Tutorial Question

(1) If X is an exponential random variable with parameter λ, and $c > 0$, show that cX is exponential with parameter λ/c.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases} \Leftarrow \text{probability density function}$$

Let $Y = cX$

CDF of Y, $F_Y(y) = P(Y \le y) = P(cX \le y) = P(X \le \frac{y}{c}) = \int_0^{\frac{y}{c}} \lambda e^{-\lambda x} \, dx$

$$= \left[-e^{-\lambda x} \right]_0^{\frac{y}{c}} = 1 - e^{-\frac{\lambda y}{c}} \qquad (y > 0)$$

PDF of Y, $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \left(1 - e^{-\frac{\lambda y}{c}} \right) = \frac{\lambda}{c} e^{-\frac{\lambda}{c} y}$

$$f_Y(y) = \begin{cases} \frac{\lambda}{c} e^{-\frac{\lambda}{c} y} & y > 0 \\ 0 & \text{otherwise} \end{cases} \Rightarrow Y \sim Exp\left(\frac{\lambda}{c}\right)$$

(3) If Y is uniformly distributed over $(0, 5)$, what is the probability that the roots of the equation $4x^2 + 4xY + Y + 2 = 0$ are both real?

$$f_Y(y) = \begin{cases} \frac{1}{5} & y \in (0,5) \\ 0 & \text{otherwise.} \end{cases}$$

$\star$ In general,

$Y \sim Unif(a, b)$

$$f_Y(y) = \begin{cases} \frac{1}{b-a} & y \in (a,b) \\ 0 & \text{else.} \end{cases}$$

Both roots are real $\iff b^2 - 4ac \geq 0 \iff (4Y)^2 - 4(4)(Y+2) \geq 0$

$$Y^2 - Y - 2 \geq 0$$

$$(Y-2)(Y+1) \geq 0$$

$$Y \leq -1 \quad \text{or} \quad Y \geq 2$$

$$P(\text{both roots are real}) = P(Y \leq -1 \text{ or } Y \geq 2) = P(Y \geq 2) = \int_2^5 \frac{1}{5}\, dt = \frac{3}{5}$$

$F(x) = P(X \leq x)$ is between 0 to 1.

(4) Let $Z = F(X)$, the CDF of a random variable X. Prove that Z has a uniform distribution on $[0, 1]$.

If $Z \sim \text{Unif}(0,1) \Rightarrow f_Z(z) = \begin{cases} 1 & z \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$ To show!

increasing, continuous for continuous RV X.

CDF of Z : $F_Z(z) = P(Z \leq z) = P(F(X) \leq z)$

$= P(F^{-1}F(X) \leq F^{-1}(z))$

$= P(X \leq F^{-1}(z))$

$= F(F^{-1}(z))$ by definition of $F(\cdot)$

$= z$

PDF of Z: $f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz}(z) = 1$

(5) Let U be uniform on $[0, 1]$, and $X = F^{-1}(U)$, where F is an increasing function on $\mathbb{R}$. Prove that F is the CDF of X.

CDF of X: $F_X(x) = P(X \leq x) = P(F^{-1}(U) \leq x)$

$$f_U(u) = \begin{cases} 1 & u \in (0,1) \\ 0 & \text{else} \end{cases}$$

$= P(F(F^{-1}(U)) \leq F(x))$

$= P(U \leq F(x)) = \int_0^{F(x)} 1 \, dt$

$= F(x)$

Practice Question

(1) Let U be a uniform random variable on $[0, 1]$ and let $V = \frac{1}{U}$. Find the probability density function of V.

$v > 1$

$$f_U(u) = \begin{cases} 1 & u \in (0,1) \\ 0 & \text{else} \end{cases}$$

CDF of V: $F_V(v) = P(V \le v) = P\left(\frac{1}{U} \le v\right) = P\left(U \ge \frac{1}{v}\right)$

$$= \int_{\frac{1}{v}}^{1} 1 \, du$$

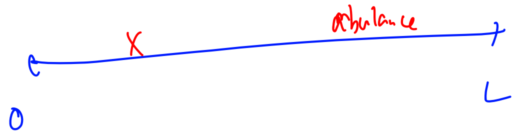
$f_U(u)$

$$= 1 - \frac{1}{v}$$

PDF of V: $f_V(v) = \frac{d}{dv} F_V(v) = \frac{d}{dv}\left(1 - \frac{1}{v}\right) = \frac{1}{v^2}$

$$f_V(v) = \begin{cases} \frac{1}{v^2} & , \; v > 1 \\ 0 & , \; \text{else} \end{cases}$$

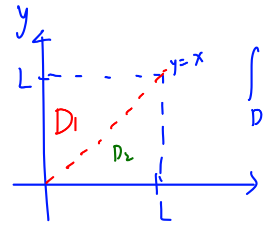
(6) An accident occurs at a point X that is uniformly distributed on a road of length L. At the time of the accident, an ambulance is at a location Y that is also uniformly distributed on the road. Assume that X and Y are independent. Find the expected distance between the ambulance and the point of the accident.



$$f_X(x) = \begin{cases} \frac{1}{L} & 0<x<L \\ 0 & \text{else.} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{L} & 0<y<L \\ 0 & \text{else.} \end{cases}$$

If X, Y are independent, $f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} \frac{1}{L^2} & 0<x,y<L \\ 0 & \text{else} \end{cases}$

Example : $\mathbb{E}[|X|] = \int_0^L |x| f_X(x)\,dx = \ldots$



$$\iint_{D_1} = \int_0^L \int_x^L dy\,dx \qquad \iint_{D_2} = \int_0^L \int_0^x dy\,dx$$

Find

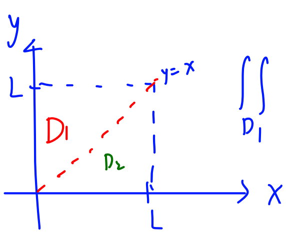
$$\mathbb{E}[|X-Y|] = \int_0^L\int_0^L |x-y|\, f_{X,Y}(x,y)\,dx\,dy = \frac{1}{L^2}\int_0^L\int_0^L |x-y|\,dx\,dy$$

$$= \frac{1}{L^2}\iint_{D_1}(y-x)\,dy\,dx + \frac{1}{L^2}\iint_{D_2}(x-y)\,dy\,dx \qquad |x-y| = \begin{cases} x-y & x>y \\ y-x & y\ge x \end{cases}$$

$$= \frac{1}{L^2}\int_0^L\int_x^L (y-x)\,dy\,dx + \frac{1}{L^2}\int_0^L\int_0^x (x-y)\,dy\,dx$$

$$= \frac{1}{L^2}\int_0^L \left[\frac{1}{2}y^2 - xy\right]_{y=x}^{y=L} dx + \frac{1}{L^2}\int_0^L \left[xy - \frac{1}{2}y^2\right]_{y=0}^{y=x} dx$$

$$= \frac{1}{L^2}\int_0^L \left(\frac{1}{2}L^2 - Lx - \frac{1}{2}x^2 + x^2\right) dx + \frac{1}{L^2}\int_0^L \left(x^2 - \frac{1}{2}x^2\right) dx$$



$$\iint_{D_1} = \int_0^L \int_x^L dy\,dx \qquad \iint_{D_2} = \int_0^L \int_0^x dy\,dx$$

$$= \frac{1}{L^2}\left[\frac{1}{2}L^2 x - \frac{1}{2}Lx^2 + \frac{1}{6}x^3\right]_0^L + \frac{1}{L^2}\left[\frac{1}{6}x^3\right]_0^L$$

$$= \frac{1}{6}L + \frac{1}{6}L = \frac{1}{3}L$$