Some Useful Properties

MH2500

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- Suppose $X_1, \ldots X_n$ are i.i.d. random variables with a CDF $F(\cdot)$ and PDF $f(\cdot)$. Let $Y_1 = \min(X_1, \ldots, X_n)$ and $Y_2 = \max(X_1, \ldots, X_n)$, then

$$P(Y_1 \leq y) = 1 - (1 - F(y))^n, \qquad P(Y_2 \leq y) = (F(y))^n.$$

$$f_{Y_1}(y) = n(1 - F(y))^{n-1} f(y), \qquad f_{Y_2}(y) = n(F(y))^{n-1} f(y).$$

- Finite geometric series:

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}.$$

- Infinite geometric series:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}, \quad |r| < 1.$$

- Binomial series:

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a + b)^n.$$

- Taylor series expansion of e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \quad \forall x \in \mathbb{R}.$$

- Gaussian Integral:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad \text{or} \quad \int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

  With a substitution $t = \frac{x-\mu}{\sqrt{2}\sigma}$, we can show that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 1.$$

- Bernoulli distribution is a special case of binomial distribution with $n = 1$.
- Suppose $X_i \sim \text{Binomial}(n_i, p)$, for $i = 1, \ldots, k$ are independent, then

$$X_1 + X_2 \ldots + X_k \sim \text{Binomial}(n_1 + \ldots + n_k, p).$$

- Suppose $X_i \sim \text{Poisson}(\lambda_i)$, for $i = 1, \ldots, k$ are independent, then

$$X_1 + X_2 \ldots + X_k \sim \text{Poisson}(\lambda_1 + \ldots + \lambda_k).$$

- Suppose $X_i \sim \text{Gamma}(\alpha_i, \beta)$, for $i = 1, \ldots, k$ are independent, then

$$X_1 + X_2 \ldots + X_k \sim \text{Gamma}(\alpha_1 + \ldots + \alpha_k, \beta).$$

- Relationship between Exp and Gamma:

$$\text{Exp}\left(\lambda\right) \sim \text{Gamma}\left(1, \lambda\right).$$

- Suppose $X_i \sim \text{Exp}(\lambda)$, for $i = 1, \ldots, k$ are independent, then

$$X_1 + \ldots + X_k \sim \text{Gamma}(k, \lambda).$$

- Suppose $X_i \sim \mathrm{N}(\mu_i, \sigma_i^2)$, for $i = 1, \ldots, k$ are independent, then

$$\sum_{i=1}^{k} a_i X_i \sim \mathrm{N}(\sum a_i \mu_i, \sum a_i^2 \sigma_i^2).$$

- Gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^t dt.$$

  $\Gamma(z+1) = z\Gamma(z) = z!$, for $z \in \mathbb{Z}^+$.