Central Moment for Normal Distribution

MH3500

March 25, 2024

In Problem 4 of Tutorial 8, we encounter the expression $E((X-\mu)^4)$, where $X \sim N(\mu, \theta)$. Now, let us show that
$$E((X-\mu)^4) = 3\theta^2.$$

Proof. Let $Y := X - \mu$, then $Y \sim N(0, \theta)$. We need to compute $E(Y^4)$. The probability density function of Y is
$$f_Y(y) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{y^2}{2\theta}\right).$$

$$\begin{aligned}
E(Y^4) &= \int_{-\infty}^{\infty} y^4 f_Y(y)dy = \frac{1}{\sqrt{2\pi\theta}} \int_{-\infty}^{\infty} y^4 \exp\left(-\frac{y^2}{2\theta}\right) dy \\
&= \frac{1}{\sqrt{2\pi\theta}} \int_{-\infty}^{\infty} y^3 \cdot y \exp\left(-\frac{y^2}{2\theta}\right) dy \\
&= \frac{1}{\sqrt{2\pi\theta}} \underbrace{\left[y^3 \cdot \left(-\theta \exp\left(-\frac{y^2}{2\theta}\right)\right)\right]_{-\infty}^{\infty}}_{=0} + \frac{1}{\sqrt{2\pi\theta}} \int_{-\infty}^{\infty} 3y^2\theta \exp\left(-\frac{y^2}{2\theta}\right) dy \\
&= 3\theta \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{y^2}{2\theta}\right) dy \\
&= 3\theta \int_{-\infty}^{\infty} y^2 f_Y(y) dy \\
&= 3\theta E(Y^2) \\
&= 3\theta^2,
\end{aligned}$$

since $E(Y^2) = Var(Y) + E(Y)^2 = \theta$. □

Extra information: We can derive a general form for $E(Y^n)$ for $n = 0, 1, 2, \ldots$, where $Y \sim N(0, \theta)$.
$$E(Y^n) = \frac{1}{\sqrt{2\pi\theta}} \int_{-\infty}^{\infty} y^n \exp\left(-\frac{y^2}{2\theta}\right) dy.$$

Let $y = \sqrt{\theta}u$,
$$E(Y^n) = \frac{\theta^{\frac{n}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^n \exp\left(-\frac{u^2}{2}\right) du = 0, \qquad \text{if } n \text{ is odd.}$$

This holds since the one-sided improper integral converges and the integrand is odd. If n is even, then the integrand is an even function, we let $u = \sqrt{2t}$
$$\begin{aligned}
E(Y^n) &= \frac{2\theta^{\frac{n}{2}}}{\sqrt{2\pi}} \int_0^{\infty} (2t)^{n/2} \exp(-t) \cdot \frac{1}{\sqrt{2t}} dt \\
&= \frac{\theta^{\frac{n}{2}}}{\sqrt{\pi}} 2^{\frac{n}{2}} \int_0^{\infty} t^{\frac{n-1}{2}} e^{-t} dt \\
&= \frac{\theta^{\frac{n}{2}}}{\sqrt{\pi}} 2^{\frac{n}{2}} \Gamma\left(\frac{n+1}{2}\right).
\end{aligned}$$

Since n is even, this gamma function is not straightforward to compute. We use the property that
$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

Let $z = \frac{n+1}{2}$, we have

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}+1\right) = 2^{-n}\sqrt{\pi}\Gamma(n+1).$$

Hence, for even n,

$$\begin{aligned} E(Y^n) &= \frac{\theta^{\frac{n}{2}}}{\sqrt{\pi}} 2^{\frac{n}{2}} \frac{2^{-n}\sqrt{\pi}\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+1\right)} \\ &= \theta^{\frac{n}{2}} \frac{n!}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!} \\ &= \theta^{\frac{n}{2}} \frac{(n-1)!}{2^{\frac{n}{2}-1}\left(\frac{n}{2}-1\right)!} \\ &= \theta^{\frac{n}{2}} (n-1)\cdot(n-3)\ldots 3\cdot 1. \end{aligned}$$

The final step can be shown via induction.