(3) Let $X_1, \ldots, X_n$ be independent Poisson random variables with mean 1, and $X = \sum_{i=1}^{n} X_i$.

(a) Use the Markov's inequality to obtain a bound on
$$\Pr\left\{\sum_{i=1}^{n} X_i \geq \frac{3}{2}n\right\}.$$

(b) Use Chebyshev's inequality to obtain a bound on the same probability in part (a).

(a) $X_i \sim Po(1)$

$X_1 \sim Po(\lambda_1)$
$X_2 \sim Po(\lambda_2)$ $\quad$ indp $\Rightarrow \quad X_1 + X_2 \sim Po(\lambda_1 + \lambda_2)$

Let $X = \sum X_i \sim Po(n)$

$$P\left(\sum X_i \geq \frac{3}{2}n\right) \leq \frac{E(\sum X_i)}{\frac{3}{2}n} = \frac{n}{\frac{3}{2}n} = \frac{2}{3}$$

Markov's Inequality

X non negative RV.

$k > 0$.

$$P(X > k) \leq \frac{E(X)}{k}$$

Chebyshev's

X is RV

$Var(X) < \infty$

$k > 0$

$$P(|X - E(x)| > k)$$
$$\leq \frac{Var(x)}{k^2}$$

(b) $P\left(\sum X_i \geq \frac{3}{2}n\right) = P\left(X - n \geq \frac{n}{2}\right)$

$$\leq P\left(|X - n| \geq \frac{n}{2}\right)$$

$$\leq \frac{Var(x)}{\left(\frac{n}{2}\right)^2} = \frac{n}{\left(\frac{n}{2}\right)^2} = \frac{4}{n}$$

if $A \subseteq B$

$\Downarrow$

$P(A) \leq P(B)$

(4) Flip a fair coin 100 times. By using Central Limit Theorem, estimate the probability of more than 55 heads.

$\underline{CLT}$: If $X_1, \ldots, X_n$ i.i.d. RV

← identically & independently distributed.

$$\text{and} \quad \mu = E(X_1), \quad \sigma^2 = Var(X_1)$$

Then for $n > 30$ (recommended)

$$\sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2) \qquad \underline{approximation}$$

Let $X_i \sim Bin(1, \frac{1}{2})$ be the RV, where $X_i = 1$ if i^{th} flip is a head.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad X_i = 0$ otherwise.

$$\mu = E(X_1) = \frac{1}{2}$$

Find $P\left(\sum X_i > 55\right)$

$$\sigma^2 = Var(X_1) = \frac{1}{4}$$

$$\approx P\left(\sum X_i \geq 55.5\right) \qquad\qquad \sum X_i \sim N\left(\frac{1}{2}n, \frac{1}{4}n\right) \quad n = 100$$

$$= P\left(Z \geq \frac{55.5 - 50}{\sqrt{25}}\right) = \underline{\qquad}$$

(5) If X is a gamma random variable with parameters $(n, 1)$, approximately how large need n be so that

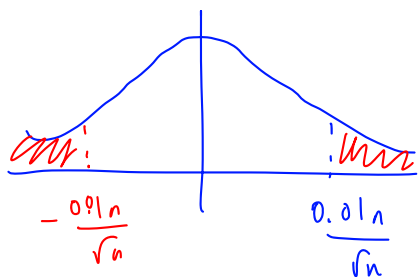$$\Pr\left\{\left|\frac{X}{n} - 1\right| > 0.01\right\} \le 0.01?$$

Fact : $X_1 \sim \text{Gamma}(\alpha_1, \beta)$
$X_2 \sim \text{Gamma}(\alpha_2, \beta)$ indp $\iff X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$

Let $X_i \sim \text{Gamma}(1, 1)$ for $i = 1, \ldots, n$ be independent RV.

$$E(X_i) = 1 \quad, \quad \text{Var}(X_i) = 1$$

$$\Rightarrow \sum X_i \sim \text{Gamma}(n, 1) \quad \text{can be approx by} \quad N(\underset{\mu}{n \cdot 1}, \underset{\sigma^2}{n \cdot 1})$$

$$0.01 \ge P\left(\left|\frac{X}{n} - 1\right| > 0.01\right) = P(|X - n| > 0.01 n)$$

$$= P(|\textstyle\sum X_i - n| > 0.01 n)$$

$$\approx P\left(|Z| > \frac{0.01 n}{\sqrt{n}}\right)$$

$$= 2\left[1 - \Phi\left(\frac{0.01 n}{\sqrt{n}}\right)\right]$$

$-\frac{0.01 n}{\sqrt{n}} \qquad \frac{0.01 n}{\sqrt{n}}$

$$0.01 \ge 2\left[1 - \Phi(0.01\sqrt{n})\right]$$

$$0.005 \ge 1 - \Phi(0.01\sqrt{n})$$

$$\Phi(0.01\sqrt{n}) \ge 0.995$$

$$0.01\sqrt{n} \ge \Phi^{-1}(0.995)$$

$$n \ge \frac{1}{0.01}\left[\Phi^{-1}(0.995)\right]^2 = \underline{}$$

(6) Civil engineers believe that W, the amount of weight (in units of 1000 pounds) that a certain span of a bridge can withstand without structural damage resulting, is normally distributed with mean 400 and standard deviation 40. Suppose that the weight (again, in units of 1000 pounds) of a car is a random variable with mean 3 and standard deviation 0.3. Approximately how many cars would have to be on the bridge span for the probability of structural damage to exceed 0.1?

$$W \sim N\left(400, 40^2\right)$$

Let X_i be the RV for the weight of car i.

$$\mathbb{E}(X_i) = 3 \quad, \quad Var(X_i) = 0.09$$

Find n, such that $P\left(\sum_{i=1}^{n} X_i > W\right) \geqslant 0.1$

$$X_1 \sim N\left(\mu_1, \sigma_1^2\right)$$
$$X_2 \sim N\left(\mu_2, \sigma_2^2\right)$$

$$X_1 \pm X_2$$
$$\sim N\left(\mu_1 \pm \mu_2, \sigma_1^2 + \sigma_2^2\right)$$

By CLT, $X = \sum X_i \sim N\left(3n, 0.09n\right)$

$$0.1 < P\left(X - W > 0\right)$$

$$X - W \sim N\left(3n - 400, 40^2 + 0.09n\right)$$

$$\approx P\left(Z > \frac{0 - (3n - 400)}{\sqrt{40^2 + 0.09n}}\right)$$

$$\Phi\left(\frac{3n - 400}{\sqrt{40^2 + 0.09n}}\right) > 0.1$$

$$\frac{3n - 400}{\sqrt{40^2 + 0.09n}} > \Phi^{-1}(0.1) \implies \text{solve for } n.$$

① Probability : Bayes Rule

Law of Total Probability

② Discrete RV : Bernoulli, Binomial, Geometric, Poisson.

③ Continuous RV : Normal, Uniform, Exponential, Gamma.

$$E(X), \quad Var(X)$$

④ Joint RV :

$$f_X(x) = \int f(x,y) \, dy$$

$$f_Y(y) = \int f(x,y) \, dx$$

$$\iint_D f(x,y) \, dy \, dx = 1$$

$$f(x,y) = f_X \cdot f_Y$$
$$\Updownarrow$$
independence.

$$Cov(X,Y) = \mathbb{E}(XY) - E(X)E(Y).$$

If X, Y indp $\Rightarrow$ $E(XY) = E(X)E(Y)$ $\Rightarrow$ $Cov(X,Y) = 0$

Converse is FALSE.

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X) Var(Y)}} \qquad -1 < \rho < 1$$

If X_i indp $\Rightarrow$ $Var(\sum X_i) = \sum Var(X_i)$ (False in general)

$$\mathbb{E}(\sum X_i) = \sum \mathbb{E}(X_i) \quad \text{(always true)}$$

⑤ Law of Total Expectation.

$$E(X) = \sum_i E(X|A_i) P(A_i)$$

$$\mathbb{E}(X) = E_Y(E_X(X|Y))$$

in terms of Y.

Conditional Variance.

$$Var(Y | X=x) = \mathbb{E}(Y^2 | X=x) - \left[E(Y | X=x)\right]^2$$

Law of Total Variance

$$Var(Y) = Var(\mathbb{E}(Y|X)) + \mathbb{E}\left[Var(Y|X)\right]$$

⑥   Markov Ineq,   Chebyshev Ineq,   CLT

Normal   Approximation , Poisson   Approx   to   Binomial