

Practice Week 6 Question 3

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Here, we will go through Practice Week 6 Question 3(a) and 3(b) in details and I will explain a few different methods to approach 3(b).

For $0 < p < 1$, we say X follows a negative binomial distribution if for $n = r, r + 1, \dots$, if

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}.$$

In short, we can write $X \sim \text{NegBin}(r, p)$. The generalized binomial theorem states that for any real numbers x, y and any real number r , we have

$$(x + y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k y^{r-k}, \quad \text{provided the infinite series converges.}$$

This in fact generalises to $r \in \mathbb{C}$. If we consider r as a non-negative integer and use the convention that $\binom{r}{k} = 0$, if $r < k$, then we have the simpler version of the binomial theorem, i.e.,

$$(x + y)^r = \sum_{k=0}^r \binom{r}{k} x^k y^{r-k}.$$

For both, we define $\binom{r}{k} = \frac{r \cdot (r-1) \cdot \dots \cdot (r-k+1)}{k!}$.

For Question 3(a), we have

$$\begin{aligned} \sum_{n=r}^{\infty} P(X = n) &= \sum_{n=r}^{\infty} \binom{n-1}{r-1} p^r (1-p)^{n-r} \\ &= \sum_{n=r}^{\infty} \binom{n-1}{n-r} p^r (1-p)^{n-r} && \text{by symmetry, } \binom{n}{k} = \binom{n}{n-k} \\ &= p^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} (1-p)^j && \text{let } j = n-r \\ &= p^r \sum_{j=0}^{\infty} \frac{(r+j-1) \cdot (r+j-2) \cdot \dots \cdot (r+1) \cdot r}{j!} (1-p)^j \\ &= p^r \sum_{j=0}^{\infty} (-1)^j \frac{(-r-j+1) \cdot (-r-j+2) \cdot \dots \cdot (-r-1) \cdot (-r)}{j!} (1-p)^j \\ &= p^r \sum_{j=0}^{\infty} (-1)^j \binom{-r}{j} (1-p)^j && \text{by definition of } \binom{-r}{j} \\ &= p^r \sum_{j=0}^{\infty} \binom{-r}{j} (p-1)^j \cdot 1^{-r-j} \\ &= p^r (p-1+1)^{-r} && \text{generalised binomial theorem, } |p-1| < 1 \\ &= 1. \end{aligned}$$

For Question 3(b), I will provide two different methods to approach it.

Method 1. We need to compute $\mathbb{E}(X)$ and $Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$. For any $k \in \mathbb{Z}$, $k \geq 1$, we can compute $\mathbb{E}(X^k)$ as follows:

$$\begin{aligned}
\mathbb{E}(X^k) &= \sum_x x^k P(X = x) \\
&= \sum_{n=r}^{\infty} n^k \binom{n-1}{r-1} p^r (1-p)^{n-r} \\
&= r \sum_{n=r}^{\infty} n^{k-1} \binom{n}{r} p^r (1-p)^{n-r} && \text{use } r \binom{n}{r} = n \binom{n-1}{r-1} \\
&= \frac{r}{p} \sum_{n=r}^{\infty} n^{k-1} \binom{n}{r} p^{r+1} (1-p)^{n-r} \\
&= \frac{r}{p} \sum_{m=r+1}^{\infty} (m-1)^{k-1} \binom{m-1}{r} p^{r+1} (1-p)^{m-r-1} && \text{let } m = n + 1 \\
&= \frac{r}{p} \cdot \mathbb{E}[(Y-1)^{k-1}] && \text{based on pmf of the distribution where } Y \sim \text{NegBin}(r+1, p).
\end{aligned}$$

Hence, we have

$$\mathbb{E}(X) = \frac{r}{p} \cdot \mathbb{E}(Y-1)^0 = \frac{r}{p}.$$

We can also compute

$$\mathbb{E}(X^2) = \frac{r}{p} \mathbb{E}(Y-1) = \frac{r}{p} (\mathbb{E}(Y) - 1) = \frac{r}{p} \left(\frac{r+1}{p} - 1 \right).$$

Hence,

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{r(1-p)}{p^2}.$$

This method is a more combinatorial approach and allows us to calculate $\mathbb{E}(X^k)$ for all k in a recursive manner. $\mathbb{E}(X^k)$ is the k -th moment of the negative binomial distribution (r, p) . The moments of a distribution essentially define the distribution itself.

Method 2. This will be a calculus-based approach where we need to analyse the convergence of infinite series under differentiation.

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{n=r}^{\infty} n \binom{n-1}{r-1} p^r (1-p)^{n-r} \\
&= \sum_{n=r}^{\infty} n \binom{n-1}{n-r} p^r (1-p)^{n-r} && \text{by symmetry} \\
&= \sum_{j=0}^{\infty} (j+r) \binom{r+j-1}{j} p^r (1-p)^j && \text{let } j = n - r \\
&= r p^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} (1-p)^j + p^r \sum_{j=0}^{\infty} \binom{r+j-1}{j} j (1-p)^j \\
&= r + p^r \sum_{j=0}^{\infty} \binom{r+j-1}{j} j (1-p)^j && \text{from (a): } p^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} (1-p)^j = 1 \\
&= r + p^r \sum_{j=0}^{\infty} \binom{-r}{j} j (p-1)^j && \text{from (a)} \\
&= r + p^r (p-1) \sum_{j=0}^{\infty} \binom{-r}{j} j (p-1)^{j-1} \\
&= r + p^r (p-1) \sum_{j=0}^{\infty} \binom{-r}{j} \frac{d}{dp} (p-1)^j \\
&= r + p^r (p-1) \frac{d}{dp} \left[\sum_{j=0}^{\infty} \binom{-r}{j} (p-1)^j \right] && \text{interchanging summation and differentiation} \\
&= r + p^r (p-1) \frac{d}{dp} (p^{-r}) && \text{from (a)} \\
&= r - \frac{r p^r (p-1)}{p^{r+1}} = \frac{r}{p}.
\end{aligned}$$

We can compute $\mathbb{E}(X^2)$ in a similar manner.

$$\begin{aligned}
\mathbb{E}(X^2) &= \sum_{n=r}^{\infty} n^2 \binom{n-1}{r-1} p^r (1-p)^{n-r} \\
&= \sum_{j=0}^{\infty} (j+r)^2 \binom{r+j-1}{j} p^r (1-p)^j \\
&= \underbrace{r^2 p^r \sum_{j=0}^{\infty} \binom{r+j-1}{j} (1-p)^j}_{=1, \text{ from (a)}} + \underbrace{2r p^r \sum_{j=0}^{\infty} j \binom{r+j-1}{j} (1-p)^j}_{=\frac{r}{p} - r, \text{ from } \mathbb{E}(X)} + p^r \sum_{j=0}^{\infty} j^2 \binom{r+j-1}{j} (1-p)^j \\
&= r^2 + 2r \left(\frac{r}{p} - r \right) + p^r \sum_{j=0}^{\infty} j(j-1) \binom{r+j-1}{j} (1-p)^j + \underbrace{p^r \sum_{j=0}^{\infty} j \binom{r+j-1}{j} (1-p)^j}_{=\frac{r}{p} - r} && \text{use } j^2 = j(j-1) + j \\
&= r^2 + (2r+1) \left(\frac{r}{p} - r \right) + p^r (1-p)^2 \sum_{j=0}^{\infty} \binom{-r}{j} j(j-1) (p-1)^{j-2} \\
&= r^2 + (2r+1) \left(\frac{r}{p} - r \right) + p^r (1-p)^2 \sum_{j=0}^{\infty} \binom{-r}{j} \frac{d^2}{dp^2} (p-1)^j \\
&= r^2 + (2r+1) \left(\frac{r}{p} - r \right) + p^r (1-p)^2 \frac{d^2}{dp^2} \left[\sum_{j=0}^{\infty} \binom{-r}{j} (p-1)^j \right]
\end{aligned}$$

Then, we have

$$\begin{aligned}\mathbb{E}(X^2) &= r^2 + (2r+1) \left(\frac{r}{p} - r \right) + p^r(1-p)^2 \frac{d^2}{dp^2}(p^{-r}) \\ &= r^2 + (2r+1) \left(\frac{r}{p} - r \right) + \frac{(1-p)^2 r(r+1)}{p^2}\end{aligned}$$

Thus,

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{r^2 p^2 + pr(2r+1)(1-p) + r(r+1)(1-p)^2}{p^2} - \frac{r^2}{p^2} = \frac{r(1-p)}{p^2}.$$

Extra information on calculus: In both the computations of $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$, we interchange differentiation and summation. This always works for a finite series, since differentiation is linear. However, it is not always the case for infinite series. Given a power series centered at $x = a$,

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad \text{for } |x-a| < R.$$

R is the radius of convergence. f is differentiable on the open interval of convergence $(a-R, a+R)$ and we can interchange summation and differentiation to do term-by-term differentiation, i.e.,

$$f'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n = \sum_{n=1}^{\infty} n c_n(x-a)^{n-1}.$$

f' also converges on $(a-R, a+R)$. In our example in $\mathbb{E}(X)$, we have to check the interval of convergence of $\sum_{j=0}^{\infty} \binom{-r}{j} (p-1)^j$.

We can use the ratio test.

$$\begin{aligned}L &= \lim_{n \rightarrow \infty} \left| \frac{\binom{-r}{n+1} (p-1)^{n+1}}{\binom{-r}{n} (p-1)^n} \right| \\ &= |p-1| \lim_{n \rightarrow \infty} \left| \frac{(-r)(-r-1) \dots (-r-n)}{(-r)(-r-1) \dots (-r-n+1)} \cdot \frac{n!}{(n+1)!} \right| \\ &= |p-1| \lim_{n \rightarrow \infty} \frac{n+r}{n+1} = |p-1|.\end{aligned}$$

By ratio test, the power series converges if and only if $|p-1| < 1$. From our assumption of the negative binomial distribution that $0 < p < 1$, this falls within the interval of convergence, hence the power series is always differentiable for $0 < p < 1$ and we are able to interchange differentiation and summation.

Method 3. Suppose you already know $\mathbb{E}(X) = r/p$, this gives an easier way to compute $\mathbb{E}(X^2)$. The same method has been applied to find the variance of geometric distribution.

$$\begin{aligned}
\mathbb{E}(X(X+1)) &= \sum_{n=r}^{\infty} n(n+1) \binom{n-1}{r-1} p^r (1-p)^{n-r} \\
&= \sum_{n=r}^{\infty} n(n+1) \frac{(n-1)!}{(r-1)!(n-r)!} p^r (1-p)^{n-r} \\
&= \sum_{n=r}^{\infty} \frac{(n+1)!}{(r-1)!(n-r)!} p^r (1-p)^{n-r} \\
&= \sum_{n=r}^{\infty} \frac{(n+1)!}{(r+1)!(n-r)!} \cdot r(r+1) \cdot p^r (1-p)^{n-r} \\
&= r(r+1) \sum_{n=r}^{\infty} \binom{n+1}{r+1} p^r (1-p)^{n-r} \\
&= \frac{r(r+1)}{p^2} \sum_{n=r}^{\infty} \binom{n+1}{r+1} p^{r+2} (1-p)^{n-r} \\
&= \frac{r(r+1)}{p^2} \sum_{n=r+2}^{\infty} \binom{n-1}{r+1} p^{r+2} (1-p)^{n-r-2} \\
&= \frac{r(r+1)}{p^2} \underbrace{\sum_{n=r+2}^{\infty} P(Y=n)}_{=1} \quad \text{where } Y \sim \text{NegBin}(r+2, p) \\
&= \frac{r(r+1)}{p^2}
\end{aligned}$$

. Hence,

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = (\mathbb{E}(X(X+1)) - \mathbb{E}(X)) - \mathbb{E}(X)^2 = \frac{r(1-p)}{p^2}.$$