Extra Exercises 2

Problem 6. The binomial theorem states that, for nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

What is the value of

$$\binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n}?$$

$$\binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k}$$

$$= (-1+1)^n = 0$$

Problem 8. Let k, n be integers such that $0 \le k \le n$ and $n \ge 1$. Determine the number of vectors $(x_1, \ldots, x_n)$, such that each $x_i \in \{0,1\}$ and

$$x_1 + x_2 + \ldots + x_n \ge k.$$

consider

$$x_1 + x_2 + \ldots + x_n = k$$

$$x_1 + x_2 + \ldots + x_n = k+1 \qquad \text{where } x_i \in \{0, 1\}$$

$$\vdots$$

$$x_1 + x_2 + \ldots + x_n = n$$

For $x_1 + \ldots + x_n = k$, out of the n summands, we need k of them to be 1.

The rest must be zero. Total combinations $= \binom{n}{k}$

Same analogy applies to all equations and we sum all the combinations to get $\binom{n}{k} + \binom{n}{k+1} + \ldots + \binom{n}{n} = \sum_{i=k}^{n} \binom{n}{i}$

Extra Exercises 3

Problem 2. For three events E, F, G, prove that

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E^cFG) - P(EF^cG) - P(EFG^c) - 2P(EFG).$$

By the Principle of Inclusion-Exclusion,

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(FG) - P(EG) + P(EFG)$$

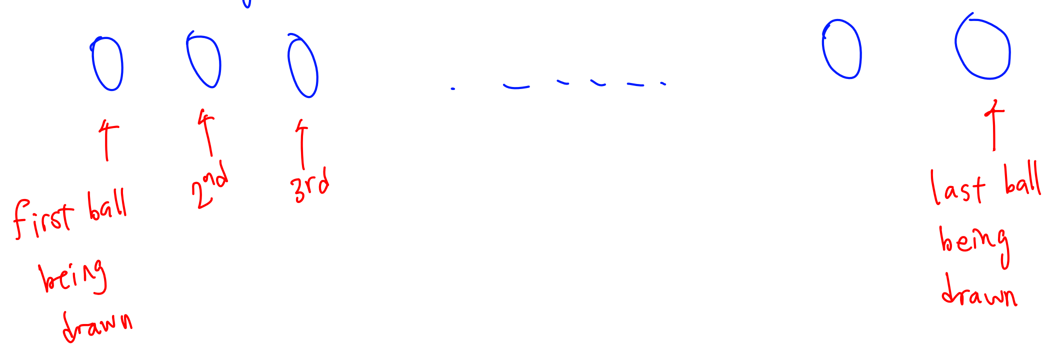
By Law of Total Probability

$$\left.\begin{aligned} P(EF) &= P(EFG) + P(EFG') \\ P(FG) &= P(EFG) + P(E'FG) \\ P(EG) &= P(EFG) + P(EF'G) \end{aligned}\right\}$$

Substitute.

Problem 4. An urn contains n white and m black balls. The balls are withdrawn one at a time until only those of the same color are left. What is the probability that they are all white?

We can arrange the $(n+m)$ balls in a line



To make sure that balls of same color are left at the end of the line and they must be white, this is as if making sure the last ball is white. For example, we can have sequences like

.... B W W W

.... W B B W

.... W B W W

All valid sequences must end with a white ball. Hence, we fix the last ball as white and rearrange the $(n-1)$ white and m black balls in front.

$$\text{Probability} = \frac{\dfrac{(n+m-1)!}{m!\,(n-1)!}}{\dfrac{(n+m)!}{n!\,m!}} = \frac{(n+m-1)!}{m!\,(n-1)!} \times \frac{n!\,m!}{(n+m)!} = \frac{n}{n+m}$$

Extra Exercises 4

Problem 2. A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose each component works independently with probability 0.5. Find the conditional probability that component 1 works given that the system is functioning. What value does the probability converge to when n is very large?

C_i : component i is working, $P(C_i) = 0.5$ $i = 1, 2, \ldots, n$

F : Whole system is functioning.

$$P(C_1 | F) = \frac{P(C_1 \cap F)}{P(F)} = \frac{P(F|C_1)P(C_1)}{1 - P(F')} = \frac{1 \cdot 0.5}{1 - P(F')}$$

$P(F')$ = probability that the whole system is not functioning.

This only happens when all components are not working simultaneously.

$$P(F') = \left(\frac{1}{2}\right)^n \Rightarrow P(C_1 | F) = \frac{\frac{1}{2}}{1 - \left(\frac{1}{2}\right)^n}$$

Remark:

Cannot do $P(F) = \sum_{j=1}^{n} P(F|C_j)P(C_j)$, $C_1, \ldots, C_n$ are not mutually exclusive

$P(\bigcup C_i) \neq 1$, they are not exhaustive.

Problem 3. Show that if $P(A|B) = 1$, then $P(B^c|A^c) = 1$.

$$P(B'|A') = \frac{P(A' \cap B')}{P(A')} = \frac{P[(A \cup B)']}{P(A')} = \frac{1 - P(A \cup B)}{1 - P(A)} = \frac{1 - (P(A) + P(B) - P(A \cap B))}{1 - P(A)}$$

Since $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = 1 \Rightarrow P(B) = P(A \cap B)$

$\therefore P(B'|A') = 1$

Similar to if B then A $\equiv$ if not A, then not B

Problem 5. Show that if $P(A) > 0$, then $P(A \cap B|A) \geq P(A \cap B|A \cup B)$.

Since $A \subseteq A \cup B \Rightarrow P(A) \leq P(A \cup B)$

$A \cap B \subseteq A \cup B \Rightarrow (A \cap B) \cap (A \cup B) = A \cap B$

$$P(A \cap B | A) = \frac{P(A \cap B)}{P(A)} \geq \frac{P(A \cap B)}{P(A \cup B)} = P(A \cap B | A \cup B)$$

Problem 6. Let $S = \{1, 2, \ldots, n\}$ and suppose that A and B are, independently, equally likely to be any of the 2^n subsets (including the null set and S itself) of S. Let $N(B)$ denote the number of elements in B. By showing that

$$P(A \subseteq B|N(B) = i) = \frac{2^i}{2^n},$$

and conditioning on $N(B) = i$, for $i = 0, \ldots, n$, compute $P(A \subseteq B)$.

Each element $x \in S$, falls into one of the cases with equal probability

$x \in A, \quad x \in B$

$x \notin A, \quad x \in B$

$x \in A, \quad x \notin B$

$x \notin A, \quad x \notin B$

$$P(A \subseteq B \cap N(B) = i) = \frac{\binom{n}{i} 2^i \cdot 1^{n-i}}{4^n}$$

$\leftarrow$ sample size (4 possible cases for each element)

choose i elements, put it in B

For each element in B, there are two possible cases, either $\in A$ or $\notin A$

Hence, 2^i. For each element not in B, only one possible case,

it must be $\notin A$ to ensure $A \subseteq B$ Hence 1^{n-i}

$$P(N(B) = i) = \binom{n}{i}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^{n-i}$$

$$\therefore P(A \subseteq B | N(B) = i) = \frac{\frac{1}{4^n}\binom{n}{i} 2^i}{\binom{n}{i}\left(\frac{1}{2}\right)^n} = \frac{2^i}{2^n}$$

$$P(A \subseteq B) = \sum_{i=0}^{n} P(A \subseteq B | N(B) = i)\, P(N(B) = i)$$

$$= \sum_{i=0}^{n} \frac{2^i}{2^n}\binom{n}{i}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^{n-i} = \frac{1}{4^n}\sum_{i=0}^{n}\binom{n}{i}2^i = \frac{1}{4^n}\cdot 3^n = \left(\frac{3}{4}\right)^n$$

$$\sum_{i=0}^{n}\binom{n}{i}2^i = \sum_{i=0}^{n}\binom{n}{i}2^i(1)^{n-i} = 3^n$$

Problem 7. A and B alternate rolling a pair of dice, stopping either when A rolls the sum 9 or when B rolls the sum 6. Assuming that A rolls first, find the probability that the final roll is made by A.

$$P(A \text{ rolls } 9) = \frac{1}{9} \qquad P(B \text{ rolls } 6) = \frac{5}{36}$$

$$\underbrace{\frac{1}{9}}_{A=9} + \underbrace{\frac{8}{9}}_{A\neq 9} \cdot \underbrace{\frac{31}{36}}_{B\neq 6} \cdot \underbrace{\frac{1}{9}}_{A=9} + \underbrace{\left(\frac{8}{9}\cdot\frac{31}{36}\right)^2}_{A\neq 9,\, B\neq 6,\, A\neq 9,\, B\neq 6} \cdot \underbrace{\frac{1}{9}}_{A=9} + \cdots = \frac{\frac{1}{9}}{1-\frac{8}{9}\cdot\frac{31}{36}} = \frac{9}{19}$$

Geometric sum : $a + ar + ar^2 + \ldots = \dfrac{a}{1-r}$, $|r|<1$

Problem 8. In successive rolls of a pair of fair dice, what is the probability of getting 2 sevens before 6 even numbers?

Rolling numbers which are not 7 or not even are meaningless. They do not contribute to the event that we are interested in.

We can condition on those rolls which are either 7 or even.

Let X be the result of a roll.

$$P(X=7 \mid X=7 \text{ or } X \text{ is even}) = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{2}} = \frac{1}{4}$$

$$P(X \text{ is even} \mid X=7 \text{ or } X \text{ is even}) = \frac{3}{4}$$

Out of the meaningful rolls, $\frac{1}{4}$ of them will result in 7. $\frac{3}{4}$ of them are even. (in terms of probability)

It is easier to consider the complement : getting 6 even numbers before two 7s.

Possible cases

E E E E E E $\Rightarrow \left(\frac{3}{4}\right)^6$

$\underbrace{7 \text{ E E E E E}}_{\text{can swap}}$ E $\Rightarrow \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^6 \cdot \frac{6!}{5!}$

We can ignore those rolls which are not even / 7, because the probabilities $\frac{1}{4}, \frac{3}{4}$ are based only on meaningful rolls.

Since this is the complement, the probability required is $1 - \left(\frac{3}{4}\right)^6 - \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^6 \cdot \frac{6!}{5!} = \frac{4547}{8192}$

Problem 10. An urn contains 4 blue balls and 3 red balls. A fair die is rolled. If the number shown on the die is i, $(i = 1, 2, \ldots, 6)$, then i balls are drawn from the urn uniformly at random without replacement. Given that all balls drawn are red, what is the probability that the remaining balls in the urn are all blue?

$$P(\text{Remaining all blue} \mid \text{Drawn are all red})$$

$$= P(i=3 \mid \text{Drawn are all red})$$

$$= \frac{P(\text{Drawn are all red} \mid i=3)\, P(i=3)}{P(\text{Drawn all red})}$$

$$= \frac{\frac{\binom{3}{3}}{\binom{7}{3}} \cdot \frac{1}{6}}{P(\text{Drawn all red} \mid i=1)\, P(i=1) + P(\text{Drawn all red} \mid i=2)\, P(i=2) + P(\text{Drawn all red} \mid i=3)\, P(i=3)}$$

$$= \frac{\frac{1}{35} \cdot \frac{1}{6}}{\frac{\binom{3}{1}}{\binom{7}{1}} \cdot \frac{1}{6} + \frac{\binom{3}{2}}{\binom{7}{2}} \cdot \frac{1}{6} + \frac{\binom{3}{3}}{\binom{7}{3}} \cdot \frac{1}{6}} = \frac{1}{21}$$