AY 15/16

QUESTION 1 **(10 marks)**

A professor, due to his inexperience, designed a confusing quiz question to determine whether students understood the lecture. For a student who understood the lecture, the probability of giving the correct answer to the quiz question is 0.75. On the other hand, for a student who did not understand the lecture, the probability of giving the correct answer is 0.1. It is estimated that 60% of the students understood the lecture and the remaining 40% did not.

Let U denote the event that a particular student understood the lecture. Let C denote the event that the student answered the question correctly. Find $P(U|C)$.

QUESTION 3 **(25 marks)**

(a) Let X be a random variable with $0 \le X \le 1$ and $E(X) = \mu$. Show that:

(i) $0 \le \mu \le 1$;

(ii) $0 \le Var(X) \le \mu(1-\mu) \le \frac{1}{4}$. [Hint: Use $X^2 \le X$.]

(b) The result in Part (a) may be generalized as follows. Let X be a random variable with $a \le X \le b$ and $E(X) = \mu$. Show that:

(i) $a \le \mu \le b$;

(ii) $0 \le Var(X) \le (\mu - a)(b - \mu) \le \frac{1}{4}(b-a)^2$.

QUESTION 4 **(25 marks)**

Suppose (X, Y) is uniform on the unit disk (centered at origin with radius 1).

(a) Find f_X and f_Y, the marginal density functions of X and Y, respectively.

(b) Show that X and Y are not independent.

(c) Prove that $Cov(X, Y) = 0$.

QUESTION 5 **(15 marks)**

Suppose $X_1, X_2, \ldots, X_{30}$ are independent Poisson random variables with $\lambda = 2$. Use the central limit theorem to approximate

$$P\left(\sum_{i=1}^{30} X_i > 50\right).$$

AY 16/17

QUESTION 1 **(20 marks)**

Let X and Y be two independent geometric random variables with parameter p.

(a) For $n \geq 1$, prove that $P(X \geq n) = (1-p)^{n-1}$.

(b) Let $Z = \min(X, Y)$ and work out $P(Z \geq n)$. What is the distribution of Z?

(c) Find $P(Y = 2 | X + Y = 4)$.

QUESTION 2 **(20 marks)**

Let T represent the interior of a right-angled triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$. (See Figure 1 below). Let X and Y be jointly continuous random variables with uniform density inside T.

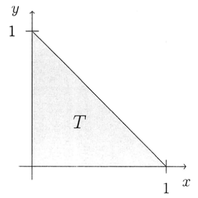


Figure 1

(a) Find the marginal cdf of X.

(b) Find the density of $X + Y$.

(c) Are X and Y independent? Justify your answer.

QUESTION 3 **(20 marks)**

Let $X_1, X_2, \ldots, X_n$ be independent random variables having Poisson distribution with mean $\lambda = 2$.

(a) Write down the moment generating function of X_1. You need not justify your answer.

(b) For any $n \geq 1$, let $Y_n = X_1 + X_2 + \cdots + X_n$. Find the moment generating function of Y_n.

(c) Find the mean and variance of Y_n.

(d) Prove that $P(Y_n \geq n^2) \leq \frac{2}{n}$.

QUESTION 4 **(25 marks)**

Let X and Y have the joint density

$$f(x, y) = \begin{cases} Ae^{-x}, & \text{if } 0 \leq y \leq x \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of A;

(b) Determine the marginal density functions of X and of Y;

(c) Find the covariance of X and Y;

(d) Find the correlation of X and Y;

(e) Find $\mathrm{E}(Y | X = \frac{1}{2})$.

QUESTION 5 **(15 marks)**

A university has 33,000 students and plans to build a new library. Once the library is ready to use, the probability of each student studying in the new library during the peak hours is estimated to be 0.05.

(a) Let X be the number of students studying in the new library during the peak hours, find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(b) To ensure that at least 99% of those students who go to the library in the peak hours can find their seats, at least how many seats should the new library prepare (refer to the cumulative normal distribution table on the next page if needed)?

AY 17/18

QUESTION 1 **(20 marks)**

Suppose that X is a random variable taking values in $\{0,1,2\}$ and Y is a random variable taking values in $\{0,1,2,3\}$. A partial joint probability table of X and Y is given below.

	$Y=0$	$Y=1$	$Y=2$	$Y=3$
$X=0$	0.10		0.16	
$X=1$	0.05	0.04	0.08	0.04
$X=2$	0.10	0.08		0.08

(a) (5 marks) Find the probability $\Pr(X > Y)$.

(b) (5 marks) Find the conditional mass function of X given that $Y = 0$.

(c) (5 marks) If the marginal distribution of Y is uniform, what is the probability $\Pr(X = 0, Y = 1)$?

(d) (5 marks) Find the probability $\Pr(X = 1)$ and determine whether X and Y are independent when the marginal distribution of Y is uniform.

QUESTION 2 **(15 marks)**

Suppose you have in your pocket a fair coin and a two-headed coin. You select one of the coins at random (with equal probability). All questions below are about this one coin; it is not replaced, and no other coin is drawn.

(a) (5 marks) When you flip it, it shows head. What is the probability that it is the fair coin?

(b) (5 marks) What is the probability that it will show a head on a second flip, given that it shows a head on the first flip?

(c) (5 marks) When you flip the coin a second time, it shows head again. What is the probability that it is the fair coin?

Leave your answers as fractions.

QUESTION 3 **(20 marks)**

A random rectangle is formed in the following way: The base X is chosen to be uniformly random on $[0,1]$, and after having generated the base, the height H is chosen to be uniform on $[0,X]$.

(a) (5 marks) Find the conditional expectation $\mathbb{E}(H|X)$.

(b) (5 marks) Find the expected area of the rectangle.

(c) (5 marks) Find the variance of the area of the rectangle.

(d) (5 marks) Find the correlation between the base and the area of the rectangle.

Leave your answers as fractions, or numerical values that are accurate up to 0.001.

QUESTION 4 **(25 marks)**

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables of the following density function:

$$f(x) = \begin{cases} Cx^{-(1+\alpha)}, & \text{if } x \geq 1; \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ is a constant and C is the normalization constant. Let Z_n be the geometric mean of $X_1, \ldots, X_n$, that is, $Z_n = (X_1 X_2 \cdots X_n)^{1/n}$.

(a) (5 marks) Determine the value of C. (The value of C may depend on α, in which case your answer should be expressed in terms of α).

(b) (10 marks) Find the density function of $Y_i = \ln X_i$.

(c) (5 marks) Determine whether Z_n converges to some constant L in probability as $n \to \infty$. If yes, find the constant L. (Hint: you may find part (b) useful).

(d) (5 marks) If your answer to part (c) is 'yes', show that

$$\Pr\left(Z_n \leq e^{\frac{1}{\alpha}} L\right) \geq 1 - \frac{1}{n}.$$

If your answer to part (c) is 'no', show that

$$\Pr(Z_n \geq e^{\frac{1}{\alpha}} n) \geq 1 - \frac{1}{n}.$$

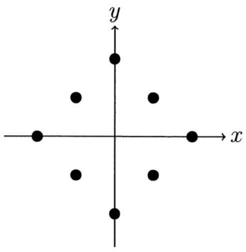
AY 2018 | 2019

QUESTION 1 **(25 marks)**

Suppose that (X, Y) is a point drawn uniformly at random from the set

$$S = \{(2,0), (1,1), (0,2), (-1,1), (-2,0), (-1,-1), (0,-2), (1,-1)\}.$$

Pictorially, (X, Y) is a uniformly random point among the eight points illustrated in the picture below.



(a) (5 marks) Find $\Pr(X^2 + Y^2 \le 3)$.

(b) (5 marks) Find the conditional probability $\Pr(Y \le X^2 | Y \ge X^2)$.

(c) (5 marks) Find the marginal distribution of $|X|$ (the absolute value of X).

(d) (5 marks) Find in the range of Y the value k that minimizes the conditional variance $\text{Var}(X^7 \mid Y = k)$.

(e) (5 marks) Are X and Y independent? State your reason.

QUESTION 2 **(15 marks)**

Suppose that X and Y are uniformly distributed on $(0, 1]$. Let $Z = X/Y$.

(a) (10 marks) Find the probability density function of Z and the median of Z.

(b) (5 marks) Find the expected value and the variance of $\sqrt{Z}$. In the case of non-existence, answer 'do not exist' and give reasons.

QUESTION 3 **(15 marks)**

Let $n \ge 2$ and $X_1, \ldots, X_n$ be i.i.d. random variables taking values in $\{-2, 1\}$ with probabilities $\Pr(X_1 = 1) = 2/3$ and $\Pr(X_1 = -2) = 1/3$. Denote $S_n = \sum_{i=1}^n X_i$.

(a) (5 marks) Find $\mathbb{E}(S_n)$, $\text{Var}(S_n)$ and $\lim_{n\to\infty} \Pr(S_n \ge \sqrt{n \ln n})$.

(b) (5 marks) Find $\lim_{n\to\infty} \Pr(S_n \ge 0)$.

(c) (5 marks) Show that there exists a constant C (which does not depend on n) such that

$$\Pr(S_n \ge \sqrt{n \ln n}) \le \frac{C}{(\ln n)^{3/2}}.$$

No points will be given for the limit in part (a) if you invoke part (c) without proving it.

QUESTION 4 **(20 marks)**

Let $X_1, \ldots, X_n$ be independent uniform random variables over $[0, 1]$ and $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be their order statistics.

(a) (8 marks) Find the expectations $\mathbb{E}(X_{(1)})$ and $\mathbb{E}(X_{(n)})$.

(b) (12 marks) Find the covariance $\text{Cov}(X_{(1)}, X_{(n)})$.

A Y 19/20

QUESTION 1 **(15 marks)**

An urn contains 4 blue balls and 3 red balls. A fair die is rolled. If the number shown on the die is i ($i = 1, 2, \ldots, 6$), then i balls are drawn from the urn uniformly at random without replacement.

(a) (5 marks) When $i = 3$, what is the probability that all balls drawn are red?

(b) (10 marks) Given that all balls drawn are red, what is the probability that the remaining balls in the urn are all blue?

QUESTION 2 **(30 marks)**

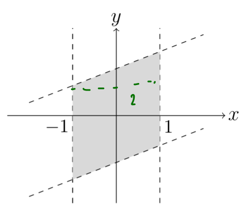
For each $\theta \in [0, \pi/2)$, we define a planar region A_θ as

$$A_\theta = \{(x, y) : |x \cos\theta - y \sin\theta| < 1\}, \quad 0 \le \theta < \frac{\pi}{2}.$$

For each $\theta \in (0, \pi/2)$, we define a planar region D_θ as

$$D_\theta = A_\theta \cap A_0, \quad 0 < \theta < \frac{\pi}{2}.$$

The shaded region on the right is a sketch of $D_{6/5}$.



Let (X, Y) be a point uniformly distributed in D_θ for some $\theta \in (0, \pi/2)$.

(a) (5 marks) Find the marginal distribution of X.

(b) (5 marks) Find the conditional density function of Y given $X = x$.

(c) (5 marks) Find the conditional expectation $\mathbb{E}(Y|X)$.

(d) (5 marks) Find $\mathbb{E}(Y)$.

(e) (5 marks) Find $\mathrm{Var}(Y)$.

(f) (5 marks) Are X and Y correlated? If they are correlated, are they positively or negatively correlated? If they are not correlated, are they independent? State your reason.

QUESTION 3 **(20 marks)**

Suppose that a fair die is rolled n times independently. For each $i = 1, \ldots, 6$, let X_i denote the number of times i appears.

(a) (2 marks) Find the probability mass function of X_2.

(b) (2 marks) Find the joint probability mass function of (X_2, X_4).

(c) (2 marks) Find the conditional probability mass function of X_2 given that $X_4 = \ell$ $(0 \le \ell \le n)$.

(d) (4 marks) Find the probability mass function of $X_2 + X_4$.

(e) (10 marks) Find $\mathrm{Cov}(X_2, X_4)$.

QUESTION 4 **(20 marks)**

For each integer $m \ge 1$, we define $2m$ numbers $b_{m,1}, \ldots, b_{m,2m}$ as follows:

$$b_{m,i} = \begin{cases} 1, & 1 \le i \le m; \\ -1, & m+1 \le i \le 2m. \end{cases}$$

Let $Y_1, Y_2, \ldots$ be i.i.d. random variables such that $\Pr\{Y_i = 0\} = \Pr\{Y_i = 1\} = 1/2$.

(a) (10 marks) Show that there exists a constant $c_1 > 0$ (which does not depend on m) such that for all $m \ge 1$,

$$\Pr\left\{\left|\sum_{i=1}^{2m} b_{m,i} Y_i\right| > c_1 \sqrt{m}\right\} \le \frac{1}{10}.$$

(b) (10 marks) Show that there exists a constant $c_2 > 0$ such that

$$\lim_{m\to\infty} \Pr\left\{\left|\sum_{i=1}^{2m} b_{m,i} Y_i\right| > c_2 \sqrt{m}\right\} > \frac{9}{10}.$$

AY 21/22

QUESTION 1. **(20 marks)**

An urn initially contains one red and one blue ball. At each stage, a ball is randomly chosen from urn and then replaced along with another of the same colour. So after the i-th round, there are $2+i$ many balls in the urn. Let X denote the selection number of the first chosen ball that is blue. For instance, if the first selection is red and the second blue, then X is equal to 2.

(a) Find $P\{X > i\}$ for each $i \geq 1$.

(b) Show that, with probability 1, a blue ball is eventually chosen. (That is, show that $P\{X < \infty\} = 1$.)

(c) Find $\mathbb{E}[X]$.

QUESTION 2. **(10 Marks)**

Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5000-mile trip, what is the probability that he or she will be able to complete the trip without having to replace the car battery?

QUESTION 3. **(15 Marks)**

Two numbers X and Y are chosen independently from the uniform distribution on the unit interval $[0, 1]$. Let Z be the maximum of the two numbers. Find the probability density function of Z, and then find its expected value and variance.

QUESTION 4. **(20 marks)**

Let X and Y be continuous random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find $P\left\{0 \leq x \leq \frac{1}{4}, 0 \leq y \leq \frac{1}{4}\right\}$.

(b) Find marginal density functions $f_X(x)$ and $f_Y(y)$, and determine whether X and Y are independent or not.

(c) Find the conditional pdf of X given $Y = 0.3$.

QUESTION 5. **(15 marks)**

Let X and Y be discrete random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } x = 3, y = 4, \\ \frac{1}{3} & \text{if } x = 3, y = 6, \\ \frac{1}{6} & \text{if } x = 5, y = 6, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find $\mathbb{E}(X), \mathbb{E}(Y), \mathbb{E}(XY)$ and $Cov(X,Y)$.

(b) Find $Var(X)$, $Var(Y)$ and the correlation coefficient of X and Y.

QUESTION 6. **(20 marks)**

Let $X_1, \cdots, X_{20}$ be independent Poisson random variables with mean 1.

(a) Use the Markov's inequality to obtain an upperbound on

$$P\left\{\sum_{i=1}^{20} X_i > 15\right\}.$$

(b) Use the central limit theorem to approximate the same probability in part (a).

AY 22/23

QUESTION 1. **(10 marks)**

Consider the discrete random variable X, with outcomes $1, 2, 3, 4, 5$ and corresponding probability distribution

x	1	2	3	4	5
$p(x)$	0.1	0.3	0.4	0.1	0.1

Find the expectation and the variance of X.

QUESTION 2. **(15 Marks)**

Let X be an exponential random variable with parameter $\lambda = 1$ and Y be a uniform random variable on $[0, 1]$. Suppose that X and Y are independent. Find the probability density function of $Z = X + Y$.

QUESTION 3. **(20 Marks)**

Let X and Y have the joint probability density function

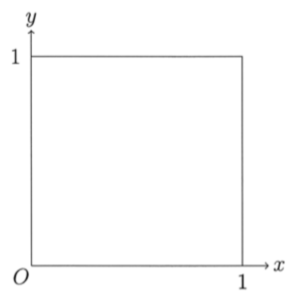
$$f_{X,Y}(x, y) = \begin{cases} 4xy, & 0 < x < 1, 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the region representing the event $X + Y \leq 1$.

(b) Find $P(X + Y \leq 1)$.

(c) Find the marginal distribution of X.

(d) Are X and Y independent? Justify your answers.



QUESTION 4. **(20 marks)**

The joint probability density function of X and Y is given by

$$f(x, y) = \frac{1}{y} e^{-(y + \frac{x}{y})} \qquad x > 0, y > 0.$$

Find $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathbb{E}(XY)$ and $Cov(X, Y)$.

QUESTION 5. **(20 marks)**

An urn contains a white and b black balls, where a and b are positive integers. One ball at a time is randomly drawn until the first white ball is drawn. Find the expected number of black balls that are drawn, by considering conditional probabilities.

QUESTION 6. **(15 marks)**

A certain component is critical to the operation of an electrical system and must be replaced immediately upon failure. If the mean lifetime of this type of component is 100 hours and its standard deviation is 30 hours, how many of these components must be in stock so that the probability that the system is in continual operation for the next 2000 hours is at least 0.95?

AY23/24

QUESTION 1. **(10 marks)**

An insurance company observed that the number of injury claims per month is modeled by a random variable X with

$$\mathbb{P}\mathrm{r}\{\mathrm{X}=\mathrm{n}\} = \frac{1}{(\mathrm{n}+1)(\mathrm{n}+2)}, \qquad \text{where } \ \mathrm{n} \geq 0.$$

Find the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

QUESTION 2. **(15 Marks)**

Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} 15y & \text{for } x^2 \leq y \leq x; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{P}\mathrm{r}\{\mathrm{X} \geq \mathrm{Y}\}$.

(b) Find the marginal distribution $g(y)$ of Y.

(c) Are X and Y independent? Justify your answers.

QUESTION 3. **(20 Marks)**

Let X be a Poisson random variable with parameter λ.

(a) Prove that $\mathbb{E}(X^n) = \lambda\,\mathbb{E}((X+1)^{n-1})$.

(b) By using the identity proved in (a), prove that $\mathbb{E}(X^3) = \lambda^3 + 3\lambda^2 + \lambda$.

QUESTION 4. **(20 marks)**

Let X be a standard normal random variable, and let I, independent of X, be such that $\mathbb{P}\mathrm{r}\{\mathrm{I}=1\} = \mathbb{P}\mathrm{r}\{\mathrm{I}=0\} = 1/2$. Now define Y by

$$Y = \begin{cases} X, & \text{if } I = 1; \\ -X, & \text{if } I = 0. \end{cases}$$

In words, Y is equally likely to equal to either X or $-X$.

(a) Are I and Y independent? Justify your answer.

(b) Prove that Y is also a standard normal variable.

(c) Prove that $\mathrm{Cov}(X,Y) = 0$.

QUESTION 5. **(20 marks)**

A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. We assume that the miner is at all times equally likely to choose any one of the doors.

(a) Find the expected length of time until he reaches safety.

(b) Find the variance of the length of time until the miner reaches safety.

QUESTION 6. **(15 marks)**

Suppose that we throw a fair die n times and after each throw, we record the score showing as X_i, $(i = 1, 2, \cdots, n)$.

(a) Prove that $\mathbb{E}(X_i) = 7/2$ and $\mathrm{Var}(X_i) = 35/12$, for each i.

(b) Let $\overline{X} = \left(\sum_{i=1}^{n} X_i\right)/n$. Prove that $\mathbb{E}(\overline{X}) = \frac{7}{2}$ and $\mathrm{Var}(\overline{X}) = \frac{35}{12n}$.

(c) Let $n = 25$.

(i) Use Chebyshev's inequality to estimate

$$\mathbb{P}\mathrm{r}\{|\overline{\mathrm{X}} - 3.5| \geq 0.5\}.$$

(ii) Use the Central Limit Theorem (CLT) to estimate

$$\mathbb{P}\mathrm{r}\{|\overline{\mathrm{X}} - 3.5| \geq 0.5\}.$$