Extra Exercises for Week 7

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Most questions are taken from the textbook: A First Course in Probability (9th edition) by Sheldon Ross.

Problem 1. Balls numbered 1 through N are in an urn. Suppose that $n, n \leq N$, of them are randomly selected without replacement. Let Y denote the largest number selected.

(a) Find the probability mass function of Y.

(b) Derive an expression for $\mathbb{E}(Y)$. You can simplify it using the combinatorial identity

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}, \quad n \geq k.$$

Problem 2. An urn contains n balls numbered 1 through n. If you withdraw m balls randomly in sequence, each time replacing the ball selected previously, find $P(X = k), k = 1, \ldots, m$, where X is the maximum of the m chosen numbers. It is a very common technique to start with $P(X = k) = P(X \leq k) - P(X < k)$.

Problem 3. Show that if X is a geometric random variable with parameter p, then

$$\mathbb{E}\left(\frac{1}{X}\right) = \frac{-p \ln p}{1-p}$$

Hint: You may need to write $\frac{a^i}{i} = \int_0^a x^{i-1} dx$ and do term-by-term integration on an infinite series. Term-by-term differentiation/integration holds within the radius of convergence of the power series.

Problem 4. Each of the members of a 7-judge panel independently makes a correct decision with probability 0.7. If the panel's decision is made by majority rule, what is the probability that the panel makes the correct decision? Given that 4 of the judges agreed, what is the probability that the panel made the correct decision?

Problem 5. Suppose that the probability mass function, p_X , of the random variable X is of the form

$$p_X(x) = \begin{cases} \frac{C}{x(x-1)(x+1)}, & x = 2, 3, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find C.

(b) Compute $\mathbb{E}(X)$.

Problem 6. A and B play the following game: A writes down either number 1 or number 2, and B must guess which one. If the number that A has written down is i and B has guessed correctly, B receives i units from A. If B makes a wrong guess, B pays 3/4 units to A. Given that B randomizes his decision by guessing 1 with probability p and 2 with probability $1-p$. We denote B_k as the random variable that represents the gain by B when A wrote down the number k. Find $\mathbb{E}(B_1)$ and $\mathbb{E}(B_2)$. Then, find

$$\max_{p \in (0,1)} \min_{k=1,2} \mathbb{E}(B_k).$$

What is the corresponding value of p that maximises $\min_{k=1,2} \mathbb{E}(B_k)$? This is the value of p that maximises the minimum expected gain of player B.

Problem 7. Let X be a standard normal random variable with the probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad -\infty < x < \infty.$$

Let Φ denote the CDF of X, i.e. $\Phi(x) = P(X \leq x)$. For $x \geq 0$, prove the following.

(a) $\Phi(x) + \Phi(-x) = 1$.

(b) $P(|X| > x) = 2 - 2\Phi(x)$.

(c) $P(|X| < x) = 2\Phi(x) - 1$.

Problem 8. Let U be a uniform $(0, 1)$ random variable, and let $a < b$ be constants.

(a) Show that if $b > 0$, then bU is uniformly distributed on $(0, b)$, and if $b < 0$, then bU is uniformly distributed on $(b, 0)$.

(b) Show that $a + U$ is uniformly distributed on $(a, 1 + a)$.

(c) What function of U is uniformly distributed on (a, b)?

(d) Show that $\min(U, 1 - U)$ is a uniform $(0, \frac{1}{2})$ random variable.

(e) Show that $\max(U, 1 - U)$ is a uniform $(\frac{1}{2}, 1)$ random variable.

Problem 9. If X is uniformly distributed over $(-1, 1)$, find

(a) $P(|X| > \frac{1}{2})$

(b) the density function of the random variable $|X|$.

Answers (Let me know if there are any discrepancies):

1(a). $\dfrac{\binom{y-1}{n-1}}{\binom{N}{n}}$

1(b). $\dfrac{n(N+1)}{n+1}$

2. $\left(\dfrac{k}{n}\right)^m - \left(\dfrac{k-1}{n}\right)^m$

4. $\displaystyle\sum_{k=4}^{7} \binom{7}{k} 0.7^k 0.3^{7-k}$, 0.7

5(a). $C = 4$

5(b). 3

6. $\mathbb{E}(B_1) = \frac{7}{4}p - \frac{3}{4}$, $\mathbb{E}(B_2) = 2 - \frac{11}{4}p$, $\frac{23}{72}$, $p = \frac{11}{18}$

8(c). $a + (b - a)U$

9(a). $\frac{1}{2}$

9(b). $|X|$ follows a uniform distribution on $(0, 1)$. So, the density function is 1 on $(0, 1)$.