Recap for Tutorial 7

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1 Limits of Sequences

A sequence $(a_n)_{n\geq 1}$ is said to converge to a limit L (or has a limit L) if and only if

$$\lim_{n\to\infty} a_n = L.$$

Formal definition for limit of sequence:

$$\lim_{n\to\infty} a_n = L \iff \forall\varepsilon > 0, \exists N \in \mathbb{N}, \text{ such that } |a_n - L| < \varepsilon, \text{ whenever } n > N.$$

If $\lim_{n\to\infty} a_n$ exists, we say that $(a_n)_{n\geq 1}$ converges, otherwise the sequence diverges.

Definition 1. Given a sequence $(a_n)_{n\geq 1}$, a subsequence of (a_n) is of the form

$$a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots,$$

where $1 \leq n_1 < n_2 < \ldots$. We can also write this subsequence as $(a_{n_k})_{k\geq 1}$

Theorem 1 (Subsequence Test)**.** If a sequence $(a_n)_{n\geq 1}$ converges to L, then every subsequence of (a_n) converges to L.

Theorem 2 (Limit Laws of Sequences)**.** Suppose $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are convergent sequences and c is a constant, then

- $$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n.$$
- $$\lim_{n\to\infty} c \cdot a_n = c \lim_{n\to\infty} a_n.$$
- $$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n.$$
- $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \text{ if } \lim_{n\to\infty} b_n \neq 0.$$
- $$\lim_{n\to\infty} (a_n)^p = \left(\lim_{n\to\infty} a_n\right)^p, \text{ if } p > 0, a_n > 0.$$

Theorem 3. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for $n \in \mathbb{Z}$, then

$$\lim_{n\to\infty} a_n = L.$$

Theorem 4. If $a_n \leq b_n \leq c_n$ for all $n > N_1$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then

$$\lim_{n\to\infty} b_n = L$$

Proof. Let $\varepsilon > 0$ be given, then there exists $N_2 \in \mathbb{N}$, such that for $n > N_2$, we have

$$-\varepsilon < a_n - L < \varepsilon \quad \text{and} \quad -\varepsilon < c_n - L < \varepsilon$$

Choose $N = \max\{N_1, N_2\}$, for all $n > N$, we have

$$-\varepsilon < a_n - L \leq b_n - L \leq c_n - L < \varepsilon \implies |b_n - L| < \varepsilon.$$

□

Theorem 5. If $\lim_{n\to\infty} a_n = L$ and $f(x)$ is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$$

Proof. Let $\varepsilon > 0$ be given. Since f is continuous at L, there exists $\delta > 0$, such that

$$|x - L| < \delta \implies |f(x) - f(L)| < \varepsilon.$$

Since $a_n \to L$, for the δ chosen above, there exists $N \in \mathbb{N}$, such that for $n > N$, $|a_n - L| < \delta$. By continuity, this exactly implies

$$|f(a_n) - f(L)| < \varepsilon.$$

□

2 Extra Exercises

Problem 1. Determine whether the sequence converges or diverges. If it converges, find the limit.

- $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\}$.
- $a_n = \ln(n + 1) - \ln n$.
- $a_n = \dfrac{(-3)^n}{n!}$.
- $a_n = \dfrac{e^n + e^{-n}}{e^{2n} - 1}$.

Problem 2. Show that, if a sequence converges to a limit, then this limit is unique.
(Hint: Prove by contradiction, suppose two different limits.)

Problem 3. Suppose a sequence $(a_n)_{n\geq 1}$ converges to a, and for each n, $a_n \geq 0$. Prove that $a \geq 0$.
(Hint: Prove by contradiction, choose $\varepsilon = -a/2$.)