MH2500 AY23/24

Solution 1. Given that

$$P(X = n) = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}.$$

Then,

$$\begin{aligned}
P(X \geq 1 | X \leq 4) &= \frac{P(X \geq 1) \cap P(X \leq 4)}{P(X \leq 4)} \\
&= \frac{P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)}{P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)} \\
&= \frac{\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{5} - \frac{1}{6}\right)}{\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{5} - \frac{1}{6}\right)} \\
&= \left(\frac{1}{2} - \frac{1}{6}\right) \div \left(\frac{1}{1} - \frac{1}{6}\right) = \frac{2}{5}
\end{aligned}$$

Solution 2.

(a) $P(X \geq Y) = 1$, otherwise the PDF $= 0$.

(b) The marginal PDF of Y is

$$g(y) = f_Y(y) = \int_y^{\sqrt{y}} 15y dx = 15y(\sqrt{y} - y), \qquad 0 \leq y \leq 1.$$

(c) The marginal PDF of X is

$$f_X(x) = \int_{x^2}^{x} 15y dy = \frac{15}{2}(x^2 - x^4), \qquad 0 \leq x \leq 1.$$

Clearly, $f(x, y) \neq f_X(x) f_Y(y)$. Hence X and Y are not independent.

Solution 3.

(a)

$$\begin{aligned}
\mathbb{E}(X^n) &= \sum_{k=0}^{\infty} k^n P(X = k) = \sum_{k=0}^{\infty} k^n e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k^{n-1} e^{-\lambda} \frac{\lambda^k}{(k-1)!} = \sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda} \frac{\lambda^{j+1}}{j!} \\
&= \lambda \sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda} \frac{\lambda^j}{j!} = \lambda \sum_{j=0}^{\infty} (j+1)^{n-1} P(X = j) = \lambda \mathbb{E}((X+1)^{n-1}).
\end{aligned}$$

(b)

$$\mathbb{E}(X^3) = \lambda \mathbb{E}((X+1)^2)) = \lambda \mathbb{E}(X^2) + 2\lambda \mathbb{E}(X) + \lambda = \lambda^2 \mathbb{E}(X+1) + 2\lambda^2 + \lambda = \lambda^3 + 3\lambda^2 + \lambda.$$

Solution 4.

(a) We first claim that if $X \sim$N$(0,1)$, then so is $-X$. Let $W=-X$. By substitution,

$$f_W(w)=\frac{d}{dw}P(W\le w)=\frac{d}{dw}\int_{-w}^{\infty}\frac{1}{2\pi}\exp\left(-\frac{x^2}{2}\right)dx=\frac{d}{dw}\int_{-\infty}^{w}\frac{1}{2\pi}\exp\left(-\frac{x^2}{2}\right)dx=\frac{1}{2\pi}\exp\left(-\frac{w^2}{2}\right).$$

Let (a,b) be any arbitrary interval on $\mathbb{R}$.

$$P(a<Y<b|I=1)=P(a<X<b)=P(a<-X<b)=P(a<Y<b|I=0).$$

Therefore, I and Y are independent.

(b)

$$\begin{aligned}P(Y\le y)&=P(Y\le y|I=0)P(I=0)+P(Y\le y|I=1)P(I=1)=\frac{1}{2}(P(-X\le y)+P(X\le y))\\&=\frac{1}{2}(P(X\le y)+P(X\le y))=\int_{-\infty}^{y}\frac{1}{2\pi}\exp\left(-\frac{x^2}{2}\right)dx.\end{aligned}$$

Therefore, the PDF of Y follows the standard normal distribution. By the fundamental theorem of calculus,

$$f_Y(y)=\frac{d}{dy}P(Y\le y)=\frac{1}{2\pi}\exp\left(-\frac{y^2}{2}\right).$$

(c)

$$\begin{aligned}Cov(X,Y)&=\mathbb{E}(XY)-\mathbb{E}(X)\mathbb{E}(Y)=\mathbb{E}(XY|I=0)P(I=0)+\mathbb{E}(XY|I=1)P(I=1)-0\\&=\frac{1}{2}(\mathbb{E}(X^2)+\mathbb{E}(-X^2))=0.\end{aligned}$$

Solution 5.

(a) Let X be the length of time until he reaches safety. Let D_i be the event that the i-th door is chosen.

$$\mathbb{E}(X)=\sum_{i=1}^{3}\mathbb{E}(X|D_i)P(D_i)=\frac{1}{3}(\mathbb{E}(X|D_1)+\mathbb{E}(X|D_2)+\mathbb{E}(X|D_3))=\frac{1}{3}(3+(5+\mathbb{E}(X))+(7+\mathbb{E}(X))$$

$$\mathbb{E}(X)=5+\frac{2}{3}\mathbb{E}(X)\implies\mathbb{E}(X)=15\text{ hours}.$$

(b)

$$\mathbb{E}(X^2|D_1)=9,\qquad\mathbb{E}(X^2|D_2)=\mathbb{E}((X+5)^2),\qquad\mathbb{E}(X^2|D_3)=\mathbb{E}((X+7)^2)$$

$$\mathbb{E}(X^2)=\sum_{i=1}^{3}\mathbb{E}(X^2|D_i)P(D_i)=\frac{1}{3}(9+(\mathbb{E}(X^2)+10\mathbb{E}(X)+25)+(\mathbb{E}(X^2)+14\mathbb{E}(X)+49))=\frac{443}{3}+\frac{2}{3}\mathbb{E}(X^2).$$

$$\mathbb{E}(X^2)=443\implies Var(X)=\mathbb{E}(X^2)-\mathbb{E}(X)^2=218\text{ hours}^2.$$

Solution 6.

(a)

$$\mathbb{E}(X_i) = \sum_{k=1}^{6} kP(X_i = k) = \frac{7}{2}.$$

$$\mathbb{E}(X_i^2) = \sum_{k=1}^{6} k^2 P(X_i = k) = \frac{91}{6}.$$

$$Var(X_i) = \mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2 = \frac{35}{12}.$$

(b)

$$\mathbb{E}(\bar{X}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(X_i\right) = \frac{1}{n}\cdot n\cdot\frac{7}{2} = \frac{7}{2}.$$

$$Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}Var\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\left(\sum_{i=1}^{n} Var\left(X_i\right) + \sum_{i\neq j}\underbrace{Cov(X_i, X_j)}_{=0}\right)$$
$$= \frac{1}{n^2}\cdot n\cdot Var(X_1) = \frac{35}{12n}.$$

(ci) By Chebyshev's inequality,

$$P(|\bar{X} - 3.5| \geq 0.5) \leq \frac{Var(\bar{X})}{0.5^2} = \frac{7}{15}.$$

(cii) By central limit theorem, as n is large, $\bar{X} \sim \mathrm{N}\big(3.5, \frac{35}{12n}\big)$

$$P(|\bar{X} - 3.5| \geq 0.5) = P\left(|Z| \geq \frac{0.5}{\sqrt{\frac{35}{12n}}}\right) = 2 - 2\Phi\left(\sqrt{\frac{15}{7}}\right). \qquad \text{(use the CDF table to get the answer)}$$

Z is the standard normal random variable and $\Phi(\cdot)$ is the CDF of Z.