MH3500 Statistics

Tutorial 6 AY2022/23 Semester 2

Problem 6.1

Let $X_1, \dots, X_n$ be an i.i.d. random sample drawn from $U(0, \theta)$ (uniform distribution on the interval$(0, \theta)$), where $\theta > 0$ is an unknown parameter. Find an unbiased estimator $\hat{\theta}_n$ for θ with $\lim_{n\to\infty} Var[\hat{\theta}_n] = 0$

Hint: Start with $\max\{X_1, \dots, X_n\}$, compute its expected value and then modify it to make it unbiased.

Let $\tilde{\theta} = \max\{X_1, \dots, X_n\}$ ← MLE for θ

$Bias(\tilde{\theta}) = E(\tilde{\theta}) - \theta$

$Var(\tilde{\theta}) = E(\tilde{\theta}^2) - E(\tilde{\theta})^2$

CDF for $\tilde{\theta}$

$$F_{\tilde{\theta}}(y) = P(\tilde{\theta} \le y) = P(\max\{X_1, \dots, X_n\} \le y)$$

$$= P(X_1 \le y, X_2 \le y, \dots, X_n \le y)$$

$$= P(X_1 \le y) P(X_2 \le y) \dots P(X_n \le y) \quad \text{indp.}$$

$X_1 \sim Unif(0, \theta)$

$$= [P(X_1 \le y)]^n = \left(\frac{y}{\theta}\right)^n$$

PDF for $\tilde{\theta}$. $f_{\tilde{\theta}}(y) = \frac{d}{dy} F_{\tilde{\theta}}(y) = \frac{d}{dy}\left(\frac{y}{\theta}\right)^n = n\left(\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} = \frac{ny^{n-1}}{\theta^n}$

$$E(\tilde{\theta}) = \int_0^\theta y f_{\tilde{\theta}}(y)\, dy = \int_0^\theta \frac{ny^n}{\theta^n}\, dy = \frac{n}{n+1}\theta \Rightarrow \tilde{\theta} \text{ is biased}$$

$$E(\tilde{\theta}^2) = \int_0^\theta y^2 f_{\tilde{\theta}}(y)\, dy = \int_0^\theta y^2 \cdot \frac{ny^{n-1}}{\theta^n} = \frac{n}{\theta^n} \cdot \frac{\theta^{n+2}}{n+2} = \frac{n}{n+2}\theta^2$$

$$Var(\tilde{\theta}) = E(\tilde{\theta}^2) - [E(\tilde{\theta})]^2 = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\right)^2\theta^2 \longrightarrow 0 \quad \text{as } n \to \infty.$$

Let $\hat{\theta} = \frac{n+1}{n}\tilde{\theta} \Rightarrow E(\hat{\theta}) = \frac{n+1}{n} E(\tilde{\theta}) = \frac{n+1}{n} \cdot \frac{n}{n+1}\theta = \theta$

unbiased est.

$Var(\hat{\theta}) \to 0$ as $n \to \infty$

Problem 6.2

Is the following statement TRUE?

If $\hat{\theta}$ is an unbiased estimator for θ, then $\hat{\theta}^2$ is an unbiased estimator for θ^2.

If yes, prove it; otherwise give a counter example.

Know : $E(\hat{\theta}) = \theta$

Is $E(\hat{\theta}^2) = \theta^2$ true ?

$$E(\hat{\theta}^2) = Var(\hat{\theta}) + [E(\hat{\theta})]^2 = Var(\hat{\theta}) + \theta^2$$

If $E(\hat{\theta}^2) = \theta^2 \Rightarrow Var(\hat{\theta}) + \theta^2 = \theta^2 \Rightarrow Var(\hat{\theta}) = 0$

$\Downarrow$

$\hat{\theta}$ is a constant estimator

In general, the statement (in the ques) is FALSE

$X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, σ^2 is known.

$\hat{\mu} = \bar{X}$ estimates μ. $\Rightarrow Var(\bar{X}) = \frac{\sigma^2}{n} \neq 0 \Rightarrow E(\hat{\mu}^2) \neq \mu^2$

$\hat{\theta}$ is a consistent estimator for θ if for all $\varepsilon > 0$, $\lim_{n\to\infty} P(|\hat{\theta} - \theta| > \varepsilon) = 0$

Sufficient condition for consistency.

$Bias(\hat{\theta}) \to 0$

$Var(\hat{\theta}) \to 0$ $\Longrightarrow$ $\hat{\theta}$ is consistent.

as $n \to \infty$

Problem 7 Let $X_1, \dots, X_n$ be iid with PMF given by

$$f(x) = \frac{1}{k} \text{ for } x = 1, \dots, k \text{ and } f(x) = 0 \text{ otherwise,}$$

where k (positive integer) is an unknown parameter.

~~(a)~~ Find the method of moments estimator for k and compute its bias and variance.

Hints: $1 + 2 + \cdots + k = k(k+1)/2$,
$1^2 + 2^2 + \cdots + k^2 = k(k+1)(2k+1)/6$.

(b) Consider the estimator $\hat{k} = 2\bar{X}$ for k, where $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Is $\hat{k}$ consistent? Justify your answer.

(c) Consider the estimator $\hat{k}_1 = \max(X_1, \dots, X_n)$ for k. Is $\hat{k}_1$ consistent? Justify your answer.

2022 Midterm.

discrete uniform

$k=1$

$P(X=1) = \frac{1}{k}$
$P(X=2) = \frac{1}{k}$
$\vdots$
$\Rightarrow P(X=1) = 1$

$\bar{X} = \frac{1}{n}\sum X_i$

(b) $\text{Bias}(\hat{k}) = E(2\bar{X}) - k = 2E(\bar{X}) - k \not\to 0$

$\text{Var}(\hat{k}) = \text{Var}(2\bar{X}) = 4\,\text{Var}(\bar{X}) \to 0$

Can't tell consistency.

$\hat{k} = 2\bar{X}$ is not consistent.

$\hat{k} = 2\bar{X}$

Show $\exists \varepsilon > 0$, $\exists k$. $\lim_{n\to\infty} P(|\hat{k} - k| > \varepsilon) \neq 0$

Choose $k=1$. $\Rightarrow X_1, X_2, \dots, X_n = 1 \Rightarrow \bar{X} = 1$

Choose $\varepsilon \in (0,1)$, $\varepsilon = \frac{1}{2}$

$$\lim_{n\to\infty} P(|\hat{k} - k| > \varepsilon) = \lim_{n\to\infty} P(|2-1| > \varepsilon) = 1 \neq 0.$$

$\hat{k}$ is inconsistent.

(c) $f(x) = \frac{1}{k}$ $\quad x = 1, \dots, k$

$\hat{k} = \max\{X_1, \dots, X_n\}$ consistent?

$\text{Bias}(\hat{k}) = E(\hat{k}) - k$
$\text{Var}(\hat{k}) = E(\hat{k}^2) - E(\hat{k})^2$

Prove / ~~disprove~~ $\lim_{n\to\infty} P(|\hat{k}-k|>\varepsilon)=0$.

$\hat{k}=\max\{X_1,\dots,X_n\}$. $X_i\in\{1,\dots,k\}$. $\Rightarrow \hat{k}\le k$

$\Rightarrow |\hat{k}-k| = k-\hat{k}$

$$
\begin{aligned}
P(|\hat{k}-k|>\varepsilon) &= P(k-\hat{k}>\varepsilon)\\
&= P(\hat{k}<k-\varepsilon)\\
&= P(\hat{k}\le k-1)\\
&= P(\max\{X_1,\dots,X_n\}\le k-1)\\
&= [P(X_1\le k-1)]^n = [P(X_1=1)+\dots+P(X_1=k-1)]^n\\
&= \left(\frac{1}{k}+\dots+\frac{1}{k}\right)^n\\
&= \left(\frac{k-1}{k}\right)^n \to 0 \text{ as } n\to\infty.
\end{aligned}
$$

$\hat{k}$ takes discrete values.

$\varepsilon\in(0,1)$

Since $P(\ )<1$

$\Rightarrow [P(\ )]^n\to 0$

$\hat{k}$ is consistent.

$X_1,\dots,X_n \sim \text{Unif}(0,\theta)$

$\hat{\theta}=X_1-X_2$ consistent? No.

Choose $\varepsilon=\frac{\theta}{2}$

$$P(|\underbrace{X_1-X_2}_{}^{\hat{\theta}}-\theta|>\varepsilon)$$

$$= P\left(|X_1 - X_2 - \theta| > \frac{\theta}{2}\right)$$

$$\geq P\left(X_1 \leq \frac{\theta}{2},\ X_2 \geq \frac{\theta}{4}\right) > 0$$

If $X_1 \leq \frac{\theta}{2}$

$X_2 \geq \frac{\theta}{4}$

$\Rightarrow X_1 - X_2 \leq \frac{\theta}{2} - \frac{\theta}{4} = \frac{\theta}{4}$

$\Rightarrow |X_1 - X_2 - \theta| \geq \frac{3\theta}{4} > \frac{\theta}{2}$

github.com/y-x-y-x/documents.