Recap for Tutorial 6

MH1101

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1 Rational Functions by Partial Fractions

Let $P(x)$ and $Q(x)$ are polynomials. To solve for $\displaystyle\int \frac{P(x)}{Q(x)} \mathrm{d}x$, we follow the steps below.

Step 1. If $\deg P(x) \geq \deg Q(x)$, then we perform polynomial long division to obtain

$$\frac{P(x)}{Q(x)} = K(x) + \frac{R(x)}{Q(x)},$$

where $K(x), R(x)$ are polynomials and $\deg R(x) < \deg Q(x)$.

Step 2. $K(x)$, as a polynomial, is easy to integrate. We are left with integrating $R(x)/Q(x)$. We now factorise $Q(x)$ as completely as possible.

- If $Q(x)$ contains the factor $(ax+b)^r$, we will have the partial fraction decomposition as

$$\frac{A_1}{(ax+b)^1} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_r}{(ax+b)^r}$$

- If $Q(x)$ contains the factor $(ax^2+bx+c)^r$, we will have the partial fraction decomposition as

$$\frac{A_1 x + B_1}{(ax^2+bx+c)^1} + \frac{A_2 x + B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_r x + B_r}{(ax^2+bx+c)^r}$$

Step 3. Equate $R(x)$ with the partial fraction decomposition and solve for constants, by comparing coefficients, A_r, B_r (where possible).

Step 4. Perform integration for each individual fraction (after decomposition).

2 Numerical Integration

In this section, we solve $\int_a^b f(x)\ \mathrm{d}x$ numerically, without obtaining the anti-derivative for $f(x)$. This will be an approximation to the actual value of the integral. First, suppose we partition the interval $[a, b]$ into n sub-intervals of equal width, i.e. $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. Let $\Delta x = \dfrac{b-a}{n}$.

Theorem 1 (Left endpoint approximation)**.**

$$\int_a^b f(x)\ \mathrm{d}x \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

Theorem 2 (Right endpoint approximation)**.**

$$\int_a^b f(x)\ \mathrm{d}x \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

Theorem 3 (Midpoint Rule)**.**

$$\int_a^b f(x)\ \mathrm{d}x \approx M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x.$$

Theorem 4 (Trapezoidal Rule)**.**

$$\int_a^b f(x)\ \mathrm{d}x \approx T_n = \frac{1}{2}(L_n + R_n) = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)).$$

Theorem 5 (Simpson's Rule)**.** n must be even.

$$\int_a^b f(x)\ \mathrm{d}x \approx S_n = \frac{1}{3}T_{n/2} + \frac{2}{3}M_{n/2} = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)).$$

3 Error Bounds

Suppose $|f''(x)| \leq K$ for $x \in [a, b]$, then

$$\left|\int_a^b f(x)\ \mathrm{d}x - T_n\right| = |E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

$$\left|\int_a^b f(x)\ \mathrm{d}x - M_n\right| = |E_T| \leq \frac{K(b-a)^3}{24n^2}.$$

Suppose $|f^{(4)}(x)| \leq K$ for $x \in [a, b]$, then

$$\left|\int_a^b f(x)\ \mathrm{d}x - S_n\right| = |E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

4 Extra Exercises

Problem 1. Evaluate the following integral.

- $\displaystyle\int x^3 \sin(x^2)\ \mathrm{d}x.$
- $\displaystyle\int \frac{e^{4x} + 2e^{2x} - e^x}{e^{2x} + 1}\ \mathrm{d}x.$
- $\displaystyle\int_e^1 \frac{1}{x\sqrt{1 + (\ln x)^2}}\ \mathrm{d}x.$
- $\displaystyle\int_0^{\pi/3} x \tan^2 x\ \mathrm{d}x.$