MH2500 AY22/23

Solution 1. $\mathbb{E}(X) = 2.8$, $Var(X) = 1.16$.

Solution 2. The PDF of X and Y are

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0; \\ 0, & x < 0, \end{cases} \qquad f_Y(y) = \begin{cases} 1, & x \in [0,1]; \\ 0, & \text{otherwise.} \end{cases}$$

Let $Z = X + Y$, then

$$F_Z(z) = P(Z \leq z) = P(X+Y \leq z) = P(Y \leq z - X) = \int_0^\infty \int_0^{z-x} f_{X,Y}(x,y)dydx = \int_0^\infty \int_0^{z-x} f_X(x)f_Y(y)dydx.$$

By fundamental theorem of calculus,

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_0^\infty f_X(x)f_Y(z-x)dx = \begin{cases} \int_0^\infty e^{-x}dx, & \text{if } 0 \leq z - x \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

When $0 \leq z \leq 1$, then we need $0 \leq x \leq z$, so that $z - x \in [0,1]$,

$$f_Z(z) = \int_0^z e^{-x}dx = 1 - e^{-z}.$$

When $z > 1$, then we need $z - 1 \leq x \leq z$, so that $z - x \in [0,1]$,

$$f_Z(z) = \int_{z-1}^z e^{-x}dx = e^{1-z} - e^{-z}.$$

Therefore,

$$f_Z(z) = \begin{cases} 1 - e^{-z}, & \text{if } 0 \leq z \leq 1; \\ e^{1-z} - e^{-z}, & \text{if } z > 1. \end{cases}$$

Solution 3.

(b)
$$P(X + Y \leq 1) = \int_0^1 \int_0^{1-x} 4xydydx = \frac{1}{6}.$$

(c)
$$f_X(x) = \int_0^1 4xydy = 2x, \qquad 0 < x < 1.$$

(d)
$$f_Y(y) = \int_0^1 4xydx = 2y, \qquad 0 < y < 1.$$

Since, $f(x,y) = f_X(x)f_Y(y)$, X and Y are independent.

Solution 4.

$$\begin{aligned}\mathbb{E}(X) &= \int_0^\infty x f_X(x)dx = \int_0^\infty x \int_0^\infty f(x,y)dydx = \int_0^\infty \int_0^\infty \frac{x}{y}\exp\left(-y-\frac{x}{y}\right)dydx \\ &= \int_0^\infty \frac{e^{-y}}{y}\int_0^\infty xe^{-\frac{x}{y}}dxdy = \int_0^\infty \frac{e^{-y}}{y}\left(\left[-xye^{-\frac{x}{y}}\right]_0^\infty + y\int_0^\infty e^{-\frac{x}{y}}dx\right)dy \\ &= \int_0^\infty \frac{e^{-y}}{y}\cdot y^2 dy = 1.\end{aligned}$$

$$\begin{aligned}\mathbb{E}(Y) &= \int_0^\infty y f_Y(y)dy = \int_0^\infty y \int_0^\infty f(x,y)dxdy = \int_0^\infty \int_0^\infty \exp\left(-y-\frac{x}{y}\right)dxdy \\ &= \int_0^\infty e^{-y}\int_0^\infty e^{-\frac{x}{y}}dxdy = \int_0^\infty ye^{-y}dy = 1.\end{aligned}$$

$$\begin{aligned}\mathbb{E}(XY) &= \int_0^\infty \int_0^\infty xyf(x,y)dxdy = \int_0^\infty e^{-y}\int_0^\infty xe^{-\frac{x}{y}}dxdy \\ &= \int_0^\infty y^2e^{-y}dy = 2.\end{aligned}$$

$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 1.$

Solution 5. Let X_b be the random variable denoting the number of black balls drawn if there are b black balls from the start. By law of total expectation,

$$\begin{aligned}\mathbb{E}(X_b) &= \mathbb{E}(X|\text{first ball is black})P(\text{first ball is black}) + \underbrace{\mathbb{E}(X|\text{first ball is not black})}_{=0}P(\text{first ball is not black}) \\ &= \mathbb{E}(X|\text{first ball is black})P(\text{first ball is black}) = (1+\mathbb{E}(X_{b-1}))\frac{b}{a+b}.\end{aligned}$$

The initial case is $b=0$ and we have that $\mathbb{E}(X_0)=0$, since there are no black balls to start with. Using the recurrence relation above, $\mathbb{E}(X_1) = \frac{1}{a+1}$. Similarly, we can see that

$$\mathbb{E}(X_2) = \left(1+\frac{1}{a+1}\right)\frac{2}{a+2} = \frac{2}{a+1}.$$

We now claim and prove by induction that $\mathbb{E}(X_b) = \frac{b}{a+1}$. The base case $b=0$ has been established. Suppose it is true for $b-1$, i.e. $\mathbb{E}(X_{b-1}) = \frac{b-1}{a+1}$. Then,

$$\mathbb{E}(X_b) = (1+\mathbb{E}(X_{b-1}))\frac{b}{a+b} = \left(1+\frac{b-1}{a+1}\right)\frac{b}{a+b} = \frac{b}{a+1}.$$

This concludes the induction and we have $\mathbb{E}(X_b) = \frac{b}{a+1}$.

Solution 6. Let X_i denote the component's lifetime, $i=1,2,\ldots,n$. Then, $\mathbb{E}(X_i)=100$, $Var(X_i)=30^2$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^N X_i$. When n is large, by central limit theorem, $\bar{X}$ converges in distribution to $\mathrm{N}\left(100, \frac{30^2}{n}\right)$. Hence,

$$P\left(\sum_{i=1}^n X_i \geq 2000\right) \geq 0.95 \implies P\left(Z \geq \frac{\frac{2000}{n}-100}{30/\sqrt{n}}\right) \geq 0.95 \implies \frac{\frac{2000}{n}-100}{30/\sqrt{n}} \leq -1.645.$$

Solve for n.