(1) One of the most important joint distributions is the **multinomial distribution**, which arises when a sequence of n independent and identical experiments is performed. Suppose that each experiment can result in any one of r possible outcomes, with respective probabilities $p_1, p_2, \cdots, p_r$, with $\sum_{i=1}^{r} p_i = 1$. Let X_i denote the number of the n experiments that result in outcome number i. Prove the following

$$\mathbb{P}r\{X_1 = n_1, \cdots, X_r = n_r\} = \frac{n!}{n_1! \cdots n_r!} p_1^{n_1} \cdots p_r^{n_r}.$$

PMF of multinomial distribution

probability of outcome 1 occurs n_1 times.

arrange all the n experiments.

r possible outcomes.

Outcome 1 occurs n_1 times.

⋮

Outcome r occurs n_r times.

In binomial:

$$P(X_1 = n_1, X_2 = n_2) = \frac{n!}{n_1!\, n_2!} p_1^{n_1} p_2^{n_2}$$

Where $p_2 = 1 - p_1$, $n_2 = n - n_1$

$$\sum_{n_1 + \cdots + n_r = n} \frac{n!}{n_1! \cdots n_r!} p_1^{n_1} \cdots p_r^{n_r} = (p_1 + p_2 + \cdots + p_r)^n = 1^n = 1$$

(2) In the multinomial distribution above, suppose we are given that n_j of the trials resulted in outcome j, for $j = k+1, \cdots, r$, where $\sum_{j=k+1}^{r} \cancel{x}\, n_j = m \le n$. Find the conditional distribution of

$$\mathbb{P}r\{X_1 = n_1, \cdots, X_k = n_k \mid X_{k+1} = n_{k+1}, \cdots, X_r = n_r\}.$$

$$n_{k+1} + n_{k+2} + \ldots + n_r = m$$

$$n_1 + n_2 + \ldots + n_r = n$$

$$= \frac{P(X_1 = n_1, \ldots, X_k = n_k, X_{k+1} = n_{k+1}, \ldots, X_r = n_r)}{P(X_{k+1} = n_{k+1}, \ldots, X_r = n_r)}$$

$$= \frac{\frac{n!}{n_1! \ldots n_r!} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}}{\sum_{n_1 + \ldots + n_k = n-m} \underbrace{P(X_{k+1} = n_{k+1}, \ldots, X_r = n_r}_{B} \mid \underbrace{X_1 = n_1, \ldots, X_k = n_k)}_{A} \underbrace{P(X_1 = n_1, \ldots, X_k = n_k)}_{A}}$$

$$P(B \mid A) \cdot P(A) = P(A \cap B)$$

$$= \frac{\frac{n!}{n_1! \ldots n_r!} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}}{\sum_{n_1 + \ldots + n_k = n-m} \frac{n!}{n_1! \ldots n_r!} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}}$$

$$\boxed{\sum_{n_1 + \ldots + n_r = n} \frac{n!}{n_1! \ldots n_r!} p_1^{n_1} \ldots p_r^{n_r} = (p_1 + p_2 + \ldots + p_r)^n}$$

$$= \frac{\frac{n!}{n_1! \ldots n_r!} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}}{\frac{1}{n_{k+1}! \ldots n_r!} p_{k+1}^{n_{k+1}} \cdots p_r^{n_r} \cdot \frac{n!}{(n-m)!} \sum_{n_1 + \ldots + n_k = n-m} \frac{(n-m)!}{n_1! \ldots n_k!} p_1^{n_1} \ldots p_k^{n_k}}$$

$$= \frac{\frac{\cancel{n!}}{n_1! \ldots \cancel{n_r!}} p_1^{n_1} p_2^{n_2} \cancel{\ldots p_r^{n_r}}}{\frac{1}{\cancel{n_{k+1}! \ldots n_r!}} \cancel{p_{k+1}^{n_{k+1}} \cdots p_r^{n_r}} \cdot \frac{\cancel{n!}}{(n-m)!} \boxed{(p_1 + \ldots + p_k)^{n-m}}}$$

$$= \frac{(n-m)!}{n_1! \ldots n_k!} \cdot \frac{p_1^{n_1} \ldots p_k^{n_k}}{(p_1 + \ldots + p_k)^{n-m}}$$

(3) Suppose that the number of people who enter a post office on a given day is a Poisson random variable with parameter λ. Prove that if each personwho enters the post office is a male with probability p and a female with probability $1-p$, then the number of males and females entering the post office are **independent** Poisson random variables with respective parameters λp and $\lambda(1-p)$.

Let X be the number of males entering post office.

Let Y be the ... females ....

continuous

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

To show $\forall i,j$ $\quad \underline{P(X=i, Y=j) = P(X=i)P(Y=j)}$

$$f_X(x) = \int f(x,y)\,dy$$

$$LHS = P(X=i, Y=j) = P(X=i, Y=j \mid X+Y=i+j)\, P(X+Y=i+j)$$

$$+ \underbrace{P(X=i, Y=j \mid X+Y \neq i+j)}_{=0}\, P(X+Y \neq i+j)$$

$$X+Y \sim Po(\lambda)$$

$$= \frac{\cancel{(i+j)!}}{i!\,j!} p^i (1-p)^j \cdot e^{-\lambda} \frac{\lambda^{i+j}}{\cancel{(i+j)!}}$$

$$\boxed{e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}}$$

$$= e^{-\lambda} \frac{(\lambda p)^i}{i!} \cdot \frac{[\lambda(1-p)]^j}{j!}$$

$$P(X=i) = \sum_{j=0}^{\infty} P(X=i, Y=j) = e^{-\lambda} \frac{(\lambda p)^i}{i!} \sum_{j=0}^{\infty} \frac{\lambda(1-p)^j}{j!} = e^{-\lambda} \frac{(\lambda p)^i}{i!} e^{\lambda(1-p)}$$

$$X \sim Po(\lambda p) \Longleftarrow = e^{-\lambda p} \frac{(\lambda p)^i}{i!}$$

$$P(Y=j) = \sum_{i=0}^{\infty} P(X=i, Y=j) = e^{-\lambda} \frac{(\lambda(1-p))^j}{j!} \sum_{i=0}^{\infty} \frac{(\lambda p)^i}{i!} = e^{-\lambda} \frac{(\lambda(1-p))^j}{j!} e^{\lambda p}$$

$$Y \sim Po(\lambda(1-p)) \Longleftarrow = e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!}$$

(6) Let X and Y be independent binomial random variables with respective parameters (n, p) and $(\cancel{n}, \cancel{q})$. Calculate the distribution of $X + Y$.

$m \quad p$

$X \in \{0, 1, \ldots, n\} \qquad Y \in \{0, \ldots, m\}$

For $k \in \{0, 1, \ldots, n+m\}$

$k > n$ or $k \leq n$

$$P(X+Y=k) = \sum_{i=0}^{n} P(Y = k - X \mid X = i)\, P(X = i)$$

$$= \sum_{i=0}^{n} P(Y = k-i)\, P(X=i) = \sum_{i=0}^{n} \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i} \cdot \binom{n}{i} p^{i} (1-p)^{n-i}$$

$$= p^{k} (1-p)^{n+m-k} \sum_{i=0}^{n} \binom{n}{i}\binom{m}{k-i}$$

$$= \binom{n+m}{k} p^{k} (1-p)^{n+m-k}$$

$$\binom{n+m}{k} = \binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \ldots + \binom{n}{k}\binom{m}{0}$$

$X + Y \sim Bin(n+m, p)$

(5) Consider n independent trials, with each trial being a success with probability p. Given a total of k successes, prove that all possible orderings of the k successes and $n - k$ failures are equally likely.

Let X denotes the number of successes.

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Let O be an arbitrary ordering of the k successes.

$$P(O \mid X=k) = \frac{P(O \cap X=k)}{P(X=k)} = \frac{p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{1}{\binom{n}{k}}$$

Any possible ordering has equal probability $\frac{1}{\binom{n}{k}}$ of occuring.