + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 x)\,\mathrm{d}x$$

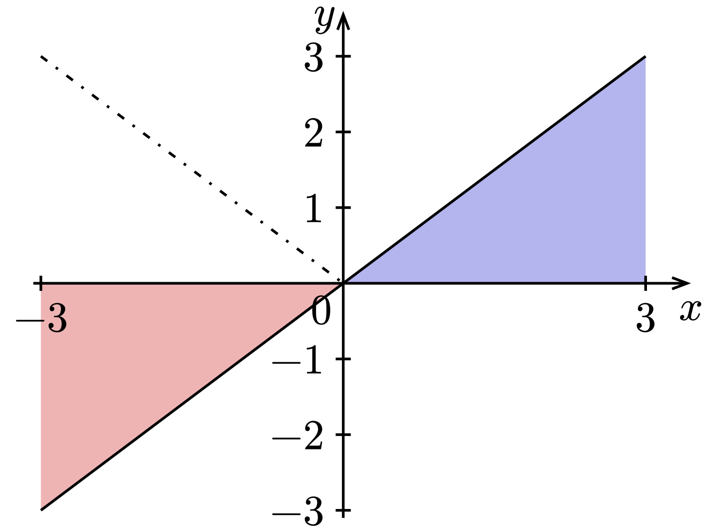
$$a_n = \frac{2}{p/2} \int_0^{p/2} f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$$

$$A = \frac{\text{highlighted area (half period)}}{\text{half period length}}$$
$$= \frac{a_0}{2} = \frac{1}{2} \cdot \frac{2}{p/2} \int_0^{p/2} f(x)\,\mathrm{d}x$$

Sines Only (Odd Function)

The function is mirrored on the y-axis then flipped on the x-axis, within one period. Example:

$$f(x) = x, -3 < x < 3$$



Properties

Truly sines only.
Because A is an even component, $A = 0$ here. It can also be proven from the fact that the area over the full period will be 0.

$$f(x) = -f(-x) = f(x+p)$$
$$= \sum_{n=1}^{\infty} b_n \sin(n\omega_0 x)$$

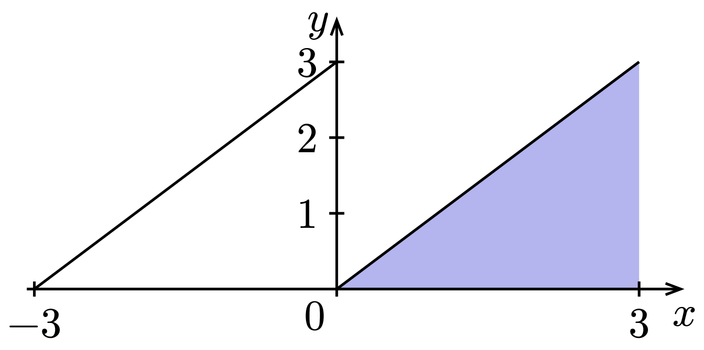
$$A = 0, a_0 = 0, a_n = 0$$

$$b_n = \frac{2}{p/2} \int_0^{p/2} f(x) \cdot \sin(n\omega_0 x)\,\mathrm{d}x$$

Even Harmonic

The function is repeated twice in one period. n is even. Example:

$$f(x) = \begin{cases} x, & 0 < x < 3 \\ x+3, & -3 < x < 0 \end{cases}$$



Properties

$$f(x) = f(x+p) = f\left(x + \frac{p}{2}\right)$$

$$= A + \sum_{n:\text{even}}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)$$

$$A = \frac{\text{highlighted area (half period)}}{\text{half period length}}$$

$$= \frac{a_0}{2} = \frac{1}{2} \cdot \frac{2}{p/2} \int_0^{p/2} f(x)\,\mathrm{d}x$$

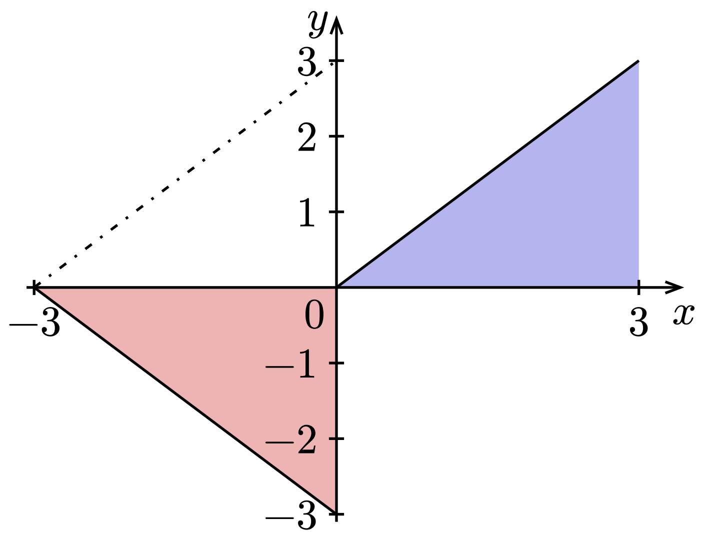
$$a_n = \frac{2}{p/2} \int_0^{p/2} f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$$

$$b_n = \frac{2}{p/2} \int_0^{p/2} f(x) \cdot \sin(n\omega_0 x)\,\mathrm{d}x$$

Odd Harmonic

The function is repeated twice in one period, but will be flipped about the x-axis every other repetition. n is odd. Example:

$$f(x) = \begin{cases} x, & 0 < x < 3 \\ -3-x, & -3 < x < 0 \end{cases}$$



Properties

$$f(x) = f(x+p) = -f\left(x + \frac{p}{2}\right)$$

$$= \sum_{n:\text{odd}}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)$$

$A = 0$ (from graph), $a_0 = 0$

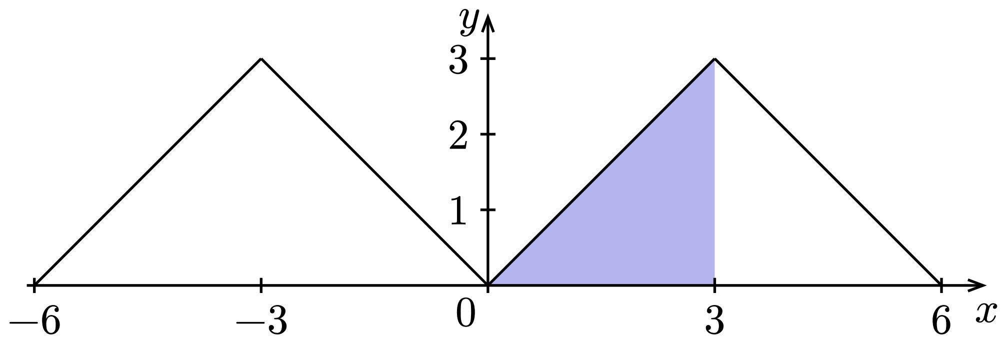
$$a_n = \frac{2}{p/2} \int_0^{p/2} f(x) \cdot \cos(n\omega_0 x)\,\mathrm{d}x$$

$$b_n = \frac{2}{p/2} \int_0^{p/2} f(x) \cdot \sin(n\omega_0 x)\,\mathrm{d}x$$

Even Cosines

Mirror about y-axis, then repeat after half-period. Example:

$$f(x) = \begin{cases} 6+x, & -6 < x < -3 \\ -x, & -3 < x < 0 \\ x, & 0 < x < 3 \\ 6-x, & 3 < x < 6 \end{cases}$$



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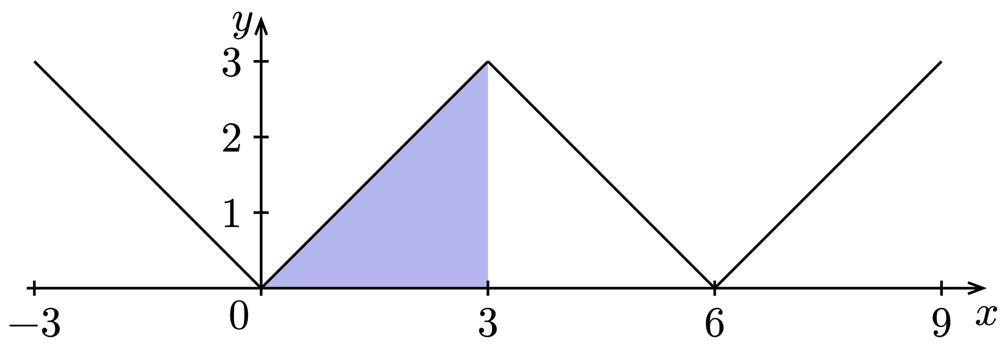


Properties

$$f(x) = f(-x) = f(x+p) = f\left(x + \frac{p}{2}\right)$$

$$= A + \sum_{n:\text{even}}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)$$

$$A = \frac{a_0}{2} = \frac{1}{2} \cdot \frac{4}{p/2} \int_0^{p/4} f(x)\,\mathrm{d}x$$

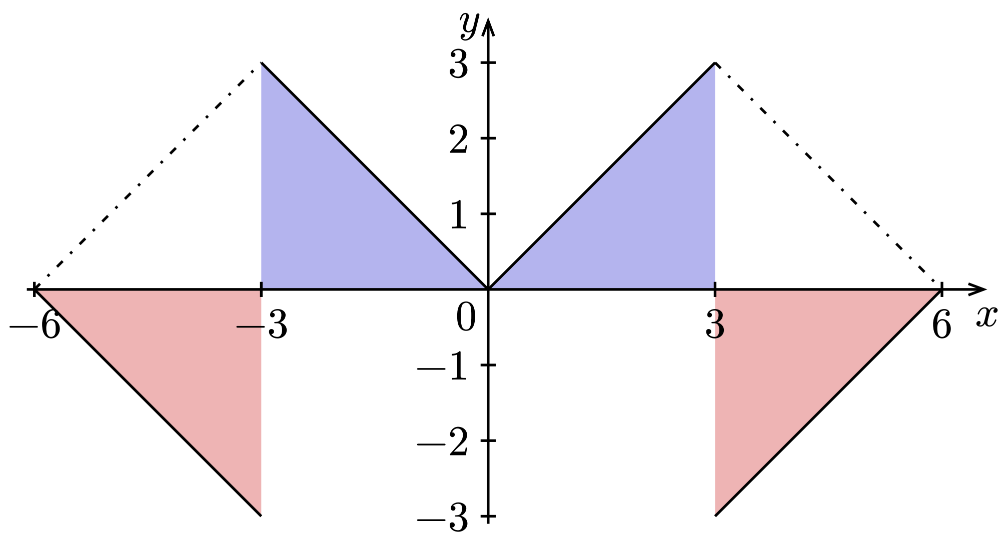
$$a_n = \frac{4}{p/2} \int_0^{p/4} f(x) \cos(n\omega_0 x)\,\mathrm{d}x$$

$$b_n = \frac{2}{p/2} \int_0^{p/4} f(x) \sin(n\omega_0 x)\,\mathrm{d}x$$

Odd Cosines

Mirror about y-axis, then repeat flipped about x-axis after half-period. Example:

$$f(x) = \begin{cases} -x-6, & -6 < x < -3 \\ -x, & -3 < x < 0 \\ x, & 0 < x < 3 \\ x-6, & 3 < x < 6 \end{cases}$$



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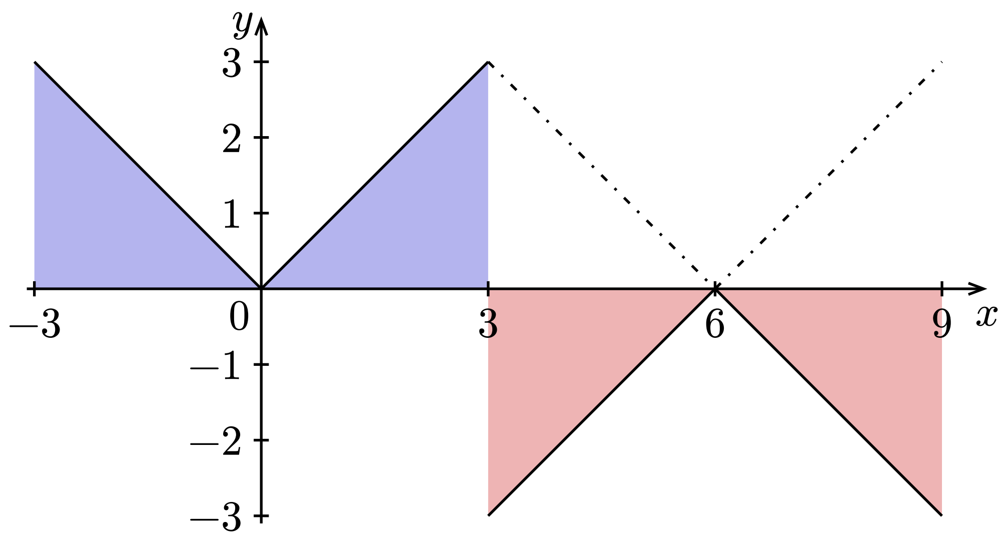


$$f(x) = f(-x) = f(x+p) = -f\left(x + \frac{p}{2}\right)$$

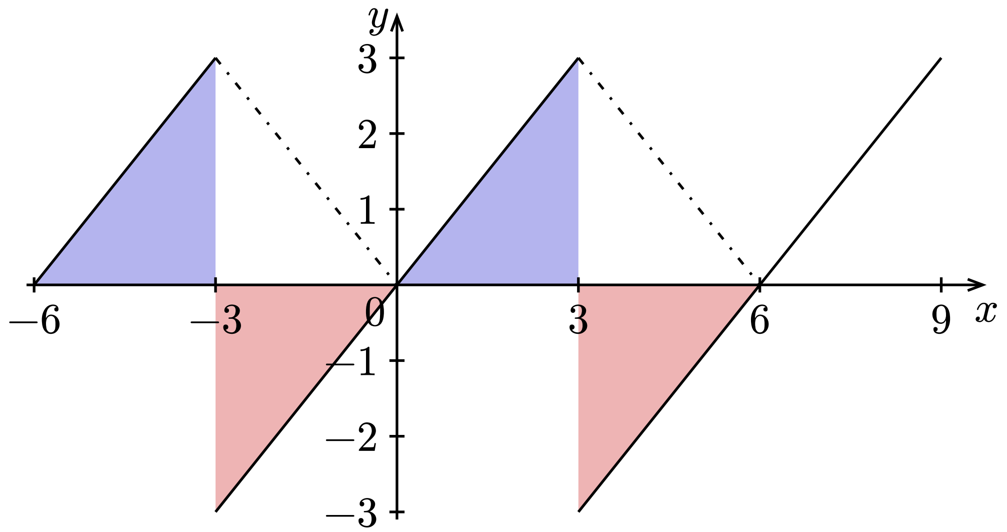
$$= \sum_{n:\text{odd}}^{\infty} a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)$$

a_n and b_n are the same as those in even cosines

Even Sines

Mirror about y-axis then flip about x-axis. Then repeat after half-period.

$$f(x) = \begin{cases} x, & 0 < x < 3 \\ x-6, & 3 < x < 9 \end{cases}$$



$$f(x) = -f(-x) = f(x+p) = f\left(x + \frac{p}{2}\right)$$

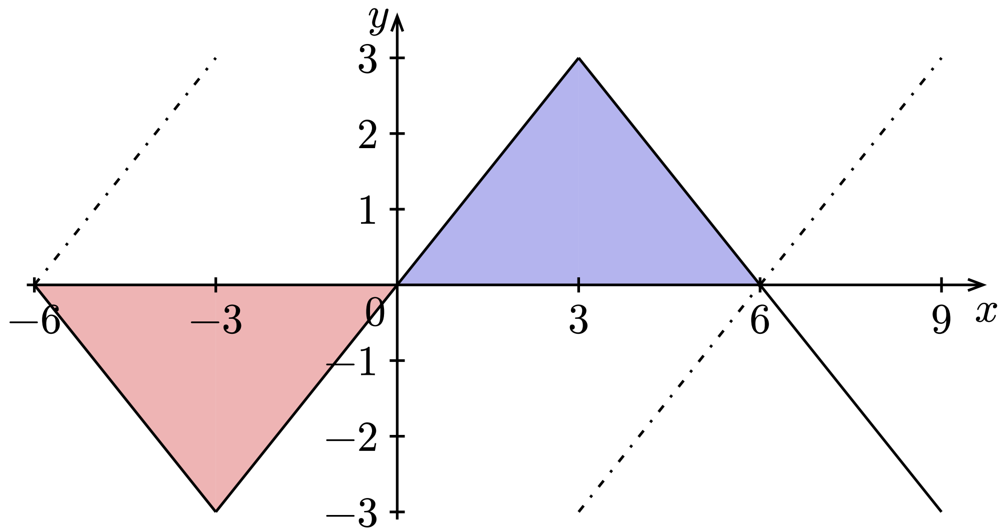
$$= \sum_{n:\text{even}}^{\infty} b_n \sin(n\omega_0 x)$$

$$b_n = \frac{4}{p/2} \int_0^{p/4} f(x) \cdot \sin(n\omega_0 x)\,\mathrm{d}x$$

Odd Sines

Mirror about y-axis, then flip about x-axis. Then repeat flipped after half-period.

$$f(x) = \begin{cases} x, & 0 < x < 3 \\ 6-x, & 3 < x < 9 \end{cases}$$



$$f(x) = -f(-x) = f(x+p) = -f\left(x + \frac{p}{2}\right)$$

$$= \sum_{n:\text{odd}}^{\infty} b_n \sin(n\omega_0 x)$$

$$b_n = \frac{4}{p/2} \int_0^{p/4} f(x) \cdot \sin(n\omega_0 x)\,\mathrm{d}x$$