MH2500 Mini-Quiz

Question 1

Each game you play is a win with probability $0 < p < 1$. You plan to play 5 games, but if you win the fifth game, then you will keep on playing until you lose. Find the expected number of games that you lose.

MH2500 Mini-Quiz

Solution 1

In the first four games, let X be the number of games that you lose.
Then $X \sim \text{Bin}(4, 1-p)$. Hence, $\mathbb{E}(X) = 4(1-p)$. From the fifth game onwards, either you keep on winning, or once you lose, that's it. The probability that you keep on winning endlessly is $\lim_{n\to\infty} p^n = 0$. Hence, you will lose for once almost surely.
The expected number of games that you lose is $4(1-p)+1$.

MH2500 Mini-Quiz

Alternative solution by one of you

Based on the question, we can say that after the fourth game, we have just one more to lose. Let X be the number of games lost.

$$P(X=1) = P(\text{losing 0 out of first 4 games}) = p^4$$

$$P(X=2) = P(\text{losing 1 out of first 4 games}) = \binom{4}{1}p^3(1-p)$$

$$P(X=3) = P(\text{losing 2 out of first 4 games}) = \binom{4}{2}p^2(1-p)^2$$

$$P(X=4) = P(\text{losing 3 out of first 4 games}) = \binom{4}{3}p(1-p)^3$$

$$P(X=5) = P(\text{losing 4 out of first 4 games}) = (1-p)^4$$

$$\mathbb{E}(X) = \sum_{k=1}^{5} kP(X=k) = 5-4p$$

MH2500 Mini-Quiz

Some suggestions

Given $X \sim \text{Bin}(n, p)$, we want to find $\mathbb{E}(X)$. Many of you tend to compute

$$\mathbb{E}(X) = \sum_{k=0}^{n} kP(X = k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}.$$

It is fine, but very time-consuming and prone to errors. Just use $\mathbb{E}(X) = np$.

Some people may not be able to identify the distribution and just compute $E(X)$ using the longer method. You should try to tell whether the scenario stated in the question follows a known distribution, this will help you in subsequent computations if the question gets longer. For those known distributions, you should also know the formula for pmf/pdf/mean/variance directly. "Known" distribution means those you see in the notes/tutorials, e.g. Bernoulli, Binomial, Poisson, Geometric, Normal, Exponential, Uniform, Gamma.

MH2500 Mini-Quiz

Question 2

Ten balls are to be distributed among 5 urns, with each ball going into urn i with probability p_i, where $\sum_{i=1}^{5} p_i = 1$. Let X_i denote the number of balls that go into urn i.
Assume that events corresponding to the locations of different balls are independent. Find $P(X_1 + X_2 + X_3 = 7)$.

MH2500 Mini-Quiz

Solution 2

Balls either go into urn $1, 2, 3$ or not. Hence, $X_1 + X_2 + X_3 \sim$ Bin$(10, p_1 + p_2 + p_3)$. Therefore,

$$P(X_1 + X_2 + X_3 = 7) = \binom{10}{7}(p_1 + p_2 + p_3)^7(p_4 + p_5)^3.$$

MH2500 Mini-Quiz

Question 3

Let X be a discrete random variable such that $a \leq X \leq b$. Use the definition of expectation to prove that $a \leq \mathbb{E}(X) \leq b$.

MH2500 Mini-Quiz

Solution 3

$$\mathbb{E}(X) = \sum_x xP(X=x) \geq \sum_x aP(X=x) = a\sum_x P(X=x) = a.$$

With the same analogy, $\mathbb{E}(X) \leq b$.