Extra Exercises for Week 8

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Most questions are taken from the textbook: A First Course in Probability (9th edition) by Sheldon Ross.

Problem 1. If X is a normal random variable with mean μ and variance σ^2 , then the random variable $Y = e^X$ is said to be a lognormal random variable with parameters μ and σ^2. Derive $\mathbb{E}(Y)$.

Problem 2. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x}, \quad x \geq 0.$$

Compute the expected lifetime of such a tube.

Problem 3. The median of a continuous random variable having cumulative distribution function F is that value m such that $F(m) = \frac{1}{2}$. That is, a random variable is just as likely to be larger than its median as it is to be smaller. Find the median of X if X has the following distribution.

(a) Uniform distribution over (a, b);

(b) Standard normal distribution, $N(0, 1)$;

(c) Exponential distribution with parameter λ;

(d) Lognormal distribution with parameters 0 and 1.

Problem 4. A standard Cauchy random variable has density function

$$f(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty.$$

Show that if X is a standard Cauchy random variable, then $1/X$ is also a standard Cauchy random variable.

Problem 5. Let X be a standard normal random variable. Show that $\mathbb{E}(X^{n+1}) = n\mathbb{E}(X^{n-1})$, for integer $n \geq 1$. Prove that

(a) $\mathbb{E}(X^k) = 0$, if k is an odd integer.

(b) $\mathbb{E}(X^k) = (k-1) \cdot (k-3) \cdot \ldots \cdot 3 \cdot 1$, if k is an even integer.

We also write $(k-1) \cdot (k-3) \cdot \ldots \cdot 3 \cdot 1 = (k-1)!!$ as a double factorial. Read more here.

Problem 6. A point is chosen at random on a line segment of length L. Find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

Problem 7. Suppose that X is a normal random variable with mean 5. If $P(X > 9) = 0.2$, approximately what is $Var(X)$? You may need to refer to the standard normal distribution table.

Problem 8. Let X be a random variable with the following density function

$$f_X(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the CDF and PDF of $Y = X^2$.

Answers (Let me know if there are any discrepancies):

1. $\sqrt{e}$
2. 2

3(a). $\frac{a+b}{2}$

3(b). 0

3(c). $\frac{\ln 2}{\lambda}$

3(d). 1

6. $\frac{2}{5}$
7. 22.6
8.

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ \frac{2\sqrt{y}}{3}, & 0 \leq y < 1, \\ \frac{1}{3}(\sqrt{y} - 1) & 1 \leq y < 4, \\ 1, & y \geq 4. \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}}, & 0 \leq y < 1, \\ \frac{1}{6\sqrt{y}}, & 1 \leq y < 4, \\ 0, & \text{otherwise.} \end{cases}$$