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Solution 1.

$$P(U|C) = \frac{P(C|U)P(U)}{P(C|U)P(U) + P(C|U')P(U')} = \frac{0.75 \cdot 0.6}{0.75 \cdot 0.6 + 0.1 \cdot 0.4} = \frac{45}{49}.$$

Solution 3. We will assume X follows a continuous distribution. Discrete distribution follows in a similar manner.

(ai)

$$0 = \int_0^1 0 \cdot f(x)dx \leq \int_0^1 x \cdot f(x)dx \leq \int_0^1 1 \cdot f(x)dx = 1 \implies 0 \leq \mu \leq 1.$$

(aii)

$$\mathbb{E}(X^2) = \int_0^1 x^2 f(x)dx \leq \int_0^1 xf(x)dx = \mu \implies Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \leq \mu - \mu^2 = \mu(1-\mu).$$

Since $0 \leq \mu \leq 1$, $\mu(1-\mu)$ is maximised at $\frac{1}{4}$. Use the first derivative to check this. We know $Var(X) \geq 0$ due to Jensen's inequality, so

$$0 \leq Var(X) \leq \mu(1-\mu) \leq \frac{1}{4}.$$

(bi) Let $Y = \frac{X-a}{b-a}$, then we have $0 \leq Yleq1$. From part (a), we can conclude that $0 \leq \mathbb{E}(Y) \leq 1$. By linearity of expectation,

$$\mathbb{E}(Y) = \frac{\mathbb{E}(X) - a}{b-a} = \frac{\mu - a}{b-a} \implies a \leq \mu \leq b.$$

(bii) We know that $Var(Y) = \frac{1}{(b-a)^2}Var(X)$. From part (a), we can conclude that $0 \leq Var(Y) \leq \mathbb{E}(Y)(1-\mathbb{E}(Y)) \leq \frac{1}{4}$. Therefore,

$$0 \leq Var(X) = (b-a)^2 Var(Y) \leq (b-a)^2 \frac{\mu-a}{b-a}\left(1 - \frac{\mu-a}{b-a}\right) = (\mu-a)(b-\mu).$$

Since $a \leq \mu \leq b$, $(\mu-a)(b-\mu)$ is maximised at $\frac{1}{4}(b-a)^2$. Use the first derivative to check this. Therefore, the conclusion follows.

Solution 4.

(a) The area of the unit disk is π. Hence the joint density function is

$$f(x,y) = \begin{cases} \frac{1}{\pi}, & x^2 + y^2 \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{1-x^2}, \qquad x \in (-1,1).$$

Similarly, by symmetry, $f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}, \qquad y \in (-1,1)$.

(b) Clearly not indepedent, there exists x, y within the unit disk such that $f_X(x)f_Y(y) \neq f(x,y)$.

(c)

$$\mathbb{E}(XY) = \iint_{x^2+y^2 \leq 1} xyf(x,y)dydx = \frac{1}{\pi}\int_{-1}^1 x \underbrace{\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} ydy}_{=0} dx = 0.$$

The function y is odd and is integrated over a symmetric interval centered at 0, hence the inner integral is 0.

$$\mathbb{E}(X) = \int_{-1}^1 x \cdot f_X(x)dx = \frac{2}{\pi}\int_{-1}^1 x\sqrt{1-x^2}dx = 0.$$

Again, the integral is 0 due to integrating an odd function over $(-1,1)$. Hence, $Cov(X,Y) = 0$.

Solution 5. Since $\mathbb{E}(X_1) = 2$ and $Var(X_1) = 2$, let $\bar{X} = \frac{1}{30}\sum_{i=1}^{30} X_i$. By central limit theorem, $\sum_{i=1}^{30} X_i \sim \mathrm{N}(60, 60)$.

$$P\left(\sum_{i=1}^{30} X_i > 50\right) \approx P\left(Z \geq \frac{50 + 0.5 - 60}{\sqrt{60}}\right) = 0.9082.$$