# Recap for Tutorial 5

MH1101

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## 1 Trigonometric Integrals

**Technique 1.** Integrating $\int \sin^m x \cos^n x \, \mathrm{d}x$.

- If $n$ is odd, make substitution $u = \sin x$.
- If $m$ is odd, make substitution $u = \cos x$.
- If $m$ and $n$ are even, use half angle formula: $\sin^2 x = \dfrac{1-\cos 2x}{2}$ or $\cos^2 x = \dfrac{1+\cos 2x}{2}$

**Technique 2.** Integrating $\int \tan^m x \sec^n x \, \mathrm{d}x$.

- If $n$ is even, make substitution $u = \tan x$.
- If $m$ is odd, make substitution $u = \sec x$.

**Technique 3.** Integrating $\int \sin(mx)\sin(nx) \, \mathrm{d}x$, $\int \cos(mx)\cos(nx) \, \mathrm{d}x$ or $\int \sin(mx)\cos(nx) \, \mathrm{d}x$.

- Use product to sum formula (provided in formula sheet).

## 2 Trigonometric Substitution

We usually perform trigonometric substitution in integration when the integrand contains a specific form. However, there are other cases (not mentioned here) where we use similar substitution to simplify our problem.

- If the integrand contains $\sqrt{a^2 - x^2}$, let $x = a\sin\theta$, where $|\theta| \le \pi/2$.
- If the integrand contains $\sqrt{a^2 + x^2}$, let $x = a\tan\theta$, where $|\theta| < \pi/2$.
- If the integrand contains $\sqrt{x^2 - a^2}$, let $x = a\sec\theta$, where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$.

## 3 Extra Exercises

**Problem 1.** Prove the reduction formula

$$\int \sin^n x \, \mathrm{d}x = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, \mathrm{d}x.$$

**Problem 2.** For $n = 0, 1, \ldots$, we define the polynomials

$$L_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$$

(a) Using integration by parts, show that $\displaystyle\int_{-1}^{1} |L_n(x)|^2 \mathrm{d}x = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n \mathrm{d}x$.

(b) Make a substitution $x = \cos t$ and apply the reduction formula in Problem 1 to conclude that

$$\int_{-1}^{1} |L_n(x)|^2 \mathrm{d}x = \frac{2}{2n+1}$$

**Extra Information:** $(L_n(x))_{n\geq 0}$ forms an orthogonal basis for the function space $L^2([-1,1])$. This is a vector space that contains functions, $f(x)$ such that $\int_{-1}^{1} |f(x)|^2 \mathrm{d}x < \infty$. These polynomials are called Legendre polynomials. Due to this orthogonality, you can also show that $\int_{-1}^{1} L_m(x)L_n(x)\mathrm{d}x = 0$, if $m \neq n$.

**Problem 3.** Evaluate the following.

- $\displaystyle\int_{\pi/2}^{\pi} \sqrt{1-\sin^2 x}\, \mathrm{d}x.$ Ans: 1
- $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{\mathrm{d}x}{x^5\sqrt{9x^2-1}}.$ Ans: $\displaystyle\frac{81}{64}(-16+7\sqrt{3}+2\pi)$
- $\displaystyle\int \frac{\mathrm{d}x}{[(ax)^2-b^2]^{3/2}}.$ Ans: $\displaystyle-\frac{x}{b\sqrt{(ax)^2-b^2}}+C$