Extra Exercises Week 9.

Problem 1. Suppose X is a random variable which is uniformly distributed in $(-\pi/2, \pi/2)$. Compute the PDF of $Y = \tan X$.

$$f_X(x) = \begin{cases} \frac{1}{\pi} & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ 0 & \text{else.} \end{cases}$$

$Y = \tan X$

CDF: $F_Y(y) = P(Y \le y) = P(\tan X \le y) = P(X \le \tan^{-1} y) = \int_{-\frac{\pi}{2}}^{\tan^{-1} y} \frac{1}{\pi}\, dx$

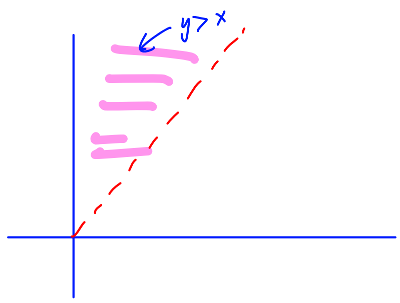
$$= \frac{1}{\pi}\left(\tan^{-1} y + \frac{\pi}{2}\right)$$

PDF: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{\pi} \frac{d}{dy}(\tan^{-1} y) = \frac{1}{\pi(1+y^2)} \qquad -\infty < y < \infty$

Problem 3. The joint probability density function of X and Y is given by

$$f(x, y) = e^{-(x+y)}, \quad 0 \le x < \infty, \quad 0 \le y < \infty.$$

Find $P(X < Y)$ and $P(X < a)$, for $a > 0$.



$$P(X<Y) = \int_0^\infty \int_x^\infty f(x,y)\, dy\, dx$$

$$= \int_0^\infty e^{-x} \int_x^\infty e^{-y}\, dy\, dx$$

$$= \int_0^\infty e^{-x} \left[-e^{-y}\right]_x^\infty dx = \int_0^\infty e^{-x} \cdot e^{-x}\, dx = \left[-\frac{1}{2} e^{-2x}\right]_0^\infty = \frac{1}{2}$$

$$P(X<a) = \int_0^a f_X(x)\, dx = \int_0^a \left[\int_0^\infty f(x,y)\, dy\right] dx$$

$$= \int_0^a e^{-x} \int_0^\infty e^{-y}\, dy\, dx$$

$$= \int_0^a e^{-x}\, dx = 1 - e^{-a}$$

Problem 4. Consider a sample of size 5 from a uniform distribution over $(0, 1)$. Let $X_{(3)}$ be the random variable referring to the median of the five samples. We call this the third order statistics, as the median is the third sample if we arrange the five samples in order. Given that the density function of $X_{(3)}$ is

$$f_{X_{(3)}}(x) = \frac{5!}{2!2!}(1 - F_X(x))^2(F_X(x))^2 f_X(x),$$

where $X \sim \text{Unif}(0, 1)$, F_X and f_X are the CDF and PDF of X respectively. Try to understand and make sense of the PDF above. Hence, find $P\left(\frac{1}{4} \leq X_{(3)} \leq \frac{3}{4}\right)$.

$$f_X(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{else} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & \text{else.} \end{cases}$$

$$f_{X_{(3)}}(x) = 30(1-x)^2 x^2 \qquad 0 \leq x \leq 1$$

$$P\left(\frac{1}{4} \leq X_{(3)} \leq \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} 30x^2(1-x)^2\,dx = 30\int_{\frac{1}{4}}^{\frac{3}{4}} x^4 - 2x^3 + x^2\,dx$$

$$= \underline{\qquad\qquad}$$

Problem 5. The joint probability density function, $f(x, y)$, of X and Y is given by

$$f(x,y) = \frac{Ke^{-y}}{y} \exp\left(-\frac{(x-y)^2}{y}\right), \quad -\infty < x < \infty, \quad y > 0,$$

where $K > 0$ is a real constant. It is known that $\int_0^\infty t^{-1/2} e^{-t} dt = \sqrt{\pi}$.

(a) Find K.

(b) Find $f_Y(y)$, the marginal probability density function of Y.

(a) $$\int_0^\infty \int_{-\infty}^\infty \frac{Ke^{-y}}{y} \exp\left(-\frac{(x-y)^2}{y}\right) dx\, dy = 1$$

$$\int_0^\infty \frac{e^{-y}}{y} \int_{-\infty}^\infty \exp\left(-\frac{(x-y)^2}{y}\right) dx\, dy = \frac{1}{K}$$

Let $\frac{x-y}{\sqrt{y}} = u \Rightarrow \frac{1}{\sqrt{y}} dx = du$

$$\int_0^\infty \frac{e^{-y}}{y} \int_{-\infty}^\infty \exp(-u^2) \sqrt{y}\, du\, dy = \frac{1}{K}$$

$$\int_0^\infty y^{-\frac{1}{2}} e^{-y} \left(\int_{-\infty}^\infty e^{-u^2} du\right) dy = \frac{1}{K}$$

$$\sqrt{\pi} \int_0^\infty y^{-\frac{1}{2}} e^{-y}\, dy = \frac{1}{K}$$

$$\sqrt{\pi}\sqrt{\pi} = \frac{1}{K} \implies K = \frac{1}{\pi}$$

(b) $$f_Y(y) = \int_{-\infty}^\infty f(x,y)\, dx$$

$$= \frac{1}{\pi} \cdot \frac{e^{-y}}{y} \int_{-\infty}^\infty \exp\left(-\frac{(x-y)^2}{y}\right) dx$$

Let $u = \frac{x-y}{\sqrt{y}}$

$$= \frac{1}{\pi} \cdot \frac{e^{-y}}{y} \int_{-\infty}^\infty \exp(-u^2) \sqrt{y}\, du$$

$$= \frac{1}{\pi} \cdot \frac{e^{-y}}{\sqrt{y}} \cdot \sqrt{\pi} = \frac{e^{-y}}{\sqrt{\pi y}} \qquad y > 0$$

Problem 6. The joint probability density function, $f(x, y)$, of X and Y is given by

$$f(x,y) = cxy^2, \quad x, y > 0, \quad x + y < 1,$$

where $c > 0$ is a real constant.

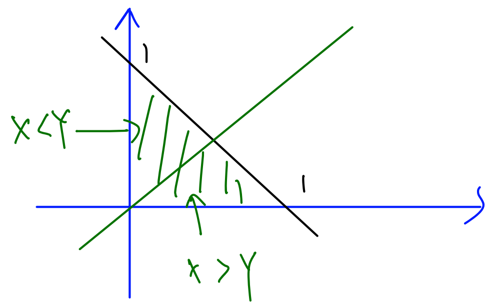
(a) Find c.

(b) Find $P(X > Y)$.

(c) Find $f_Y(y)$, the marginal probability density function of Y.

(d) Find the conditional probability density function of X, given that $Y = y$, where $0 < y < 1$.

(a) $\int_0^1 \int_0^{1-x} cxy^2 \, dy \, dx = 1$



$\frac{1}{c} = \int_0^1 x \left[\frac{1}{3} y^3\right]_0^{1-x} dx$

$\frac{1}{c} = \frac{1}{3}\int_0^1 x(1-x)^3 \, dx$

$\frac{3}{c} = \int_0^1 x^3(1-x)\, dx = \frac{1}{4} - \frac{1}{5} \quad \Rightarrow \quad c = 60$

(b) $P(X>Y) = \int_0^{\frac{1}{2}} \int_y^{1-y} cxy^2 \, dx \, dy = 60 \int_0^{\frac{1}{2}} y^2 \left[\frac{1}{2}x^2\right]_y^{1-y} dx$

$= 30 \int_0^{\frac{1}{2}} y^2\left[(1-y)^2 - y^2\right] dy = 30 \int_0^{\frac{1}{2}} y^2(1-2y)\, dy$

$= 30\left[\frac{1}{3}\cdot\frac{1}{8} - \frac{1}{2}\cdot\frac{1}{16}\right] = \frac{5}{16}$

(c) $f_Y(y) = \int_0^{1-y} cxy^2 \, dx = 60y^2\left[\frac{1}{2}x^2\right]_0^{1-y} = 30y^2(1-y)^2$

$0 < y < 1$

(d) $f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{60xy^2}{30y^2(1-y)^2} = \frac{2x}{(1-y)^2} \quad 0 < x < 1-y$

Problem 7. The joint probability density function, $f(x, y)$, of X and Y is given by

$$f(x,y) = C(y-x)e^{-y}, \quad -y < x < y, \quad 0 < y < \infty.$$

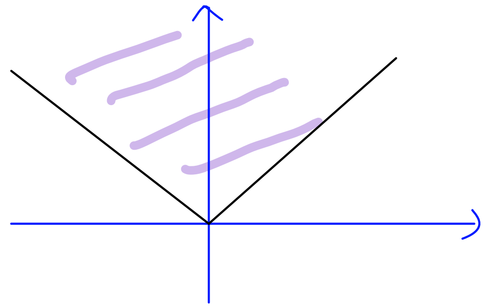
(a) Find C.

(b) Find $f_X(x)$, the marginal probability density function of X.

(c) Find $f_Y(y)$, the marginal probability density function of Y.

(d) Find $\mathbb{E}(X)$.

(e) Find $\mathbb{E}(Y)$.



(a) $$\int_0^\infty \int_{-y}^{y} C(y-x)e^{-y}\,dx\,dy = 1$$

$$\int_0^\infty \left[xye^{-y} - \frac{1}{2}x^2e^{-y}\right]_{x=-y}^{x=y} dy = \frac{1}{C}$$

$$\int_0^\infty \frac{1}{2}y^2e^{-y} + y^2e^{-y} + \frac{1}{2}y^2e^{-y}\,dy = \frac{1}{C}$$

$$\frac{1}{2C} = \int_0^\infty y^2e^{-y}\,dy = 2 \Rightarrow C = \frac{1}{4}$$

(b) When $x > 0$,

$$f_X(x) = \int_x^\infty C(y-x)e^{-y}\,dy = \frac{1}{4}\int_x^\infty ye^{-y}\,dy - \frac{1}{4}x\int_x^\infty e^{-y}\,dy$$

$$= \frac{1}{4}\left[-ye^{-y}\right]_x^\infty + \frac{1}{4}\int_x^\infty e^{-y}\,dy + \frac{1}{4}x\left[e^{-y}\right]_x^\infty$$

$$= \frac{1}{4}xe^{-x} + \frac{1}{4}e^{-x} - \frac{1}{4}xe^{-x} = \frac{1}{4}e^{-x}$$

When $x < 0$

$$f_X(x) = \int_{-x}^\infty C(y-x)e^{-y}\,dy = \frac{1}{4}\left[-ye^{-y}\right]_{-x}^\infty + \frac{1}{4}\int_{-x}^\infty e^{-y}\,dy + \frac{1}{4}x\left[e^{-y}\right]_{-x}^\infty$$

$$= -\frac{1}{4}xe^{x} + \frac{1}{4}\left[-e^{-y}\right]_{-x}^\infty + \frac{1}{4}x(0-e^{x})$$

$$= \frac{1}{4}e^{x} - \frac{1}{2}xe^{x}$$

(c)

$$f_Y(y) = \int_{-y}^{y} c(y-x)e^{-y}\,dx = \frac{1}{4}\int_{-y}^{y} ye^{-y}\,dx - \frac{1}{4}e^{-y}\underbrace{\int_{-y}^{y} x\,dx}_{=0}$$

$$= \frac{1}{4}\cdot 2y^2 e^{-y}$$

$$= \frac{1}{2}y^2 e^{-y} \qquad y>0.$$

(d)

$$\mathbb{E}(X) = \int_0^\infty x\cdot\frac{1}{4}e^{-x}\,dx + \int_{-\infty}^{0} x\left(\frac{1}{4}e^{x} - \frac{1}{2}xe^{x}\right)dx$$

$$= \frac{1}{4} + \int_{\infty}^{0} -x\left(\frac{1}{4}e^{-x} + \frac{1}{2}xe^{-x}\right)(-dx)$$

$$= \frac{1}{4} - \frac{1}{4}\int_0^\infty xe^{-x}\,dx - \frac{1}{2}\int_0^\infty x^2 e^{-x}\,dx = -1$$

(e)

$$\mathbb{E}(Y) = \int_0^\infty y\cdot\frac{1}{2}y^2 e^{-y} = \frac{1}{2}\cdot\int_0^\infty y^3 e^{-y} = 3$$