MH2500 AY23/24

Solution 1. Given that

$$P(X=n)=\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}-\frac{1}{n+2}.$$

Then,

$$\begin{aligned}
P(X\geq 1|X\leq 4) &= \frac{P(X\geq 1)\cap P(X\leq 4)}{P(X\leq 4)}\\
&= \frac{P(X=1)+P(X=2)+P(X=3)+P(X=4)}{P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)}\\
&= \frac{\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{5}-\frac{1}{6}\right)}{\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{5}-\frac{1}{6}\right)}\\
&= \left(\frac{1}{2}-\frac{1}{6}\right)\div\left(\frac{1}{1}-\frac{1}{6}\right)=\frac{2}{5}
\end{aligned}$$

Solution 2.

(a) $P(X\geq Y)=1$, otherwise the PDF $=0$.

(b) The marginal PDF of Y is

$$g(y)=f_Y(y)=\int_y^{\sqrt{y}}15y\,dx=15y(\sqrt{y}-y),\qquad 0\leq y\leq 1.$$

(c) The marginal PDF of X is

$$f_X(x)=\int_{x^2}^{x}15y\,dy=\frac{15}{2}(x^2-x^4),\qquad 0\leq x\leq 1.$$

Clearly, $f(x,y)\neq f_X(x)f_Y(y)$. Hence X and Y are not independent.

Solution 3.

(a)

$$\begin{aligned}
\mathbb{E}(X^n) &= \sum_{k=0}^{\infty}k^nP(X=k)=\sum_{k=0}^{\infty}k^ne^{-\lambda}\frac{\lambda^k}{k!}=\sum_{k=1}^{\infty}k^{n-1}e^{-\lambda}\frac{\lambda^k}{(k-1)!}=\sum_{j=0}^{\infty}(j+1)^{n-1}e^{-\lambda}\frac{\lambda^{j+1}}{j!}\\
&= \lambda\sum_{j=0}^{\infty}(j+1)^{n-1}e^{-\lambda}\frac{\lambda^j}{j!}=\lambda\sum_{j=0}^{\infty}(j+1)^{n-1}P(X=j)=\lambda\mathbb{E}((X+1)^{n-1}).
\end{aligned}$$

(b)

$$\mathbb{E}(X^3)=\lambda\mathbb{E}((X+1)^2))=\lambda\mathbb{E}(X^2)+2\lambda\mathbb{E}(X)+\lambda=\lambda^2\mathbb{E}(X+1)+2\lambda^2+\lambda=\lambda^3+3\lambda^2+\lambda.$$

Solution 4.

(a) We first claim that if $X \sim$N$(0,1)$, then so is $-X$. Let $W=-X$. By substitution,

$$f_W(w) = \frac{d}{dw}P(W \le w) = \frac{d}{dw}\int_{-w}^{\infty} \frac{1}{2\pi}\exp\left(-\frac{x^2}{2}\right)dx = \frac{d}{dw}\int_{-\infty}^{w} \frac{1}{2\pi}\exp\left(-\frac{x^2}{2}\right)dx = \frac{1}{2\pi}\exp\left(-\frac{w^2}{2}\right).$$

Let (a,b) be any arbitrary interval on $\mathbb{R}$.

$$P(a<Y<b|I=1) = P(a<X<b) = P(a<-X<b) = P(a<Y<b|I=0).$$

Therefore, I and Y are independent.

(b)

$$\begin{aligned} P(Y \le y) &= P(Y \le y|I=0)P(I=0) + P(Y \le y|I=1)P(I=1) = \frac{1}{2}(P(-X \le y) + P(X \le y)) \\ &= \frac{1}{2}(P(X \le y) + P(X \le y)) = \int_{-\infty}^{y} \frac{1}{2\pi}\exp\left(-\frac{x^2}{2}\right)dx. \end{aligned}$$

Therefore, the PDF of Y follows the standard normal distribution. By the fundamental theorem of calculus,

$$f_Y(y) = \frac{d}{dy}P(Y \le y) = \frac{1}{2\pi}\exp\left(-\frac{y^2}{2}\right).$$

(c)

$$\begin{aligned} Cov(X,Y) &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}(XY|I=0)P(I=0) + \mathbb{E}(XY|I=1)P(I=1) - 0 \\ &= \frac{1}{2}(\mathbb{E}(X^2) + \mathbb{E}(-X^2)) = 0. \end{aligned}$$

Solution 5.

(a) Let X be the length of time until he reaches safety. Let D_i be the event that the i-th door is chosen.

$$\mathbb{E}(X) = \sum_{i=1}^{3}\mathbb{E}(X|D_i)P(D_i) = \frac{1}{3}(\mathbb{E}(X|D_1) + \mathbb{E}(X|D_2) + \mathbb{E}(X|D_3)) = \frac{1}{3}(3 + (5+\mathbb{E}(X)) + (7+\mathbb{E}(X))$$

$$\mathbb{E}(X) = 5 + \frac{2}{3}\mathbb{E}(X) \implies \mathbb{E}(X) = 15 \text{ hours}.$$

(b)

$$\mathbb{E}(X^2|D_1) = 9, \qquad \mathbb{E}(X^2|D_2) = \mathbb{E}((X+5)^2), \qquad \mathbb{E}(X^2|D_3) = \mathbb{E}((X+7)^2)$$

$$\mathbb{E}(X^2) = \sum_{i=1}^{3}\mathbb{E}(X^2|D_i)P(D_i) = \frac{1}{3}(9 + (\mathbb{E}(X^2) + 10\mathbb{E}(X) + 25) + (\mathbb{E}(X^2) + 14\mathbb{E}(X) + 49)) = \frac{443}{3} + \frac{2}{3}\mathbb{E}(X^2).$$

$$\mathbb{E}(X^2) = 443 \implies Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 218 \text{ hours}^2.$$

Solution 6.

(a)

$$\mathbb{E}(X_i) = \sum_{k=1}^{6} kP(X_i = k) = \frac{7}{2}.$$
$$\mathbb{E}(X_i^2) = \sum_{k=1}^{6} k^2 P(X_i = k) = \frac{91}{6}.$$
$$Var(X_i) = \mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2 = \frac{35}{12}.$$

(b)

$$\mathbb{E}(\bar{X}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(X_i\right) = \frac{1}{n}\cdot n \cdot \frac{7}{2} = \frac{7}{2}.$$

$$Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}Var\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\left(\sum_{i=1}^{n} Var\left(X_i\right) + \sum_{i\neq j}\underbrace{Cov(X_i, X_j)}_{=0}\right)$$
$$= \frac{1}{n^2}\cdot n \cdot Var(X_1) = \frac{35}{12n}.$$

(ci) By Chebyshev's inequality,

$$P(|\bar{X} - 3.5| \geq 0.5) \leq \frac{Var(\bar{X})}{0.5^2} = \frac{7}{15}.$$

(cii) By central limit theorem, as n is large, $\sum_{i=1}^{n} X_i \sim \mathrm{N}\big(3.5n, \frac{35}{12}n\big)$

$$P(|\bar{X} - 3.5| \geq 0.5) = P(|\sum X_i - 3.5n| \geq 0.5n) \approx P\left(|Z| \geq \frac{0.5n - 0.5}{\sqrt{\frac{35}{12}n}}\right)$$

Z is the standard normal random variable and $\Phi(\cdot)$ is the CDF of Z.