Problem 7.1

Let $D_\theta, 0 \le \theta \le 1$, be the discrete distribution with the following PMF:

x	0	1	2	3
$f(x)$	$\frac{2}{3}\theta$	$\frac{1}{3}\theta$	$\frac{2}{3}(1-\theta)$	$\frac{1}{3}(1-\theta)$

and $f(x) = 0$ otherwise. Let $X_1, \ldots, X_n$ be an i.i.d. random sample drawn from D_θ.

a) Find the MLE $\hat{\theta}$ for θ based on $X_1, \ldots, X_n$.
b) Find the MME $\tilde{\theta}$ for θ based on $X_1, \ldots, X_n$.

Let n_0 be the number of samples i.e. $X_i = 0$
n_1 - - - - - $X_i = 1$
n_2 - - - $X_i = 2$
n_3 - - - - $X_i = 3$

(a)

$n_0 + n_1 + n_2 + n_3 = n$

$$L(\theta) = \prod_{i=1}^{n} f(X_i \mid \theta) = \left(\frac{2}{3}\theta\right)^{n_0}\left(\frac{1}{3}\theta\right)^{n_1}\left(\frac{2}{3}(1-\theta)\right)^{n_2}\left(\frac{1}{3}(1-\theta)\right)^{n_3}$$
$$= \frac{2^{n_0+n_2}}{3^n}\,\theta^{n_0+n_1}(1-\theta)^{n_2+n_3}$$

Find $\hat{\theta} = \arg\max_{0 \le \theta \le 1} \ln L(\theta)$

$$\ln L(\theta) = \ln\left(\frac{2^{n_0+n_2}}{3^n}\right) + (n_0+n_1)\ln\theta + (n_2+n_3)\ln(1-\theta)$$

Verify standard conditions for $L(\theta)$.

(i) $L(\theta) > 0$ for $\theta \in (0,1)$

(ii) L is differentiable.

(iii) $\lim_{\theta \to 0^+} L(\theta) = 0$ and $\lim_{\theta \to 1^-} L(\theta) = 1$

$n_0 + n_1 > 0$
$n_2 + n_3 > 0$.

Case 1

For $\theta \in (0,1)$, $\hat{\theta} = \arg\max \ln L(\theta)$ exists.

$$\frac{d}{d\theta}\ln L(\theta) = \frac{n_0+n_1}{\theta} - \frac{n_2+n_3}{1-\theta} = 0 \Rightarrow \hat{\theta} = \frac{n_0+n_1}{n_0+n_1+n_2+n_3} = \frac{n_0+n_1}{n}$$

Case 2 : $n_0 + n_1 = 0$

$$L(\theta) = \frac{2^{n_0+n_2}}{3^n}\,\theta^{n_0+n_1}(1-\theta)^{n_2+n_3} = \frac{2^{n_0+n_2}}{3^n}(1-\theta)^{n_2+n_3}$$

$\theta \in [0,1]$

$L(\theta)$ is maximized when $\theta = 0$ $\Rightarrow$ $\hat{\theta}_{MLE} = 0 = \dfrac{n_0 + n_1}{n}$

Case 3: $n_2 + n_3 = 0$

$$L(\theta) = \frac{2^{n_0+n_2}}{3^n}\,\theta^{n_0+n_1}$$

$L(\theta)$ is maximized when $\theta = 1$

$\Rightarrow$ $\hat{\theta}_{MLE} = 1 = \dfrac{n_0+n_1}{n}$

$$n_0 + n_1 + \overbrace{n_2 + n_3}^{0} = n$$

$$n_0 + n_1 = n$$

Problem 7.2

Let $X_1, \ldots, X_n$ be an i.i.d. sample drawn from a uniform distribution $U(0,\theta)$ distribution. It was shown in lecture that $\hat{\theta}_n = \max(X_1, \ldots, X_n)$ is the MLE for θ based on $X_1, \ldots, X_n$. Show that

$$P(|\hat{\theta}_n - \theta| > \varepsilon) = \left(\frac{\theta - \varepsilon}{\theta}\right)^n, \qquad \text{for } 0 < \varepsilon < \theta$$

Is $\hat{\theta}_n$ a consistent estimator for θ?

Consistent estimator,

For all $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|\hat{\theta}_n - \theta| > \varepsilon) = 0$$

$$\lim_{n \to \infty} \left(\frac{\theta - \varepsilon}{\theta}\right)^n = 0$$

Because $\frac{\theta - \varepsilon}{\theta} < 1$

Consistent

$\forall \varepsilon > 0$

$X_i \sim \mathrm{Unif}(0, \theta)$

$\hat{\theta}_n = \max[X_1, \ldots, X_n]$

$$\begin{aligned}
P(|\hat{\theta}_n - \theta| > \varepsilon) &= P(\hat{\theta}_n - \theta > \varepsilon \text{ or } \hat{\theta}_n - \theta < -\varepsilon) \\
&= P(\hat{\theta}_n > \theta + \varepsilon \text{ or } \hat{\theta}_n < \theta - \varepsilon) \\
&= P(\hat{\theta}_n < \theta - \varepsilon) \\
&= P(\max(X_1, \ldots, X_n) < \theta - \varepsilon) \\
&= P\left(\bigcap_{i=1}^{n} \{X_i < \theta - \varepsilon\}\right) \\
&= \prod P(X_i < \theta - \varepsilon) \\
&= [P(X_1 < \theta - \varepsilon)]^n \\
&= \left(\frac{\theta - \varepsilon}{\theta}\right)^n
\end{aligned}$$

indp

identical

CDF of Unif$(0, \infty)$

$$P\left(\sum_{i=1}^{n} X_i \geq k\right) \leq P\left(\bigcup \{X_i \geq \tfrac{k}{n}\}\right)$$

Problem 7.3

Let $X_1, \dots, X_n$ be an i.i.d. sample drawn from a distribution with PDF of given as

$$f(x|\theta) = \frac{\theta}{\sqrt{2\pi}} e^{-\theta^2 x^2/2} \quad \text{for all } x \in \mathbb{R}$$

where $\theta \in (0, \infty)$ is an unknown parameter.

a) Find the maximum likelihood estimator $\hat{\theta}$ (MLE) for θ based on $X_1, \dots, X_n$, under the assumption that $\sum_{i=1}^n X_i^2 > 0$.

b) Show that no MLE exists if $\sum_{i=1}^n X_i^2 = 0$.

(a) $L(\theta) = \prod_{i=1}^{n} f(X_i | \theta) = \prod_{i=1}^{n} \frac{\theta}{\sqrt{2\pi}} \exp\left(-\frac{\theta^2 X_i^2}{2}\right)$

$= \frac{\theta^n}{(2\pi)^{n/2}} \exp\left(-\frac{\theta^2}{2} \sum X_i^2\right)$

Standard conditions : $L(\theta) > 0$, L is diff.

$\lim_{\theta \to 0^+} L(\theta) = 0$ and $\lim_{\theta \to \infty} L(\theta) = 0$ (L'Hopitals Rule)

$\ln L(\theta) = n \ln \theta - \frac{n}{2} \ln(2\pi) - \frac{\theta^2}{2} \sum X_i^2$

$\frac{d}{d\theta} \ln L(\theta) : \frac{n}{\theta} - \theta \sum X_i^2 = 0 \Rightarrow \hat{\theta}_{MLE} = \sqrt{\frac{n}{\sum X_i^2}}$

(b) What if $\sum X_i^2 = 0$?

$L(\theta) = \frac{\theta^n}{(2\pi)^{n/2}} \exp\left(-\frac{\theta^2}{2} \sum X_i^2\right) = \frac{\theta^n}{(2\pi)^{n/2}}$ $\quad \theta \in (0, \infty)$

maximum of $L(\theta)$ does not exist. $\Rightarrow$ $\hat{\theta}_{MLE}$ does not exist

Problem 7.4

Let $X_1, \dots, X_n$ be i.i.d. $\sim D_\theta$, where θ is an unknown parameter. Let $g(.)$ be a one-to-one function.

a) Suppose that $\hat{\theta}$ is an MLE for θ based on $X_1, \dots, X_n$. Show that $g(\hat{\theta})$ is an MLE for $g(\theta)$.

Let $g(\theta) = \alpha$, R be the range of θ.

Let $S = \{g(\theta) : \theta \in R\}$ be the range of α.

Since g is injective, there exists g^{-1}, s.t. $g^{-1}g(\theta) = \theta$.

$\hat{\theta}$ is MLE for $\theta \Rightarrow \hat{\theta} = \underset{\theta \in R}{\arg\max}\ L(\theta)$, where $L(\theta)$ is the likelihood func. in terms of θ.

$g(\theta) = \alpha \Rightarrow g^{-1}g(\theta) = g^{-1}(\alpha) \Rightarrow \theta = g^{-1}(\alpha)$

$L(\theta) = L(g^{-1}(\alpha)) \Rightarrow L \circ g^{-1}$ is the likelihood func. in terms of $\alpha = g(\theta)$

MLE for $g(\theta) = \underset{\alpha \in S}{\arg\max}\ \underbrace{L(g^{-1}(\alpha))}^{\text{likelihood func.}} = g(\hat{\theta})$

Because $L(g^{-1}(g(\hat{\theta})) = \underbrace{L(\hat{\theta})}_{\text{maximum.}} \geq L(\theta) \quad \forall\ \theta \in R$.

b) Suppose that $\widetilde{\theta}$ is the method of moments estimator (MME) for θ based on $X_1, \ldots, X_n$. Show that $g(\widetilde{\theta})$ is an MME for $g(\theta)$.

There exists a unique function h and least t , st.

$$\theta = h(\mu_1, \ldots, \mu_t) \qquad \mu_i = E[X^i]$$

$$\Downarrow$$

$$\tilde{\theta}_{MME} = h(S_1, \ldots, S_t) \qquad S_i = \frac{1}{n}\sum_{j=1}^{n} x_j^i$$

$$g(\theta) = g(h(\mu_1, \ldots, \mu_t))$$

If $g \circ h$ uniquely represents $g(\theta)$ in terms of $\mu_1, \ldots, \mu_t$. (least t),

then by WLLN, MME for $g(\theta) = g(h(S_1, \ldots, S_t)) = g(\tilde{\theta})$.

Uniqueness

Suppose $\exists\, k \neq g \circ h$ that represents $g(\theta)$

$$\Rightarrow \quad g(\theta) = k(\mu_1, \ldots, \mu_t) \Rightarrow \theta = g^{-1} \circ k(\mu_1, \ldots, \mu_t)$$

$$\Downarrow$$

$g^{-1} \circ k$ is another representation for θ.

$$\Downarrow$$

contradiction ($\because \theta = h(\mu_1, \ldots, \mu_t)$).

Minimum 't'

Suppose $v < t$, such that $g(\theta) = g(h(\mu_1, \ldots, \mu_v))$

$$\Downarrow$$

$$g^{-1}(g(\theta)) = g^{-1}(g(h(\mu_1, \ldots, \mu_v)))$$

$\theta = h(\mu_1, \ldots, \mu_v)$. $\Rightarrow$ contradicts minimum t for θ.

MH3500 Statistics

Tutorial 7 AY2022/23 Semester 2

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