

同济大学课程考核试卷 (期中试卷评分标准)

2022—2023 学年第二学期

命题教师签名:

审核教师签名:

课号: 122005

课名: 高等数学 B 下

考试考查:

此卷选为: 期中考试(√)、期终考试()、重考()试卷

年级_____专业_____学号_____姓名_____任课教师_____

题号	一 (30)	二 (10)	三 (8)	四 (12)	五 (10)	六 (10)	七 (10)	八 (10)	总分
得分									

(注意: 本试卷共八大题, 三大张, 满分 100 分. 考试时间为 100 分钟. 要求写出解题过程, 否则不予计分)

一、填空选择题(每小题 3 分, 共 30 分)

1. 已知 $(\vec{a} \times \vec{b}) \cdot \vec{c} = 2$, 则 $[(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})] \cdot (\vec{c} + \vec{a}) = \underline{4}$.2. 将 xOy 坐标面上的曲线 $4x^2 - y^2 = 1$ 绕 x 轴旋转一周, 所生成的旋转曲面方程为 $\underline{4x^2 - (y^2 + z^2) = 1}$.3. 若 $a \neq 0$, 则 $\lim_{(x,y) \rightarrow (\infty,a)} \left(1 + \frac{1}{xy}\right)^{\frac{x^2}{x+y}} = \underline{e^{\frac{1}{a}}}$.4. 函数 $u = xy + \sin \frac{y}{2} + e^{xz}$ 在点 $(1, \pi, 0)$ 处的全微分为 $\underline{du = \pi dx + dy + dz}$.5. 函数 $u = xyz$ 在点 $(5, 1, 2)$ 处沿从点 $(5, 1, 2)$ 到点 $(9, 4, 14)$ 的方向的方向导数为 $\underline{\frac{98}{13}}$.6. 曲线 $x = \int_0^t e^u \cos u du$, $y = 2 \sin t + \cos t$, $z = 1 + e^{3t}$ 在点 $(0, 1, 2)$ 处的切线方程为 $\underline{\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}}$.7. 方程 $F(x, y, z) = 0$ 在点 (x_0, y_0, z_0) 的某个邻域内恒能唯一确定一个连续且有连续偏导数的函数 $y = f(x, z)$, 需要的充分条件是: (1) $F(x, y, z)$ 在点 (x_0, y_0, z_0) 的某个邻域内一阶偏导数连续; (2) $F(x_0, y_0, z_0) = 0$; (3) $F_y(x_0, y_0, z_0) \neq 0$.8. 函数 $z = f(x, y)$ 在某点处偏导数存在是其在该点处连续的 D 条件.

A. 充分不必要 B. 必要不充分 C. 充分必要 D. 无关

9. 将二次积分 $\int_0^2 dy \int_{y^2}^{2y} f(x, y) dx$ 交换积分次序后, 得 A.A. $\int_0^4 dx \int_{\frac{x}{2}}^{\sqrt{x}} f(x, y) dy$ B. $\int_0^2 dx \int_{\frac{x}{2}}^{\sqrt{x}} f(x, y) dy$ C. $\int_0^4 dx \int_{2x}^{\sqrt{x}} f(x, y) dy$ D. $\int_0^2 dx \int_{2x}^{\sqrt{x}} f(x, y) dy$ 10. 设 D 是以 $(1, 1)$, $(-1, 1)$ 和 $(-1, -1)$ 为顶点的三角形区域, D_1 是 D 在第一象限的部分, 则 $\iint_D (x \sin y + y \cos x) dx dy = \underline{B}$.A. $2 \iint_{D_1} x \sin y dx dy$ B. $2 \iint_{D_1} y \cos x dx dy$ C. $4 \iint_{D_1} (x \sin y + y \cos x) dx dy$ D. 0二、(10 分) 设直线 $L_1: \frac{x}{4} = \frac{y}{1} = \frac{z}{1}$, $L_2: \begin{cases} z - 5x = -6, \\ z - 4y = 3, \end{cases}$ $L_3: \begin{cases} y - 2x = 4, \\ z - 3y = 5, \end{cases}$ 求平行于 L_1 并分别与 L_2 , L_3 都相交的直线 L 的方程.解 1: 设过 L_2 的平面束: $z - 5x + 6 + k(z - 4y - 3) = 0$, 则

$$4 \cdot (-5) + 1 \cdot (-4k) + 1 \cdot (1 + k) = 0, \text{ 解得 } k = -\frac{19}{3},$$

故 L_2 与 L 确定的平面为: $15x - 76y + 16z - 75 = 0$,设过 L_3 的平面束: $y - 2x - 4 + l(z - 3y - 5) = 0$, 则

$$4 \cdot (-2) + 1 \cdot (1 - 3l) + 1 \cdot l = 0, \text{ 解得 } l = -\frac{7}{2},$$

故 L_3 与 L 确定的平面为: $4x - 23y + 7z - 27 = 0$,

$$\text{所求直线 } L: \begin{cases} 15x - 76y + 16z - 75 = 0, \\ 4x - 23y + 7z - 27 = 0. \end{cases}$$

解 2: 设 L 与 L_2 , L_3 的交点分别为 $\left(\frac{z+6}{5}, \frac{z-3}{4}, z\right)$, $(x, 2x+4, 6x+17)$,

$$\text{则 } \frac{x - \frac{z+6}{5}}{4} = \frac{2x+4 - \frac{z-3}{4}}{1} = \frac{6x+17-z}{1}, \text{ 解得: } x = \frac{-107}{41}, \text{ (或 } z = \frac{99}{41}, \text{)}$$

$$\text{所求直线 } L: \frac{x + \frac{107}{41}}{4} = \frac{y + \frac{50}{41}}{1} = \frac{z - \frac{55}{41}}{1}. \text{ (或 } \frac{x - \frac{69}{41}}{4} = \frac{y + \frac{6}{41}}{1} = \frac{z - \frac{99}{41}}{1}. \text{)}$$

三、(8 分) 设 $u = f(xyz, x^2 + y^2 + z^2)$, 其中 f 具有连续的二阶偏导数, 求 $\frac{\partial^2 u}{\partial x \partial z}$.

解: $\frac{\partial u}{\partial x} = yzf_1' + 2xf_2'$,

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial z} &= yf_1' + yz[xyf_{11}'' + 2zf_{12}'] + 2x[xyf_{21}'' + 2zf_{22}'] \\ &= yf_1' + xy^2zf_{11}'' + 2y(x^2 + z^2)f_{12}'' + 4xzf_{22}''.\end{aligned}$$

四、(12 分) 设 $z = z(x, y)$ 是由方程 $x^2 - 6xy + 10y^2 - 2yz - z^2 + 18 = 0$ 确定的隐函数, 试问函数 $z = z(x, y)$ 是否存在极值? 若存在极值, 请计算出极值并指出是极大值还是极小值.

解: 存在极值.

$$\frac{\partial z}{\partial x} = \frac{x-3y}{y+z},$$

$$\frac{\partial z}{\partial y} = \frac{-3x+10y-z}{y+z},$$

$$A = \frac{\partial^2 z}{\partial x^2} = \frac{1-(\frac{\partial z}{\partial x})^2}{y+z}, \quad B = \frac{\partial^2 z}{\partial x \partial y} = \frac{-3-\frac{\partial z}{\partial x} \frac{\partial z}{\partial y}}{y+z}, \quad C = \frac{\partial^2 z}{\partial y^2} = \frac{10-2\frac{\partial z}{\partial y}-(\frac{\partial z}{\partial y})^2}{y+z},$$

解得驻点: (9,3) 和 (-9,-3),

判别 $AC - B^2$ 的符号都大于零,

在 (9,3) 取极小值 3, 在 (-9,-3) 取极大值 -3.

注: 二阶导数也可:

$$\frac{\partial^2 z}{\partial x^2} = \frac{(y+z)-(x-3y)z_x}{(y+z)^2}, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{-3(y+z)-(x-3y)z_y}{(y+z)^2},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(10-z_y)(y+z)-(-3x+10y-z)(1+z_y)}{(y+z)^2}.$$

五、(10 分) 在曲面 $z = xy$ 上求一点, 使该点的法线垂直于平面 $x + 3y + z = 0$, 并写出该点处的切平面方程和法线方程.

解: 设切点为 (x_0, y_0, z_0) , 则法线向量为 $(y_0, x_0, -1)$,

由已知得 $\frac{y_0}{1} = \frac{x_0}{3} = \frac{-1}{1},$

解得 $y_0 = -1, x_0 = -3, z_0 = 3,$

故切平面方程为 $x + 3y + z + 3 = 0,$

法线方程为 $\frac{x+3}{1} = \frac{y+1}{3} = \frac{z-3}{1}.$

六、(10 分) 求圆周 $(x+1)^2 + y^2 = 1$ 上的点与定点 (0,1) 的距离的最大值和最小值.

解: 点 (x, y) 与定点 (0,1) 的距离 $\sqrt{x^2 + (y-1)^2},$

令 $L(x, y, \lambda) = x^2 + (y-1)^2 + \lambda[(x+1)^2 + y^2 - 1]$

$$\begin{cases} 2x + 2\lambda(x+1) = 0 \\ 2(y-1) + 2\lambda y = 0 \\ (x+1)^2 + y^2 = 1 \end{cases}$$

解得驻点: $(\frac{\sqrt{2}}{2} - 1, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2} - 1, -\frac{\sqrt{2}}{2}),$

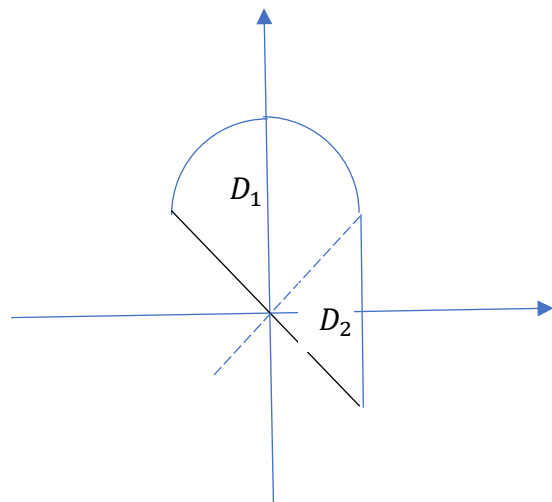
在点 $(\frac{\sqrt{2}}{2} - 1, \frac{\sqrt{2}}{2})$ 取得最小值 $\sqrt{2} - 1$ (或 $\sqrt{3 - 2\sqrt{2}}$),

在点 $(-\frac{\sqrt{2}}{2} - 1, -\frac{\sqrt{2}}{2})$ 取最大值 $\sqrt{2} + 1$ (或 $\sqrt{3 + 2\sqrt{2}}$).

七、(10 分) 计算二重积分 $\iint_D x \ln(y + \sqrt{1+y^2}) dx dy$, 其中 D 是由曲线 $y = 4 - x^2$,

直线 $y = -3x$ 及直线 $x = 1$ 围成的位于直线 $x = 1$ 左边的部分.

解: 作直线 $y = 3x$, 得 $D = D_1 + D_2$,



D_1 关于 y 轴对称, D_2 关于 x 轴对称,

$x \ln(y + \sqrt{1+y^2})$ 关于 x 和 y 都是奇函数,

$$\iint_{D_1} x \ln(y + \sqrt{1+y^2}) dx dy = 0,$$

$$\iint_{D_2} x \ln(y + \sqrt{1+y^2}) dx dy = 0,$$

$$\text{得 } \iint_D x \ln(y + \sqrt{1+y^2}) dx dy.$$

$$= \iint_{D_1} x \ln(y + \sqrt{1+y^2}) dx dy + \iint_{D_2} x \ln(y + \sqrt{1+y^2}) dx dy$$

$$= 0.$$

八、(10 分) 设 $u = u(x, y)$ 有连续的二阶偏导数, 且 $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \equiv 0$,

(1) 证明: 在变换 $\begin{cases} \xi = x - y \\ \eta = x + y \end{cases}$ 之下, $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \equiv 0$ 可变换成 $\frac{\partial^2 u}{\partial \xi \partial \eta} \equiv 0$;

(2) 若 $u(x, 2x) = x$, $u_x(x, 2x) = x^2$, 求 $u = u(x, y)$.

(1) 证明: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta})}{\partial x} = \frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial x} \\ &= \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}, \end{aligned}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} (-1) + \frac{\partial u}{\partial \eta},$$

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{\partial(-\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta})}{\partial y} = -\frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial y} - \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta \partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial y} \\ &= \frac{\partial^2 u}{\partial \xi^2} - 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}, \end{aligned}$$

$$\text{代入 } \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \equiv 0 \text{ 得: } \frac{\partial^2 u}{\partial \xi \partial \eta} \equiv 0.$$

(2) 由 $\frac{\partial^2 u}{\partial \xi \partial \eta} \equiv 0$ 得 $u(\xi, \eta) = f(\xi) + g(\eta)$,

$$\text{由 } \begin{cases} \xi = x - y \\ \eta = x + y \end{cases}, \text{ 得 } \begin{cases} u(x, y) = f(x - y) + g(x + y), \\ u_x(x, y) = f'(x - y) + g'(x + y), \end{cases}$$

又由 $u(x, 2x) = x$, $u_x(x, 2x) = x^2$, 代入上式, 可得 $\begin{cases} f(-x) + g(3x) = x, \\ f'(-x) + g'(3x) = x^2, \end{cases}$

$$\text{解得 } \begin{cases} f(-x) = \frac{1}{4}(x - x^3 - 3C), \\ g(3x) = \frac{1}{4}(3x + x^3 + 3C), \end{cases} \text{ 或者 } \begin{cases} f(x) = \frac{1}{4}(-x + x^3 - 3C), \\ g(x) = \frac{1}{4}(x + \frac{x^3}{27} + 3C), \end{cases}$$

$$u(x, y) = f(x - y) + g(x + y) = \frac{(x + y)^3}{108} + \frac{(x - y)^3}{4} + \frac{y}{2}.$$