MH3500 Midterm Recap

MH3500

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1 Discrete Random Variables

(a) $X \sim$ Bernoulli(p).
- $P(X = x) = (1-p)^{1-x}p^x, \quad$ for $x = 0, 1$.
- $E(X) = p, \quad Var(X) = p(1-p), \quad M_X(t) = pe^t + 1 - p$

(b) $X \sim$ Binomial(n, p).
- $P(X = x) = \binom{n}{p}(1-p)^{1-x}p^x, \quad$ for $x = 0, 1, \ldots, n$.
- $E(X) = np, \quad Var(X) = np(1-p), \quad M_X(t) = (pe^t + 1 - p)^n$

(c) $X \sim$ Geometric(p).
- $P(X = x) = p(1-p)^{x-1}, \quad$ for $x = 1, 2, \ldots$.
- $E(X) = \frac{1}{p}, \quad Var(X) = \frac{1-p}{p^2}, \quad M_X(t) = \frac{pe^t}{1-(1-p)e^t}, \quad$ for $t < -\ln(1-p)$

(d) $X \sim$ Poisson(λ).
- $P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad$ for $x = 0, 1, \ldots$.
- $E(X) = \lambda, \quad Var(X) = \lambda, \quad M_X(t) = e^{\lambda(e^t - 1)}$

2 Continuous Random Variables

(a) $X \sim$ Uniform(a, b).
- $f_X(x) = \frac{1}{b-a}, \quad$ for $x \in [a, b]$.
- $E(X) = \frac{a+b}{2}, \quad Var(X) = \frac{(b-a)^2}{12}, \quad M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}, \quad$ for $t \neq 0$

(b) $X \sim$ N(μ, σ^2).
- $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad$ for $x \in \mathbb{R}$.
- $E(X) = \mu, \quad Var(X) = \sigma^2, \quad M_X(t) = \exp\left(t\mu + \frac{\sigma^2 t^2}{2}\right)$

(c) $X \sim$ Exp(λ).
- $f_X(x) = \lambda e^{-\lambda x}, \quad$ for $x > 0$.
- $E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}, \quad M_X(t) = \frac{\lambda}{\lambda - t}, \quad$ for $t < \lambda$

(d) $X \sim$ Gamma(α, β).
- $f_X(x) = \frac{x^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{x}{\beta}}, \quad$ for $x > 0$.
- $E(X) = \alpha\beta, \quad Var(X) = \alpha\beta^2, \quad M_X(t) = (1 - \beta t)^{-\alpha}, \quad$ for $t < \frac{1}{\beta}$

(e) Chi-squared distribution: If $X_1, \ldots, X_n \sim N(0, 1)$, then $X_1^2 + \ldots X_n^2 \sim \chi_n^2 \sim$ Gamma $\left(\frac{n}{2}, 2\right)$.

(f) t-distribution: If $X \sim N(0, 1)$ and $Y \sim \chi_k^2$, then $\frac{X}{\sqrt{Y/k}} \sim t_k$.

(g) F-distribution: If $U \sim \chi_m^2$ and $V \sim \chi_n^2$, then $\frac{U/m}{V/n} \sim F(m, n)$.

3 Moment Generating Functions

Let X be a random variable. The moment generating function (MGF) of X is $M_X(t) = E(e^{tX})$.

- If $M_X(t) = M_Y(t)$ for all $t \in [-a, a]$, then X and Y has the same distribution.
- If $Y = aX + b$ for $a, b \in \mathbb{R}$, $M_Y(t) = e^{tb} M_X(at)$.
- If X and Y are independent random variables, then $M_{X+Y}(t) = M_X(t) M_Y(t)$.

If $M_X(t)$ exists in an open interval containing zero, then we can generate the moments by differentiating the MGF. The n-th moment of X is

$$E(X^n) = \frac{d^n}{dt^n} M_X(t)|_{t=0}.$$

The n-th central moment is defined as $E((X - E(X))^n)$.

4 Sample Mean and Sample Variance

Let $X_1, \dots X_n$ be i.i.d. random variables with population mean $E(X_1) = \mu$ and population variance $Var(X_1) = \sigma^2$. We define the following statistics:

$$\text{Sample mean, } \bar{X} := \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \text{and} \qquad \text{Sample variance, } s^2 := \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

In general, $E(\bar{X}) = \mu$, $Var(\bar{X}) = \frac{\sigma^2}{n}$ and $E(s^2) = \sigma^2$.

Next, we restrict $X_i \sim N(\mu, \sigma^2)$, for all $i = 1, \dots, n$. Then,

$$\left[\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \implies \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0,1)\right] \qquad \text{and} \qquad \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}.$$

Suppose we want to approximate μ with a given set of n samples.

- If σ^2 is known, use $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$.
- If σ^2 is unknown, use s^2 to approximate σ^2 and for $n > 30$, use $\frac{\sqrt{n}(\bar{X}-\mu)}{s} \sim N(0,1)$.
- If σ^2 is unknown, use s^2 to approximate σ^2 and for $n \leq 30$, use $\frac{\sqrt{n}(\bar{X}-\mu)}{s} \sim t_{n-1}$.

5 Important Definitions and Theorems

Definition 1 (Convergence in Probability)**.** A sequence of random variables $\{X_n\}$ is said to converge in probability towards a random variable X if for all $\varepsilon > 0$

$$\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0.$$

Definition 2 (Convergence in Distribution)**.** A sequence of random variables $\{X_n\}$ is said to converge in distribution towards a random variable X if

$$\lim_{n \to \infty} |P(X_n \leq x) - P(X \leq x)| = 0,$$

for all $x \in \mathbb{R}$ at which $P(X \leq x)$ is continuous.

Theorem 1 (Lévy's Continuity Theorem)**.** Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables with corresponding MGF $M_n(t)$ and CDF F_n. Suppose there is a random variable X with MGF $M(t)$ and CDF F.
If for all $t \in \mathbb{R}$,

$$\lim_{n \to \infty} M_n(t) = M(t),$$

then

$$\lim_{n \to \infty} F_n(x) = F(x),$$

for $x \in \mathbb{R}$, where F is continuous. This means that $\{X_n\}$ converges in distribution to X.

Theorem 2 (Central Limit Theorem)**.** Let $X_1, \ldots, X_n$ be i.i.d. r.v. with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2 < \infty$. Let $Z \sim N(0,1)$ and write

$$Z_n = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma},$$

then Z_n converges in distribution to Z. Furthermore, the convergence is uniform, i.e.,

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |P(Z_n \leq x) - P(Z \leq x)| = 0.$$

Theorem 3 (Weak Law of Large Numbers)**.** Let $X_1, \ldots, X_n$ be i.i.d. r.v. with $E(X_1) = \mu$, then $\bar{X}$ converges to μ in probability, i.e., for all $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|\bar{X} - \mu| > \varepsilon) = 0.$$

Theorem 4 (Chebyshev's Inequality)**.** Let X be a random variable with mean μ and finite non-zero variance σ^2. For any real number $k > 0$,

$$P(|X - \mu| > k) \leq \frac{\sigma^2}{k^2}.$$

This is a special case of Markov's Inequality involving the second central moment of X.

6 Method of Moments Estimators

Let $X_1, \ldots, X_n$ be i.i.d. random variables of distribution $\mathscr{D}_\theta$, where θ is unknown. The m-th sample moment is denoted as

$$s_m = \frac{1}{m} \sum_{i=1}^{n} X_i^m.$$

The m-th moment is denoted as $\mu_m = E(X_1^m)$. It can be shown using the Weak LLN, that S_m converges in probability to μ_m. We aim to write $\theta = h(\mu_1, \ldots, \mu_m)$ for some function h. Then, the MME for θ is $\hat{\theta}_{MME} = h(s_1, \ldots, s_m)$.

7 Maximum Likelihood Estimators

Let $X_1, \ldots, X_n$ be i.i.d. random variables of distribution $\mathscr{D}_\theta$, where θ is unknown. Let $\mathbb{S} = (a, b)$ be the range of parameters of θ. Let $f(X_i|\theta)$ be the mass/density function for X_i. The likelihood function for $X_1, \ldots, X_n$ is

$$L(X_1, \ldots, X_n|\theta) = \prod_{i=1}^{n} f(X_i|\theta).$$

The MLE for θ is defined as

$$\hat{\theta}_{MLE} = \arg\max_{\theta \in \mathbb{S}} L(X_1, \ldots, X_n|\theta).$$

To ensure $\hat{\theta}$ exists, we need to verify the standard conditions for $L(\theta)$, for $\theta \in (a, b)$, i.e.,

- $L(\theta) > 0$, for all $\theta \in (a, b)$.
- $L'(\theta)$ exists for all $\theta \in (a, b)$.
- $\lim_{\theta \to a^+} L(\theta) = \lim_{\theta \to b^-} L(\theta) = 0$.

If the maximizer exists, we solve $\frac{d}{d\theta} L(\theta) = 0$ for θ. However, if this is complicated to solve, we can resolve to the log-likelihood function, that is to solve $\frac{d}{d\theta} \ln L(\theta) = 0$ for θ to obtain $\hat{\theta}_{MLE}$.

8 Bias, Variance, Consistency

Let $X_1, \ldots, X_n$ be i.i.d. random variables of distribution $\mathscr{D}_\theta$, where θ is unknown. Let $\hat{\theta}$ be an estimator for θ based on the n samples. We have the following quantities:

- $\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$.
- $\text{Var}(\hat{\theta}) = E((\hat{\theta} - E(\hat{\theta}))^2)$.
- Standard Error: $\text{SE}(\hat{\theta}) = \sqrt{Var(\hat{\theta})}$.

- Mean Squared Error: $\text{MSE}(\hat{\theta}) = E((\hat{\theta}-\theta)^2)$=$\text{Bias}(\hat{\theta})^2 + Var(\hat{\theta})$.

$\hat{\theta}$ can be computed explicitly based on a given set of observations. For this set of observations, the performance of $\hat{\theta}$ can be measured via the estimated standard error, $\widehat{SE}(\hat{\theta})$, the standard error evaluated based on observations.

Definition 3 (Consistency)**.** $\hat{\theta}$ is said to be a consistent estimator for θ if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|\hat{\theta}-\theta| > \varepsilon) = 0.$$

Theorem 5 (Sufficient Condition for Consistency)**.** Let $\hat{\theta}$ be an estimator for θ. If

$$\lim_{n\to\infty} \text{Bias}(\hat{\theta}) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \text{Var}(\hat{\theta}) = 0,$$

then $\hat{\theta}$ is a consistent estimator.

Theorem 6 (MME Consistency)**.** Let $\hat{\theta} = h(s_1, \ldots, s_m)$ be the method of moments estimator for θ. If h is continuous in all $s_1, \ldots, s_m$, then $\hat{\theta}$ is a consistent estimator.

Theorem 7 (MLE Consistency)**.** Let $\hat{\theta}$ be the maximum likelihood estimator for θ. Under appropriate regularity assumptions on $L(\theta)$, $\hat{\theta}$ is a consistent estimator.

9 Fisher Information

Let $X_1, \ldots, X_n$ be i.i.d. random variables of distribution $\mathscr{D}_\theta$. The likelihood function for $X_1, \ldots, X_n$ is

$$L(X_1, \ldots, X_n|\theta) = \prod_{i=1}^{n} f(X_i|\theta).$$

The Fisher information of $X_1, \ldots, X_n$ at θ_0 is

$$I_{X_1,\ldots,X_n}(\theta_0) = E\left[\left(\frac{d}{d\theta}\ln L(X_1, \ldots, X_n|\theta)|_{\theta=\theta_0}\right)^2\right].$$

If condition $(*)$ is satisfied, i.e.,

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial\theta} f(x|\theta)dx = 0 \quad \text{(for PDF)} \quad \text{or} \quad \sum_x \frac{\partial}{\partial\theta} f(x|\theta)dx = 0 \quad \text{(for PMF)},$$

then $I_{X_1,\ldots,X_n}(\theta) = nI_{X_1}(\theta)$.

Furthermore, if we have

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial\theta^2} f(x|\theta)dx = 0 \quad \text{(for PDF)} \quad \text{or} \quad \sum_x \frac{\partial^2}{\partial\theta^2} f(x|\theta)dx = 0 \quad \text{(for PMF)},$$

then we have the alternative formula for $I_{X_1}(\theta)$,

$$I_{X_1}(\theta) = E\left[\left(\frac{d}{d\theta}\ln L(X_1|\theta)\right)^2\right] = -E\left[\frac{d^2}{d\theta^2}\ln L(X_1|\theta)\right]$$

10 Cramer-Rao Lower Bound (CRLB)

Theorem 8. Let $(X_1, \ldots, X_n)$ be any random sample with a joint PDF $f(X_1, \ldots, X_n|\theta)$. Let $T = T(X_1, \ldots, X_n)$ be an estimator of θ and $E(T)$ is differentiable with respect to θ. Suppose $f(x_1, \ldots, x_n|\theta)$ satisfies

$$\frac{\partial}{\partial\theta}\int_{\mathbb{R}^n} h(x_1, \ldots, x_n) f(x_1, \ldots, x_n|\theta) dx_1 \ldots dx_n = \int_{\mathbb{R}^n} h(x_1, \ldots, x_n)\frac{\partial}{\partial\theta} f(x_1, \ldots, x_n|\theta) dx_1 \ldots dx_n,$$

for any function h with $E(|h(x)|) < \infty$. Then

$$Var(T) \geq \frac{\left(\frac{\partial}{\partial\theta}E(T)\right)^2}{E\left[\left(\frac{d}{d\theta}\ln L(X_1, \ldots, X_n|\theta)|\right)^2\right]} =: CRLB.$$

Corollary 1. Suppose $X_1, \ldots, X_n$ are i.i.d. random variables of distribution $\mathscr{D}_\theta$, with PDF $f(x|\theta)$. Suppose T is an unbiased estimator of θ, then condition $(*)$ holds by choosing $h(x_1, \ldots, x_n) = 1$ in the theorem above and we have

$$Var(T) \geq \frac{1}{nI_{X_1}(\theta)}.$$

Definition 4. An estimator T for θ is efficient if T is unbiased and $Var(T) = (nI_{X_1}(\theta))^{-1}$.

Theorem 9 (Asymptotic Normality of MLE)**.** Suppose $X_1, \ldots, X_n$ are i.i.d. random variables of distribution $\mathscr{D}_\theta$, with PDF or PMF $f(x|\theta)$, where the true parameter θ is unknown. Let $\hat{\theta}$ be the MLE for θ, under some regularity conditions, $\sqrt{nI_{X_1}(\theta)}(\hat{\theta} - \theta)$ converges in distribution to $N(0, 1)$.

The theorem above describes the asymptotic behaviour of MLE and $\frac{1}{nI_{X_1}(\theta))}$ is the asymptotic variance of $\hat{\theta}$.

To conclude, under some regularity conditions, MLE is

consistent, **asymptotically unbiased**, **asymptotically normal** and **asymptotically efficient**.

The last point is justified since the asymptotic variance is the CRLB.

11 Confidence Interval

Definition 5 (Confidence Interval)**.** Suppose $X_1, \ldots, X_n$ are i.i.d. random variables of distribution $\mathscr{D}_\theta$, where the true parameter θ is unknown. Suppose $L(X_1, \ldots, X_n)$ and $U(X_1, \ldots, X_n)$ are statistics such that

$$P(L \leq \theta \leq U) = 1 - \alpha,$$

then $(L(X_1, \ldots, X_n), U(X_1, \ldots, X_n))$ is an exact $100(1-\alpha)\%$ confidence interval for θ.

Definition 6 (Pivotal Quantity)**.** A random variable $Q(X_1, \ldots, X_n, \theta)$ is a pivotal quantity if the distribution of Q does not depend on the unknown parameter θ.

To verify that a random variable $Q(X_1, \ldots, X_n, \theta)$ is a pivotal quantity, check that the PDF/CDF/MGF of Q does not involve any unknown parameter θ.

Example 1. Suppose $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ are i.i.d. Let $\bar{X}$ and s^2 denote the sample mean and sample variance respectively.

- $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ is a pivotal quantity to construct the C.I. for σ.
- $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0, 1)$ is a pivotal quantity to construct the C.I. for μ, if σ is known.
- $\frac{\sqrt{n}(\bar{X}-\mu)}{s} \sim t_{n-1}$ is a pivotal quantity to construct the C.I. for μ, if σ is unknown.

Due to the asymptotical normality of MLE, i.e. $\sqrt{nI_{X_1}(\theta)}(\hat{\theta} - \theta)$ converges in distribution to $N(0, 1)$, this means that $\sqrt{nI_{X_1}(\theta)}(\hat{\theta} - \theta)$ serves as an asymptotical pivotal quantity to construct a confidence interval for θ. The behaviour is only significant for large n. If we define z_α as the number such that $P(Z > z_\alpha) = \alpha$, where $Z \sim N(0, 1)$, then the approximate $100(1-\alpha)\%$ C.I. for θ is

$$\left[\hat{\theta} - \frac{z_{\alpha/2}}{\sqrt{nI_{X_1}(\hat{\theta})}}, \hat{\theta} + \frac{z_{\alpha/2}}{\sqrt{nI_{X_1}(\hat{\theta})}}\right].$$

12 Other Useful Properties

- Suppose $X_1, \ldots X_n$ are i.i.d. random variables with a CDF F. Let $Y_1 = \min(X_1, \ldots, X_n)$ and $Y_2 = \max(X_1, \ldots, X_n)$, then
$$P(Y_1 \leq y) = 1 - (1 - F(y))^n \qquad P(Y_2 \leq y) = (F(y))^n.$$
- If X and Y are independent random variables, then $E(XY) = E(X)E(Y)$. The converse is **TRUE** if X, Y are normally distributed. Generally, the converse is **FALSE**.
- Bernoulli distribution is a special case of binomial distribution with $n = 1$.

- Suppose $X_i \sim \text{Binomial}(n_i, p)$, for $i = 1, \ldots, k$ are pairwise independent, then

$$X_1 + X_2 \ldots + X_k \sim \text{Binomial}(n_1 + \ldots + n_k, p).$$

- Suppose $X_i \sim \text{Poisson}(\lambda_i)$, for $i = 1, \ldots, k$ are pairwise independent, then

$$X_1 + X_2 \ldots + X_k \sim \text{Poisson}(\lambda_1 + \ldots + \lambda_k).$$

- Suppose $X_i \sim \text{Gamma}(\alpha_i, \beta)$, for $i = 1, \ldots, k$ are pairwise independent, then

$$X_1 + X_2 \ldots + X_k \sim \text{Gamma}(\alpha_1 + \ldots + \alpha_k, \beta).$$

- Suppose $X_i \sim \text{N}(\mu_i, \sigma_i^2)$, for $i = 1, \ldots, k$ are pairwise independent, then

$$\sum_{i=1}^{k} a_i X_i \sim \text{N}(\sum a_i \mu_i, \sum a_i^2 \sigma_i^2).$$

- Suppose $X \sim \chi^2_m$ and $Y \sim \chi^2_n$ are independent random variables, then

$$X + Y \sim \chi^2_{m+n}$$

- Gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^t dt.$$

$\Gamma(z+1) = z\Gamma(z) = z!$, for $z \in \mathbb{Z}^+$.

- Beta function:

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

- Relationship between Exp, Gamma and χ^2:

$$\text{Exp}\,(\lambda) \sim \text{Gamma}\left(1, \frac{1}{\lambda}\right), \qquad \chi^2_n \sim \text{Gamma}\left(\frac{n}{2}, 2\right).$$