(1) The joint density of X and Y is given by

$$f(x,y) = \begin{cases} xe^{-(x+y)}, & x>0, y>0 \\ 0, & \text{otherwise.} \end{cases}$$

Are X and Y independent? If, instead, $f(x,y)$ were given by

$$f(x,y) = \begin{cases} 2, & 0<x<y, 0<y<1; \\ 0, & \text{otherwise,} \end{cases}$$

would X and Y be independent?

marginal PDF

X, Y are independent $\iff f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$

$$f_X(x) = \int_0^\infty f(x,y)\,dy = \int_0^\infty xe^{-(x+y)}\,dy = xe^{-x}\int_0^\infty e^{-y}\,dy = xe^{-x}\left[-e^{-y}\right]_0^\infty$$

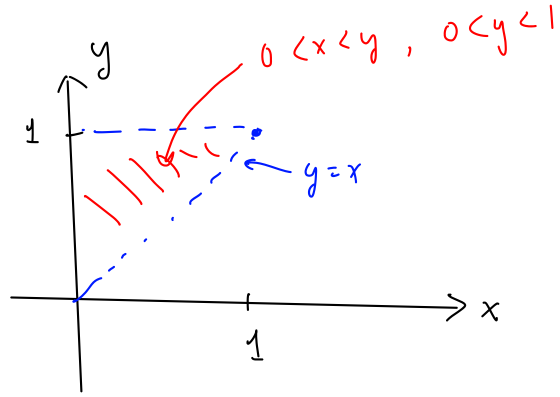
$$= xe^{-x} \quad (x>0)$$

$$f_Y(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty xe^{-(x+y)}\,dx = e^{-y}\underbrace{\int_0^\infty xe^{-x}\,dx}_{1} = e^{-y}\left\{\left[-xe^{-x}\right]_0^\infty + \int_0^\infty e^{-x}\,dx\right\}$$

$$= e^{-y}\left[0+1\right] = e^{-y} \quad (y>0)$$

Since $f(x,y) = f_X(x) f_Y(y)$ for $x>0$, $y>0$ $\Rightarrow$ X, Y are indp.

$$f(x,y) = \begin{cases} 2 & 0<x<y, \ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$



$$f_X(x) = \int_x^1 f(x,y)\,dy = \int_x^1 2\,dy = 2(1-x) \qquad 0<x<1$$

Check $\int_0^1 2(1-x)\,dx = 1$

$$f_Y(y) = \int_0^y f(x,y)\,dx = \int_0^y 2\,dx = 2y \qquad 0<y<1$$

Check $\int_0^1 2y\,dy = 1$

Since $f(x,y) \neq f_X(x) f_Y(y)$ $\Rightarrow$ X, Y are not independent

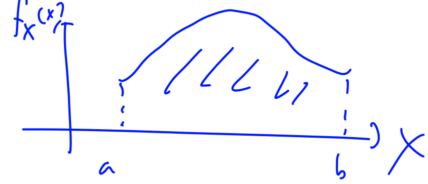
on $0<x<y$, $0<y<1$

(2) Let $z = 24xy$ $\quad y \le 1-x$

$$f(x,y) = 24xy, \qquad 0 \le x \le 1, 0 \le y \le 1, 0 \le x+y \le 1$$

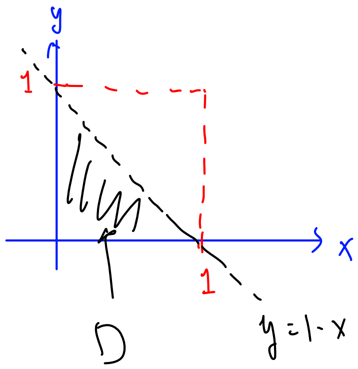
and let it be 0 otherwise.

(a) Show that $f(x,y)$ is a joint density function.
(b) Find $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.
(c) Show that X and Y are not independent.



$$\int_a^b f_X(x)\,dx = 1$$

(a) Show $\iint_D (24xy)\,dA = 1$

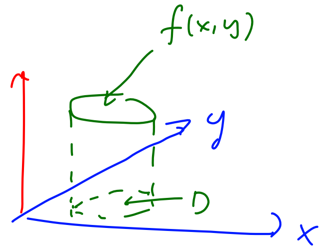


$$\int_0^1 \int_0^{1-x} (24xy)\,dy\,dx$$

$$= \int_0^1 24x \left[\frac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 24x \left[\frac{1}{2}(1-x)^2\right] dx$$

$$= 12 \int_0^1 x(1-x)^2\,dx = \ldots = \underline{1}$$



$$\iint_D f(x,y)\,dA = 1$$

(b) $\mathbb{E}(X) = \int_0^1 x f_X(x)\,dx$

$$= \int_0^1 x \cdot 12x(1-x)^2\,dx$$

$$= \frac{2}{5}$$

$$f_X(x) = \int_0^{1-x} 24xy\,dy = 24x\left[\frac{1}{2}y^2\right]_0^{1-x} = 12x(1-x)^2$$

$$f_Y(y) = \int_0^{1-y} 24xy\,dx = 12y(1-y)^2$$

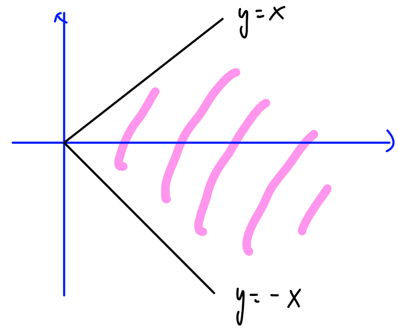
$$\mathbb{E}(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 y \cdot 12y(1-y)^2\,dy = \frac{2}{5}$$

(c) $f_{X,Y}(x,y) \neq f_X(x) f_Y(y) \Rightarrow$ not independent

(3) The joint density of X and Y is

$$f(x,y) = c(x^2 - y^2)e^{-x}, \quad 0 \le x < \infty, \; -x \le y \le x.$$

Find the conditional CDF of Y, given $X = x$.



Conditional PDF $= f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{\cancel{c}(x^2-y^2)\cancel{e^{-x}}}{\int_{-x}^{x} \cancel{c}(x^2-y^2)\cancel{e^{-x}}\,dy}$

$$= \frac{x^2-y^2}{\left[x^2y - \frac{1}{3}y^3\right]_{y=-x}^{y=x}} = \frac{x^2-y^2}{\frac{4}{3}x^3} \qquad -x \le y \le x$$

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{x^2-y^2}{\frac{4}{3}x^3} & -x \le y \le x \\ 0 & \text{otherwise.} \end{cases}$$

Conditional CDF of $Y \mid X=x$ $= F_{Y|X}(y|x) = \displaystyle\int_{-x}^{y} \frac{x^2-t^2}{\frac{4}{3}x^3}\,dt = \frac{3}{4x^3}\left[x^2t - \frac{1}{3}t^3\right]_{t=-x}^{t=y}$

$$= \frac{3}{4x^3}\left[x^2y - \frac{1}{3}y^3 + x^3 - \frac{1}{3}x^3\right]$$

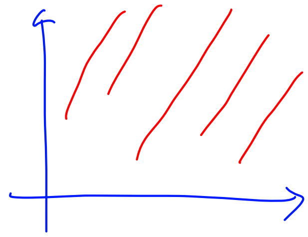
$$= \boxed{\frac{3y}{4x} - \frac{y^3}{4x^3} + \frac{1}{2}} \qquad -x \le y \le x$$

$$F_{Y|X}(y|x) = \begin{cases} 0 & y < -x \\ & -x \le y \le x \\ 1 & y > x \end{cases}$$

(4) The joint density function of X and Y is given by

$$f(x,y) = xe^{-x(y+1)}, \quad x>0, y>0$$

(a) Find the conditional density of X, given $Y=y$, and that of Y, given $X=x$.
(b) Find the density function of $Z=XY$.



(a)

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}\,dx}$$

$$= \frac{xe^{-x(y+1)}}{\left[-\frac{x}{y+1}e^{-x(y+1)}\right]_0^\infty + \frac{1}{y+1}\int_0^\infty e^{-x(y+1)}\,dx}$$

$$= \frac{xe^{-x(y+1)}}{\frac{1}{y+1}\cdot\left[-\frac{1}{y+1}e^{-x(y+1)}\right]_0^\infty} = (y+1)^2 xe^{-x(y+1)} \qquad x>0$$

$$f_{Y|X} = \frac{f(x,y)}{f_X(x)} = \frac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}\,dy} = \frac{xe^{-x(y+1)}}{xe^{-x}\int_0^\infty e^{-xy}\,dy} = \frac{e^{-xy}}{\left[-\frac{1}{x}e^{-xy}\right]_{y=0}^{y=\infty}} = xe^{-xy}$$

$$y>0$$

(b) PDF of $Z=XY$ $\qquad f(x,y) = xe^{-x(y+1)} \quad x>0,\ y>0.$

CDF of $(Z=XY)$: $F_Z(z) = P(XY\le z) = P\left(Y\le \frac{z}{X}\right)$

$$= \int_0^\infty \int_0^{\frac{z}{x}} f(x,y)\,dy\,dx$$

$$= \int_0^\infty \int_0^{\frac{z}{x}} xe^{-x(y+1)}\,dy\,dx$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$= \int_0^\infty xe^{-x}\left[-\frac{1}{x}e^{-xy}\right]_{y=0}^{y=\frac{z}{x}}\,dx$$

$$= \int_0^\infty e^{-x}\left(e^{-z} - 1\right) dx$$

$$= 1 - e^{-z} \qquad z > 0$$

PDF of Z : $f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} e^{-z} & z > 0 \\ 0 & \text{otherwise.} \end{cases}$

Test 2

Discrete RV :

- Binomial $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ $\quad X = 0, 1, 2, \ldots, n$
- Poisson $P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$ $\quad X = 0, 1, 2, \ldots$
- Geometric $P(X=k) = (1-p)^{k-1} p$ $\quad X = 1, 2, \ldots$

Continuous RV

- Unif (a,b) $\quad f_X(x) = \begin{cases} \frac{1}{b-a} & x \in (a,b) \\ 0 & \text{else.} \end{cases}$
- Normal $\quad f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad -\infty < x < \infty.$

Approximation.

$n \geq 100$, $np \leq 10$: Poisson to approx Binomial.

$$X \sim Bin(n,p) \xrightarrow{\text{Approx}} X \sim Po(np)$$

$np(1-p) \geq 10$: $X \sim Bin(n,p) \xrightarrow{\text{Approx}} X \sim N(\overset{\mu}{np}, \overset{\sigma^2}{np(1-p)})$

Binomial $\qquad$ Normal

Continuity correction.

$$P(X=k) \approx P(k-0.5 \leq X \leq k+0.5)$$

$$P(X>k) \approx P(X \geq k+0.5)$$

$$P(X \geq k) \approx P(X \geq k-0.5)$$

$P(X < k) \approx P(X \leq k - 0.5)$

$P(X \leq k) \approx P(X \leq k + 0.5)$

Geometric Series

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad |r| < 1 \qquad \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

Binomial Series

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$$

Taylor expansion for e^x

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad x \in \mathbb{R}$$

$X_1, X_2, \ldots, X_n$. having same CDF F

$$Y = \min\{X_1, \ldots, X_n\}$$

CDF of Y = $F_Y(y) = P(Y \leq y) = P(\min\{X_1, \ldots, X_n\} \leq y)$

$$= 1 - P(\min\{\quad\} \geq y)$$

$$= 1 - P(X_1 \geq y)\, P(X_2 \geq y) \ldots P(X_n \geq y)$$

$$= 1 - (1 - F(y))^n$$

$$Z = \max\{X_1, \ldots, X_n\}.$$

$$F_Z(z) = (F(z))^n.$$