Recap for Tutorial 1

MH1101

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1 Antiderivatives

Definition 1. Suppose f is a real-valued function on $\mathbb{R}$. A function $F(x)$ is an antiderivative of $f(x)$ on an interval (a, b) if

$$F'(x) = f(x) \quad \forall x \in (a, b).$$

Theorem 1. Suppose f is a real-valued function on $\mathbb{R}$. All antiderivatives of f differ by a constant.

Remark. Antiderivatives are not unique. The general antiderivate of f is the indefinite integral of f and is denoted by

$$\int f(x)\ \mathrm{d}x = F(x) + C,$$

where $F(x)$ is the antiderivative of $f(x)$ and C is an arbitrary constant.

2 Definite Integrals

Suppose $f : [a, b] \to \mathbb{R}$ is a function and we divide the interval $[a, b]$ into n sub-intervals of equal width $\Delta x = \frac{b-a}{n}$,

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n] \quad (x_0 = a, x_n = b).$$

Let $x_i^* \in [x_{i-1}, x_i]$ be sample points and then we have

$$\int_a^b f(x)\ \mathrm{d}x := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided the limit above exists and gives the same value for all sample points chosen.

When we only consider a finite number of terms in the summation, it gives an approximation to the definite integral. If each sample point are chosen as the midpoint of the sub-interval, this gives rise to the Midpoint Rule. We will revisit these ideas in Chapter 3.

Definition 2. The average value of f on $[a, b]$ is defined as

$$f_{ave} := \frac{1}{b-a} \int_a^b f(x)\ \mathrm{d}x.$$

Theorem 2 (Mean Value Theorem for Integrals)**.** If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\ \mathrm{d}x.$$

2.1 Properties of Definite Integral

- $$\int_a^b 1\ \mathrm{d}x = b - a$$

- $$\int_a^b (f(x) \pm g(x))\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x \pm \int_a^b g(x)\ \mathrm{d}x$$

- $$\int_a^b cf(x)\ \mathrm{d}x = c\int_a^b f(x)\ \mathrm{d}x$$

- $$\int_a^b f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x \qquad (c \text{ need not be in between } a \text{ and } b).$$

- If $f(x) \geq 0$ on $[a, b]$, then
$$\int_a^b f(x)\ \mathrm{d}x \geq 0.$$

- If $f(x) \geq g(x)$ on $[a, b]$, then
$$\int_a^b f(x)\ \mathrm{d}x \geq \int_a^b g(x)\ \mathrm{d}x.$$

- If $m \leq f(x) \leq M$ on $[a, b]$, then
$$m(b-a) \leq \int_a^b f(x)\ \mathrm{d}x \leq M(b-a).$$

3 Extra Exercises

Problem 1. Some questions from Calculus 1.

(a) $\displaystyle\lim_{x\to 1^+} \frac{(x^2-1)e^x}{\sqrt{x}-1}$

(b) $\displaystyle\lim_{x\to 0} \frac{\sin x}{\sqrt{x+20}+\sin\frac{1}{x}}$

(c) $\displaystyle\lim_{x\to 0^+} x^{x^2}$

Problem 2. Use Riemann sums to show that if $f(x) \leq 0$ and continuous on $[a, b]$, then
$$\int_a^b f(x)\ \mathrm{d}x \leq 0.$$

Problem 3. Write the following as an integral,
$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n}\right).$$

Problem 4. Write the following as an integral,
$$\lim_{n\to\infty}\frac{\sqrt[3]{n+1}+\sqrt[3]{n+2}+\ldots+\sqrt[3]{2n}}{n^{4/3}}.$$

Problem 5. Write the following as an integral,
$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\ldots+\frac{1}{\sqrt{n+n}}\right).$$