Teh Yu Xuan

yuxuan006@e.ntu.edu.sg.

Class participation : 10% (wk 3 onwards)
–13

11 classes (attend $\geq 9$)

**MH2500 PROBABILITY AND INTRODUCTION TO STATISTICS**

**NANYANG TECHNOLOGICAL UNIVERSITY**

Practice Questions Week 2, 2024

We will discuss several questions from Chapter 1 in Ross' book.

(1) How many outcome sequences are possible when a die is rolled four times, where we say, for instance, that the outcome is 3, 4, 3, 1 if the first roll landed with 3, the second with 4, the third with 3, and the fourth with 1?

Multiplication rule

$6 \times 6 \times 6 \times 6 = 6^4$

(2) A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

B . . . B R R R R W B

The arrangement is the same if you swap blocks of the same colour. (they are indistinguishable).

$12! = 12 \times 11 \times 10 \times \cdots \times 1$

Total permutations of 6 black blocks : $6!$

4 red blocks : $4!$

Total arrangements : $\frac{12!}{6!\,4!}$

(3) A student has to sell 2 books from a collection of 6 math, 7 science, and 4 economics books. How many choices are possible if

(a) both books are on the same subject?
(b) the books are on different subjects?

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

(a) $\binom{6}{2} + \binom{7}{2} + \binom{4}{2}$

(b) Total choices $-$ (a)

$= \binom{17}{2} - \binom{6}{2} - \binom{7}{2} - \binom{4}{2}$

Alt way

1 math 1 science $= \binom{6}{1} \times \binom{7}{1}$

1 sci 1 econ $= \binom{7}{1} \times \binom{4}{1}$

1 math 1 econ $= \binom{6}{1} \times \binom{4}{1}$

94

(4) Ten weight lifters are competing in a team weightlifting contest. Of the lifters, 3 are from the United States, 4 are from Russia, 2 are from China, and 1 is from Canada. If the scoring takes account of the countries that the lifters represent, but not their individual identities, how many different outcomes are possible in terms of scores?

$$\frac{10!}{3!\,4!\,2!}$$

(5) (a) In how many ways can 3 boys and 3 girls sit in a row?
(b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?
(c) In how many ways if only the boys must sit together?
(d) In how many ways if no two people of the same sex are allowed to sit together?

(a) _ _ _ _ _ _

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$

(b) B B B G G G $= 3! \times 3!$
(boys) (girls)

or

G G G B B B $= 3! \times 3!$

Ans : $3! \times 3! \times 2$

(c) G $\boxed{BBB}$ G G  $\boxed{BBB}$ G G G  G G $\boxed{BBB}$ G

$4! \times 3!$
(boys)

(d) $3! \times 3! \times 2$ (try)

(6) Give a combinatorial proof of the following identity:

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}.$$

This implies

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Let $m=n$, $r=n$
Use $\binom{n}{k} = \binom{n}{n-k}$

immediately.

Pick r balls among n blue balls and m red balls.

Total combination : $\binom{n+m}{r}$

Case 1 : Pick 0 blue, r red $\rightarrow \binom{n}{0}\binom{m}{r}$

2 : Pick 1 blue, r-1 red $\rightarrow \binom{n}{1}\binom{m}{r-1}$

⋮

r blue, 0 red

(7) Consider the following combinatorial identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}.$$

(a) Give a combinatorial argument for this identity by considering a set of $n$ people and determining, in two ways, the number of possible selections of a committee of any size and a chairperson for the committee.

First way : If size $= k$ , $\binom{n}{k} \times \binom{k}{1}$ (Sum $k=1$ to $k=n$)

$$\sum_{k=1}^{n} \binom{n}{k} \times \binom{k}{1} = \sum_{k=1}^{n} k\binom{n}{k}$$

Second way : Choose a chairperson : $\binom{n}{1}$ $\quad \binom{n}{1} \times 2^{n-1} = n \cdot 2^{n-1}$

The other $(n-1)$ people : $2^{n-1}$

(b) Give a combinatorial argument for the following identity:

$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2} n(n+1).$$

A set of $n$ people, num of possible selections of a comm of any size and a chairperson and a secretary for the committee
(can be same person)

First way : If size $= k$ $\quad \binom{n}{k} \times \underbrace{\binom{k}{1} \times \binom{k}{1}}_{k^2}$ (sum $k=1$ to $k=n$)

Second way : If chairperson and sec are same person . $\Rightarrow \binom{n}{1} \times 2^{n-1}$

If ... are diff $\Rightarrow \binom{n}{2} \times 2! \times 2^{n-2}$

(c) By using (a) and (b), prove

$$\sum_{k=1}^{n} \binom{n}{k} k^3 = 2^{n-3} n^2 (n+3).$$

A set of $n$ people, number of possible selections of comm of any size and a chairperson, a secretary and a treasurer for the committee
(one person can take more than one position)

First way : $\sum_{k=1}^{n} \binom{n}{k}\binom{k}{1}\binom{k}{1}\binom{k}{1} = \sum_{k=1}^{n} \binom{n}{k} k^3$

2nd way : Three positions by the same person : $\binom{n}{1} \cdot 2^{n-1}$ $\quad +$

Three positions by 2 people : $\binom{n}{2} \cdot \binom{3}{2} 2! \cdot 2^{n-2}$ $\quad +$

Three positions by 3 people : $\binom{n}{3}\ 3! \cdot 2^{n-3}$

(8) Fix $k \geq n$ as positive integers. Find the number of $n$-tuples $(x_1, \cdots, x_n)$ such that each $x_i$ is a positive integer and

$$x_1 + x_2 + \cdots + x_n \leq k.$$

$$\binom{k}{n}$$

Know that if $x_1 + \ldots + x_n = k$ $(x_i \geq 1)$

Total combinations $= \binom{k-1}{n-1}$

First method

Consider all cases : $x_1 + x_2 + \ldots + x_n = n \longrightarrow \binom{n-1}{n-1}$

$x_1 + x_2 + \ldots + x_n = n+1 \longrightarrow \binom{n}{n-1}$

$\vdots$

$x_1 + x_2 + \ldots + x_n = k \longrightarrow \binom{k-1}{n-1}$

$$\text{Total combinations} = \binom{n-1}{n-1} + \binom{n}{n-1} + \binom{n+1}{n-1} + \ldots\ldots + \binom{k-1}{n-1}$$

$$= \binom{n-1}{n-1} + \left[\cancel{\binom{n+1}{n}} - \binom{n}{n}\right] + \left[\cancel{\binom{n+2}{n}} - \cancel{\binom{n+1}{n}}\right] + \ldots + \left[\binom{k}{n} - \cancel{\binom{k-1}{n}}\right]$$

$$= \binom{n-1}{n-1} - \binom{n}{n} + \binom{k}{n} = \binom{k}{n}$$

Use $\binom{k}{r} = \binom{k+1}{r+1} - \binom{k}{r+1}$

## Second Method

$$x_1 + \dots + x_n \leq k \qquad (x_i \geq 1)$$

Introduce a new variable $x_{n+1}$, such that

$$x_1 + \dots + x_n + x_{n+1} = k \qquad (x_{n+1} \geq 0)$$

We just need to find the number of combinations $(x_1, \dots, x_{n+1})$

Truncating $x_{n+1}$, this gives us valid combinations of $(x_1, \dots, x_n)$

Let $y_{n+1} = x_{n+1} + 1$. Since $x_{n+1} \geq 0 \Rightarrow y_{n+1} \geq 1$

$$\therefore \quad x_1 + \dots + x_n + y_{n+1} - 1 = k$$

$$\Rightarrow \quad x_1 + \dots + x_n + y_{n+1} = k+1 \qquad (x_1, \dots, x_n, y_{n+1} \geq 1)$$

$$\text{Total combinations} = \binom{k+1-1}{n+1-1} = \binom{k}{n}.$$