Problem 10.1

Let $X_1, \dots, X_n$ be an i.i.d. sample drawn from a normal distribution $N(\mu, 1)$ where $\mu \in \mathbb{R}$ is an unknown parameter. The following observations $X_1, \dots, X_{10}$ are given.

$$-1.0, 4.3, 2.3, 1.3, 6.4, \quad -2.3, 5.3, 3.5, 7.1, 3.1$$

a) Consider a test for $H_0: \mu = 0$ against $H_1: \mu \neq 0$ based on the test statistic $T = \sum_{i=1}^{n} X_i$. What is the p-value of the above observations?

b) Consider a test for $H_0: \mu = 0$ against $H_1: \mu > 0$ based on the test statistic $T = \sum_{i=1}^{n} X_i$. What is the p-value of the above observations? Determine the critical value for the test based on T with 0.05 level of significance.

(a) $T = \sum X_i \sim N(10\mu, 10)$ $\quad t = \sum x_i = 30$

p-value $= P(|T| > 30 \mid H_0) = P(T > 30 \mid \mu = 0) + P(T < -30 \mid \mu = 0)$

$= \underset{T \sim N(0,10)}{P(T > 30)} + \underset{T \sim N(0,10)}{P(T < -30)}$

$X \sim N(\mu, \sigma^2)$

$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$

$= P\left(Z > \frac{30}{\sqrt{10}}\right) + P\left(Z < -\frac{30}{\sqrt{10}}\right)$

≈ 0

(b) p-value $= P(T > 30 \mid H_0) = P\left(Z > \frac{30}{\sqrt{10}}\right) \approx 0 < 0.05$

Reject H_0,

Critical value. $T \sim N(10\mu, 10)$

t_α : critical value.

$P(T > t_\alpha \mid \underset{\mu = 0}{H_0}) = 0.05$

$\underset{T \sim N(0,10)}{P}(T > t_\alpha) = 0.05$

$P\left(Z < \frac{t_\alpha}{\sqrt{10}}\right) = 0.95$

$\Phi\left(\frac{t_\alpha}{\sqrt{10}}\right) = 0.95 \Rightarrow t_\alpha = \sqrt{10}\, \Phi^{-1}(0.95)$

Problem 10.2

Let $X_1, \dots, X_n$ be an i.i.d. sample drawn from $Exp(\lambda)$ distribution, where $\lambda \in (0, \infty)$ is an unknown parameter. Consider a test for $H_0: \lambda = 1$ against $H_1: \lambda > 1$ based on the test statistic $T = \sum_{i=1}^{n} X_i$. Suppose $n = 10$ and that the test has size $\alpha = 0.05$.

a) Find the critical value for the rejection of H_0 based on T.
b) What is the power of the test at $\lambda = \lambda_1 > 1$ obtained in Part a)?
c) Is H_0 rejected for the following observations?

$$0.1, 0.2, 0.1, 0.3, 0.5, \quad 0.01, 1.2, 0.05, 0.001, 0.1$$

$X_i \sim Exp(\lambda)$

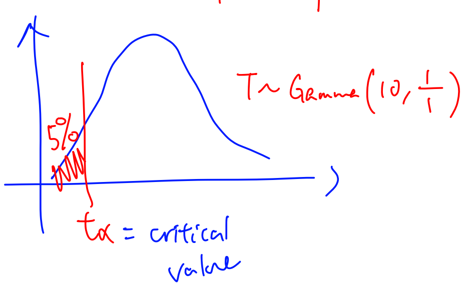
$X_i \sim Gamma\left(1, \frac{1}{\lambda}\right)$

$\sum X_i \sim Gamma\left(n, \frac{1}{\lambda}\right)$

(a) $T = \sum X_i \sim Gamma\left(10, \frac{1}{\lambda}\right) \Rightarrow E[T] = \frac{10}{\lambda}$

$\lambda \uparrow \Rightarrow E[T] \downarrow$

Small values of T provides evidence against $H_0: \lambda = 1$



$P(T < t_\alpha \mid H_0) = 0.05$

$F(t_\alpha) = 0.05$

$\Rightarrow t_\alpha = F^{-1}(0.05)$, F is CDF of $Gamma(10, 1)$

$= 5.425$

(b) Power of test at λ_1: $P(\text{Reject } H_0 \mid H_1)$

$= P(T < 5.425 \mid \lambda = \lambda_1 > 1)$ $\quad T \sim Gamma\left(10, \frac{1}{\lambda_1}\right)$

$= F_1(5.425)$, where F_1 is CDF of $Gamma\left(10, \frac{1}{\lambda_1}\right)$

Problem 10.2

Let $X_1, \ldots, X_n$ be an i.i.d. sample drawn from $\mathrm{Exp}(\lambda)$ distribution, where $\lambda \in (0, \infty)$ is an unknown parameter. Consider a test for $H_0: \lambda = 1$ against $H_1: \lambda > 1$ based on the test statistic $T = \sum_{i=1}^{n} X_i$. Suppose $n = 10$ and that the test has size $\alpha = 0.05$.

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c) Is H_0 rejected for the following observations?

$$0.1, 0.2, 0.1, 0.3, 0.5, \quad 0.01, 1.2, 0.05, 0.001, 0.1$$

a) $t_\alpha = 5.425$

b) Power : $F_1(5.425)$

(c) Reject $H_0 \iff T < 5.425$, $T = \sum X_i$

$\sum x_i = \_\_ < 5.425 \Rightarrow$ Reject $H_0 : \lambda = 1$.

Problem 10.3

Let $X_1, \ldots, X_5$ be an i.i.d. sample drawn from $Bernoulli(p)$ distribution, where $p \in [0,1]$ is an unknown parameter. Consider a test for $H_0: p = 0.2$ against $H_1: p = 0.5$ which rejects H_0 if and only if $\sum_{i=1}^{5} X_i > 2$.

$P(X_i = 1) = p$
$P(X_i = 0) = 1 - p$

a) Compute the probability for Type-I and Type-II errors.
b) Find the size and the power of the test.

Test statistic $T = \sum_{i=1}^{5} X_i \sim \text{Binomial}(5, p)$

PMF for Binomial $P(T = k) = \binom{5}{k} p^k (1-p)^{5-k}$ $\quad k = 0, 1, \ldots, 5$

(a) $P(\text{Type 1 error}) = P(\text{Reject } H_0 \mid H_0) = P(T > 2 \mid p = 0.2)$

$= P(T=3) + P(T=4) + P(T=5)$

size of test $= \alpha = \sum_{k=3}^{5} \binom{5}{k} 0.2^k \, 0.8^{5-k}$

$P(\text{Type 2 error}) = P(\text{Dont reject } H_0 \mid H_1) = P(T \le 2 \mid p = 0.5)$

$= \_\_\_\_ = \beta$

Power of test $= 1 - \beta$.

QUESTION 5. (20 marks) 2018/19

Let X_1, X_2 be iid $\sim \mathrm{N}(\mu, \sigma^2)$, where μ and σ are unknown parameters. Write $\bar{X} = (X_1 + X_2)/2$ and

$$S = \sqrt{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2}.$$

$\Rightarrow \; S^2 = (X_1-\bar{X})^2 + (X_2-\bar{X})^2$

$n=2$

$= \frac{1}{n-1} \sum (X_i - \bar{X})^2$

Compute the bias of S as an estimator for σ.

Hint: The PDF of χ^2_1 is

$$f(x) = \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} e^{-\frac{x}{2}}$$

for $x > 0$ and $f(x) = 0$ otherwise.

$X_1, \dots, X_n \sim N(\mu, \sigma^2)$

$\Rightarrow \; \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$n=2 \Rightarrow X = \frac{S^2}{\sigma^2} \sim \chi^2_1$

$\boxed{S = \sigma\sqrt{X} \text{ where } X \sim \chi^2_1}$

Bias of $S = E(S) - \sigma$

$= E(\sigma\sqrt{X}) - \sigma$

$= \sigma\, E(\sqrt{X}) - \sigma$

$= \sigma \int_0^\infty \sqrt{x}\, f_X(x)\, dx - \sigma$

$= \sigma \int_0^\infty \cancel{\sqrt{x}} \cdot \frac{1}{\sqrt{2\pi}} \cancel{x^{-\frac{1}{2}}} e^{-\frac{x}{2}}\, dx - \sigma$

$= \sigma \left[\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x}{2}}\, dx - 1 \right]$

$X \sim N(0,1)$

$\Downarrow$

$Y = X^2 \sim \chi^2_1$

If $Y = \chi^2_1 \; \cancel{\overset{?}{\Rightarrow}} \; \sqrt{Y} \sim N(0,1)$

$\sqrt{X^2} = |X| \neq X$ for all $X \in \mathbb{R}$

MH3500 Statistics

Tutorial 10 AY2022/23 Semester 2

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a) Compute the probability for Type-I and Type-II errors.

b) Find the size and the power of the test.

$(a_1 + a_2 + \dots + a_n)^2$

$= a_1^2 + a_2^2 + \dots + a_n^2 + 2\sum_{i<j} a_i a_j$

$i \neq j$

$(a_1 + a_2)^2 = a_1^2 + a_2^2 + a_1 a_2 + a_2 a_1$

$X_i \sim N(\mu, \sigma^2)$

$E\left[(X_i - \mu)^2\right] = \sigma^2$ ← Variance for X_i

$E(X_i - \mu)^2 = \sigma^2$

$$\prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda}$$

$$= \frac{\lambda^{x_1} \cdot \lambda^{x_2} \cdot \dots \cdot \lambda^{x_n}}{x_1!\, x_2! \dots x_n!} \underbrace{e^{-\lambda} e^{-\lambda} \dots e^{-\lambda}}_{n \text{ times}}$$

$$= \frac{\lambda^{\sum x_i}}{\prod (x_i!)}$$

condition (a)

$$I_{x_1, \dots x_n}(\alpha) = n I_{x_1}(\alpha)$$

By def : $I_{x_1}(\alpha) = E\left[\left(\overbrace{\frac{\partial}{\partial \alpha} \ln f(x|\alpha)}^{V}\right)^2\right]$

$\int \frac{\partial^2}{\partial \alpha^2} = 0$ Alt $I_{x_1}(\alpha) = -E\left[\frac{\partial^2}{\partial \alpha^2} \ln f(x|\alpha)\right]$