(1) The joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{y} e^{-x/y-y}, & 0 < x, y < \infty; \\ \\ 0, & \text{otherwise.} \end{cases}$$

We have seen how to calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathbb{E}(XY)$, and also $Cov(X,Y)$ from last tutorial. Now find $\mathbb{P}r\{X > 1 | Y = y\}$ for $y > 0$.

Need to find conditional PDF of $f(x|y)$

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\cancel{\frac{1}{y}} e^{-\frac{x}{y} \cancel{-y}}}{\int_0^\infty \cancel{\frac{1}{y}} e^{-\frac{x}{y} \cancel{-y}}\, dx} = \frac{e^{-\frac{x}{y}}}{\left[-y e^{-\frac{x}{y}}\right]_{x=0}^{x=\infty}} = \frac{e^{-\frac{x}{y}}}{0+y}$$

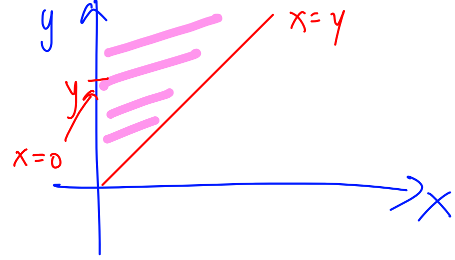
$$= \frac{1}{y} e^{-\frac{x}{y}}$$

$$P(X>1 \,|\, Y=y) = \int_1^\infty f(x|y)\, dx = \frac{1}{y} \int_1^\infty e^{-\frac{x}{y}}\, dx = \frac{1}{y}\left[y e^{-\frac{x}{y}}\right]_1^\infty = e^{-\frac{1}{y}}$$

(2) The joint density of X and Y is given by

$$f(x,y) = \frac{1}{y} e^{-y}, \quad 0 < x < y < \infty.$$

Find $\mathbb{E}[X^3 | Y = y]$



$$\mathbb{E}[X^3 | Y = y] = \int_0^y x^3 f(x|y)\, dx$$

For a given $Y=y$, x changes from $x=0$ to $x=y$

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\cancel{\frac{1}{y} e^{-y}}}{\int_0^y \cancel{\frac{1}{y} e^{-y}}\, dx} = \frac{1}{y}$$

$$\mathbb{E}[X^3 | Y = y] = \int_0^y x^3 \cdot \frac{1}{y}\, dx = \frac{1}{4} y^3$$

(3) The joint density of X and Y is given by

$$f(x,y) = \begin{cases} e^{-(x+y)}, & 0 < x, y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Find the density function of the random variable $Z = X/Y$.

CDF of Z : $F_Z(z) = P(Z \le z) = P\left(\frac{X}{Y} \le z\right) = P(X \le Yz)$

$$= \int_0^\infty \int_0^{yz} \underbrace{e^{-x-y}}_{f(x,y)} \, dx \, dy \qquad (z > 0)$$

Fundamental Thm of Calc.

$$\frac{d}{dx} \int_{\text{const.}}^{x} f(t)\,dt = f(x)$$

PDF of Z

$$: f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \int_0^\infty \frac{d}{dz}\left(\int_0^{yz} e^{-x-y} \, dx\right) dy$$

Let $w = yz$

$$= \int_0^\infty \frac{d}{dw}\left(\int_0^{w} e^{-x-y} \, dx\right) \cdot \frac{\partial w}{\partial z} \, dy$$

$$= \int_0^\infty e^{-w-y} \cdot y \, dy$$

$$= \int_0^\infty y \, e^{-y(1+z)} \, dy$$

Int by parts

$$= \underbrace{\left[-\frac{y}{1+z} e^{-y(1+z)}\right]_0^\infty}_{=0} + \frac{1}{1+z} \int_0^\infty e^{-y(1+z)} \, dy$$

$$= \frac{1}{z+1} \left[-\frac{1}{z+1} e^{-y(1+z)}\right]_0^\infty = \frac{1}{(z+1)^2}$$

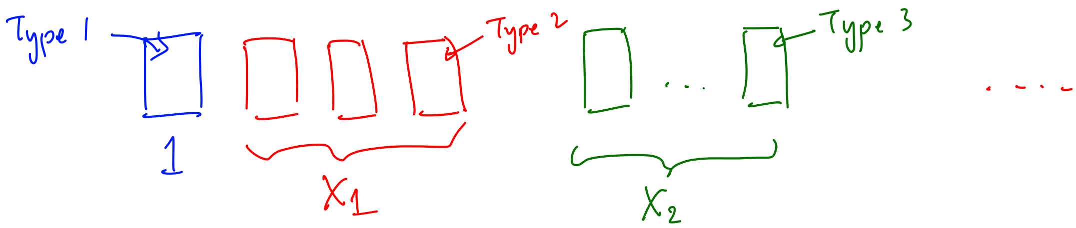
$$f_Z(z) = \begin{cases} \frac{1}{(z+1)^2} & z > 0 \\ 0 & \text{else.} \end{cases} \qquad Z = \frac{X}{Y}$$

(4) In the coupon collector problem, show that the variance of the number of coupons needed to amass a full set is equal to

$$\sum_{i=1}^{N-1} \frac{iN}{(N-i)^2}.$$

When N is large, this can be shown to be approximately equal (in the sense that their ratio approaches 1 as $N \to \infty$) to $N^2\pi^2/6$.

Let X_i be the number of additional steps to see the $(i+1)$th type after having collected distinct i types of coupons.



$\ldots\quad X_i \sim \mathrm{Geom}\left(\frac{N-i}{N}\right)$

Let X be the total number of coupons collected,

$$X = 1 + X_1 + X_2 + \ldots + X_{N-1}$$

$$\begin{aligned}
Var(X) &= Var(1 + X_1 + \ldots + X_{N-1}) \\
&= Var\left(\sum_{i=1}^{N-1} X_i\right) \\
&= \sum_{i=1}^{N-1} Var(X_i) \qquad (X_i\text{'s are indp}). \\
&= \sum_{i=1}^{N-1} \frac{1-\left(\frac{N-i}{N}\right)}{\left(\frac{N-i}{N}\right)^2} \qquad \left(\text{Var of } \mathrm{Geo}\left(\frac{N-i}{N}\right)\right) \\
&= \sum_{i=1}^{N-1} \frac{iN}{(N-i)^2} \\
&= \sum_{i=1}^{N-1} \frac{N^2}{(N-i)^2} - \sum_{i=1}^{N-1} \frac{N\cancel{(N-i)}}{(N-i)^{\cancel{2}}} \qquad \text{Write } iN = N^2 - N(N-i) \\
&= N^2 \sum_{i=1}^{N-1} \frac{1}{(N-i)^2} - N \sum_{i=1}^{N-1} \frac{1}{N-i} = N^2 \sum_{i=1}^{N-1} \frac{1}{i^2} - N \sum_{i=1}^{N-1} \frac{1}{i}
\end{aligned}$$

$N \to \infty \qquad \sum_{i=1}^{N} \frac{1}{i^2} \to \frac{\pi^2}{6} \qquad \sum_{i=1}^{N} \frac{1}{i} \to \ln N + \gamma$ ← Euler's constant 0.577...

(5) Let X and Y be independent uniform random variables on $(0,1)$. Find the density function of $Z = X + Y$.

$$f_X(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \end{cases}$$

$$f_Y(y) = \begin{cases} 1 & 0<y<1 \\ 0 & \end{cases}$$

CDF of Z : $F_Z(z) = P(X + Y \le z)$

$$= P(Y \le z - X)$$

$$= \int_0^1 \int_0^{z-x} f(x,y)\, dy\, dx$$

PDF of Z : $f_Z(z) = \frac{d}{dz} F_Z(z) = \int_0^1 \left[\frac{d}{dz} \int_0^{z-x} f(x,y)\, dy \right] dx$

$$= \int_0^1 f(x, z-x)\, dx$$

$0<z<2$

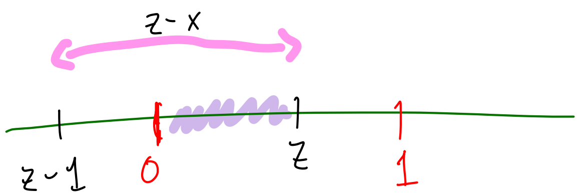
convolution product

$$= \boxed{\int_0^1 \underbrace{f_X(x)}_{=1} f_Y(z-x)\, dx} = \int_0^1 f_Y(z-x)\, dx$$

z is fixed but unknown

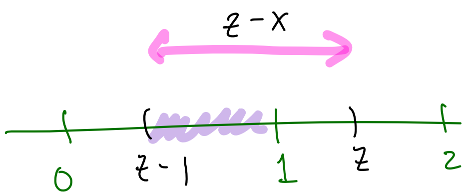
For $f_Y(z-x) = 1$, we need $z - x \in (0,1)$ while $0<x<1$

x changes from 0 to 1.



when $0<z<1$

$$f_Z(z) = \int_0^1 f_Y(z-x)\, dx = \int_0^z 1\, dx = z$$



When $1<z<2$

$$f_Z(z) = \int_0^1 f_Y(z-x)\, dx = \int_{z-1}^1 1\, dx = 2 - z$$

(6) Suppose that $X_1, \ldots, X_n$ are i.i.d. $\text{Exp}(\lambda)$ random variables. Prove that $X_1 + \cdots + X_n \sim \text{Gamma}(n, \lambda)$.

$\text{Exp}(\lambda)$ is same as $\text{Gamma}(1, \lambda)$

It is sufficient to show if $X_1 \sim \text{Gamma}(\alpha, \lambda)$, $X_2 \sim \text{Gamma}(\beta, \lambda)$, then $X_1 + X_2 \sim \text{Gamma}(\alpha+\beta, \lambda)$

$$f_{X_1+X_2}(z) = \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(z-x)\, dx$$

$$= \int_0^z \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \cdot \frac{\lambda^\beta}{\Gamma(\beta)} (z-x)^{\beta-1} e^{-\lambda(z-x)}\, dx$$

$$= \frac{\lambda^\alpha \lambda^\beta}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda z} \int_0^z x^{\alpha-1} (z-x)^{\beta-1}\, dx$$

Let $u = \frac{x}{z}$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda z} \int_0^1 (uz)^{\alpha-1} (z-uz)^{\beta-1} \cdot z\, du$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda z} z^{\alpha+\beta-1} \underbrace{\int_0^1 u^{\alpha-1} (1-u)^{\beta-1}\, du}_{\text{Beta function, } B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)} e^{-\lambda z} z^{\alpha+\beta-1}$$

$\therefore X_1 + X_2 \sim \text{Gamma}(\alpha+\beta, \lambda)$

Proof of $\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty e^{-x} x^{\alpha-1}\,dx \cdot \int_0^\infty e^{-y} y^{\beta-1}\,dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x+y)} x^{\alpha-1} y^{\beta-1}\,dx\,dy$$

Let $x = st$, $y = s(1-t)$

$$dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{vmatrix} dt\,ds$$

$$= \begin{vmatrix} s & t \\ -s & 1-t \end{vmatrix} dt\,ds$$

$$= s \cdot dt\,ds$$

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^1 e^{-s} (st)^{\alpha-1} (s(1-t))^{\beta-1} \cdot s\,dt\,ds$$

$$= \int_0^\infty e^{-s} \cdot s^{\alpha+\beta-1}\,ds \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

$$= \Gamma(\alpha+\beta) \cdot \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt.$$