MH2500 AY16/17

Solution 1.

(a)

$$P(X \geq n) = \sum_{i=n}^{\infty}(1-p)^{i-1}p = p\sum_{k=0}^{\infty}(1-p)^{k+n-1} = p(1-p)^{n-1}\sum_{k=0}^{\infty}(1-p)^{k} = p(1-p)^{n-1}\frac{1}{1-(1-p)} = (1-p)^{n-1}.$$

(b)

$$P(Z \geq n) = P(X \geq n)P(Y \geq n) = (1-p)^{2n-2}.$$
$$P(Z = n) = P(Z \geq n) - P(Z \geq n+1) = (1-p)^{2n-2} - (1-p)^{2n} = p(2-p)(1-p)^{2n-2}.$$

(c)

$$P(Y = 2|X+Y = 4) = \frac{P(2,2)}{P(2,2)+P(1,3)+P(3,1)} = \frac{((1-p)p)^2}{((1-p)p)^2 + (1-p)^2p^2 + (1-p)^2p^2} = \frac{1}{3}.$$

Solution 2.

(a) Area of $T = \frac{1}{2}$, hence the joint pdf of X, Y is

$$f(x,y) = \begin{cases} 2, & \text{if } (x,y) \in T; \\ 0, & \text{otherwise }. \end{cases}$$

The marginal pdf and cdf of X are

$$f_X(x) = \int_0^{1-x} 2dy = 2(1-x), \quad \text{for } 0 \leq x \leq 1 \implies F_X(x) = \begin{cases} 0, & \text{if } x < 0; \\ \int_0^x 2(1-t)dt = 2x - x^2., & \text{if } 0 \leq x \leq 1; \\ 1, & \text{if } x > 1. \end{cases}$$

(b) Let $Z = X + Y$,

$$F_Z(z) = P(X+Y \leq z) = P(Y \leq z - X) = \int_0^z \int_0^{z-x} 2dydx = z^2 \implies f_Z(z) = 2z, \quad 0 \leq z \leq 1.$$

(c) The marginal pdf of Y is

$$f_Y(y) = \int_0^{1-y} 2dx = 2 - 2y.$$

Choose $x = y = \frac{1}{2}$, and we have $f_X(\frac{1}{2})f_Y(\frac{1}{2}) \neq f(\frac{1}{2}, \frac{1}{2})$. Hence, X, Y are not independent.

Solution 3.

(a)

$$M_{X_1}(t) = \mathbb{E}(e^{tX_1}) = e^{2(e^t-1)}.$$

(b)

$$M_{Y_n}(t) = \prod_{i=1}^{n} M_{X_i}(t) = e^{2n(e^t-1)}.$$

(c)

$$\mathbb{E}(Y_n) = \sum_{=1}^{n}\mathbb{E}(X_i) = 2n, \qquad Var(Y_n) = \sum_{i=1}^{n} Var(X_i) = 2n. \quad \text{(due to independence)}$$

(d) By Markov's inequality,

$$P(Y_n \geq n^2) \leq \frac{\mathbb{E}(Y_n)}{n^2} = \frac{2}{n}.$$

Solution 4.

(a)

$$1 = \int_0^\infty \int_0^x Ae^{-x} dydx = \int_0^\infty Axe^{-x} dx = A.$$

(b)

$$f_X(x) = \int_0^x e^{-x} dy = xe^{-x}, \qquad \text{for } x \geq 0.$$

$$f_Y(y) = \int_y^\infty e^{-x} dx = e^{-y}, \qquad \text{for } y \geq 0.$$

(c)

$$\mathbb{E}(X) = \int_0^\infty x^2 e^{-x} dx = 2, \qquad \mathbb{E}(Y) = \int_0^\infty ye^{-y} dy = 1.$$

$$\mathbb{E}(XY) = \int_0^\infty \int_0^x xye^{-x} dydx = \frac{1}{2} \int_0^\infty x^3 e^{-x} dx = 3.$$

Hence, $Cov(X, Y) = 3 - 2(1) = 1$.

(d)

$$\mathbb{E}(X^2) = \int_0^\infty x^3 e^{-x} dx = 6, \qquad \mathbb{E}(Y^2) = \int_0^\infty y^2 e^{-y} dy = 2, \qquad Var(X) = 2, \qquad Var(Y) = 1.$$

Hence, $\rho(X, Y) = \frac{1}{\sqrt{2}}$.

(e)

$$\mathbb{E}\left(Y|X = \frac{1}{2}\right) = \int_0^{\frac{1}{2}} yf(Y|X = 0.5)dy = \int_0^{\frac{1}{2}} y\frac{f(0.5, y)}{f_X(0.5)} dy = \int_0^{\frac{1}{2}} 2ydy = \frac{1}{4}.$$

Solution 5.

(a)

$$\mathbb{E}(X) = 33000 \cdot 0.05 = 1650, \qquad Var(X) = 33000 \cdot 0.05 \cdot 0.95 = 1567.5.$$

(b)

$$0.99 \leq P(X \leq n) \approx P\left(Z \leq \frac{n + 0.5 - 1650}{\sqrt{1567.5}}\right) \implies \frac{n + 0.5 - 1650}{\sqrt{1567.5}} \geq 2.33 \implies n \geq 1649.5 + 2.33\sqrt{1567.5}.$$