Recap for Tutorial 11

MH1101

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1 Power Series

Definition 1 (Power Series)**.** A power series in x is of the form

$$\sum_{n\geq 0} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots .$$

Definition 2 (Power Series centered at $x = a$)**.** The power series at $x = a$ is of the form

$$\sum_{n\geq 0} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots .$$

Theorem 1. Let R be the radius of convergence. Given a power series $\sum c_n(x-a)^n$, there are only three possibilities:

- The series converges only at $x = a$. ($R = 0$)
- The series converges everywhere. ($R = \infty$)
- The series converges if $|x-a| < R$ and diverges if $|x-a| > R$, for $R > 0$.

Remark 1: The last possibility does not tell us the convergence of the series at the endpoint of the interval, i.e. when $x = a - R$ or $x = a + R$. To examine this, you will need to use some other convergence tests.

Remark 2: Also note that the power series will not diverge everywhere, it does not make sense because it will definitely converge at $x = a$ and the sum is just 0.

Theorem 2. Suppose that the power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

has radius of convergence $R > 0$. Then, $f(x)$ is differentiable on the interval $(a - R, a + R)$, and its derivative and antiderivative may be computed term by term, i.e.

- $$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n,$$

- $$\int f(x) = \sum_{n=0}^{\infty} c_n \int (x-a)^n dx.$$

In simple terms, you can switch the order of summation and differentiation/integration only when the series converges. The result will not be the same at points where the summation diverges.

2 Extra Exercises

Problem 1. Find the radius of convergence and its interval of convergence for the following series.

- $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot \ldots \cdot 2n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$

Problem 2. Show that $\displaystyle\int_0^{1/2} \frac{dx}{x^2-x+1} = \frac{\pi}{3\sqrt{3}}$. Hence, or otherwise, show that

$$\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2} \right).$$

Hint: Use the fact that $x^3+1 = (x+1)(x^2-x+1)$, then express $\dfrac{1}{x^3+1}$ as a power series.