Problem 11.1

Let $X_1, \dots, X_n$ be an i.i.d. sample drawn from a normal distribution $N(\mu, 1)$ where $\mu \in \mathbb{R}$ is an unknown parameter.

Consider a test for $H_0: \mu \leq 1$ against $H_1: \mu > 1$ which rejects H_0 if and only if $\bar{X} > 2$.

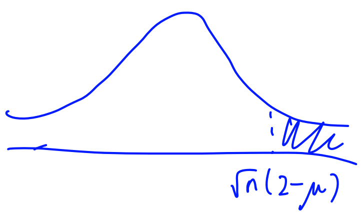
a) Compute the size of this test.
b) Determine the power of this test as a function of μ, for $\mu > 1$.

(a) Size of test, $\alpha = P(\text{Reject } H_0 \mid H_0)$

$= \sup_{\mu \leq 1} P(\bar{X} > 2)$ test statistic

$\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$

$= \sup_{\mu \leq 1} P(Z > \sqrt{n}(2-\mu))$ $Z \sim N(0,1)$ $\Rightarrow \sqrt{n}(\bar{X}-\mu) \sim N(0,1)$



$= P(Z > \sqrt{n})$ $(\mu = 1)$

$= 1 - \Phi(\sqrt{n})$, where Φ is CDF of $N(0,1)$

(b) Power : $P(\text{Reject } H_0 \mid H_1) = P(\bar{X} > 2 \mid H_1)$

$H_0: \mu \leq 1$, $H_1: \mu > 1$

$= P(Z > \sqrt{n}(2-\mu))$

Power $(\mu = 2) = 0.5$

$= 1 - \Phi(\sqrt{n}(2-\mu))$

Power $(\mu = 3) = 1 - \Phi(-\sqrt{n}) > 0.5$

Problem 11.2

Let X_1, X_2 be i.i.d. $U(\theta, \theta+1)$. Show that the CDF of $Y = X_1 + X_2$ is,

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 2\theta \\ \frac{1}{2}(y-2\theta)^2, & \text{if } 2\theta \le y < 2\theta+1 \\ 1 - \frac{1}{2}(2\theta+2-y)^2, & \text{if } 2\theta+1 \le y < 2\theta+2 \\ 1, & \text{if } y \ge 2\theta+2 \end{cases}$$

Example

$f_Y(y) = 2y$

$F_Y(y) = \int_{-\infty}^{y} 2t\, dt$

PDF of Y: $f_Y(y) = \int_\theta^{\theta+1} f_{X_2}(y-x) f_{X_1}(x)\, dx$

CDF of Y: $F_Y(y) = \int_{-\infty}^{y} \int_\theta^{\theta+1} f_{X_2}(t-x) \boxed{\underbrace{f_{X_1}(x)}_{}}\, dx\, dt$ $\quad // \ 1$

$X_1 \sim \text{Unif}(\theta, \theta+1)$

$$f_{X_1}(x) = \begin{cases} 1 & x \in [\theta, \theta+1] \\ 0 & \text{else.} \end{cases}$$

$$= \int_{2\theta}^{y} \boxed{\int_\theta^{\theta+1} f_{X_2}(t-x)\, dx}\, dt$$

$f_{X_2}(t-x) = 1$

when $\theta \le t-x \le \theta+1$

$$= \int_{2\theta}^{y} (t-2\theta)\, dt$$

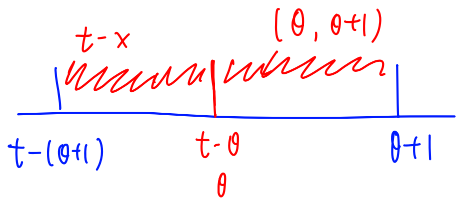
$$= \underline{\bigstar} \quad y \le 2\theta+1$$

$\int_\theta^{\theta+1} f_{X_2}(t-x)dx$ = integrate 1

when $\theta \le t-x \le \theta+1$

$= \ell\big((t-x) \cap (\theta, \theta+1) : \theta \le x \le \theta+1\big)$

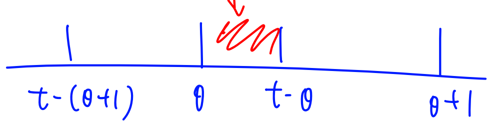
$F_Y(y) \quad y > 2\theta+1$



$$= \int_{2\theta+1}^{y} \boxed{\underbrace{\int_\theta^{\theta+1} f_{X_2}(t-x)\, dx}_{\text{length of intersection}}}\, dt + \int_{2\theta}^{2\theta+1} \int_\theta^{\theta+1} f_{X_2}(t-x)\, dx\, dt$$

t starts from 2θ.

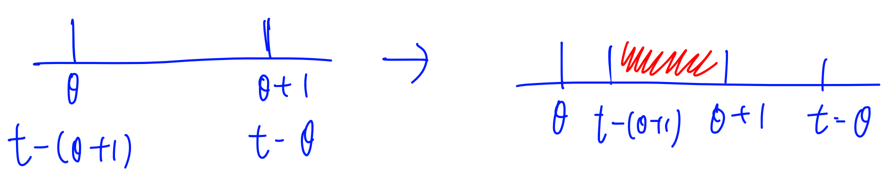
intersection $\Rightarrow$ length $= t - 2\theta$



$$= \int_{2\theta+1}^{y} (2\theta+2-t)\, dt + \frac{1}{2}$$

overlap when $t = 2\theta + 1$

$$= \underline{\bigstar\bigstar}$$



t starts from $2\theta+1$.

length $= 2\theta+2-t$

For testing $H_0: \theta = 0$ against $H_1: \theta > 0$, we have two competing tests:

$$\phi_1(X_1): \text{ Reject } H_0 \text{ if } X_1 > 0.92$$

$$\phi_2(X_1, X_2): \text{ Reject } H_0 \text{ if } X_1 + X_2 > C$$

a) Find the value of C so that ϕ_2 has the same size as ϕ_1.
b) Calculate the power function of the test ϕ_1. Draw a graph of this power function.

Let X_1, X_2 be i.i.d. $U(\theta, \theta+1)$. Show that the CDF of $Y = X_1 + X_2$ is,

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 2\theta \\ \frac{1}{2}(y-2\theta)^2, & \text{if } 2\theta \le y < 2\theta+1 \\ 1 - \frac{1}{2}(2\theta+2-y)^2, & \text{if } 2\theta+1 \le y < 2\theta+2 \\ 1, & \text{if } y \ge 2\theta+2 \end{cases}$$

(a) Size of ϕ_1

$= P(\text{Reject } H_0 \mid H_0 \text{ is true})$

$= P(X_1 > 0.92 \mid \theta = 0)$

$X_1 \sim Unif(0,1)$

$= 0.08$

Size of $\phi_2 = 0.08$

$\|$

$P(X_1 + X_2 > c \mid \theta = 0)$

$= 1 - F_Y(c)$ $\quad Y = X_1 + X_2$

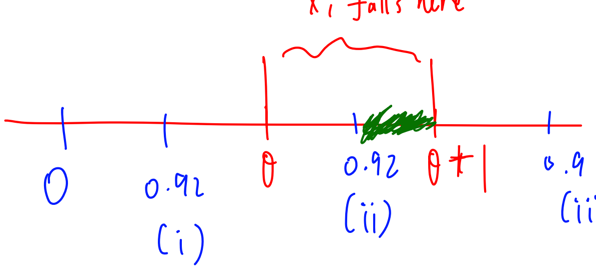
$\Rightarrow 1 - F_Y(c) = 0.08 \Rightarrow F_Y(c) = 0.92 = P(Y \le c)$

$\Rightarrow 1 - \frac{1}{2}(\cancel{2\theta} + 2 - c)^2 = 0.92$

$(2-c)^2 = 0.16 \qquad c = 1.6$

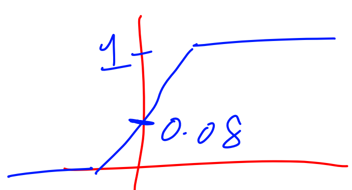
(b) ϕ_1: Reject H_0 if $X_1 > 0.92$ $\qquad X_1 \sim Unif(\theta, \theta+1)$

Power of $\phi_1 = P(\text{Reject } H_0 \mid H_1) = P(X_1 > 0.92 \mid \theta > 0)$



$$= \begin{cases} 1 & \text{(i): } 0.92 < \theta \\ \dfrac{\theta+1-0.92}{\theta+1-\theta} & \text{(ii): } \theta \le 0.92 \le \theta+1 \\ 0 & \text{(iii): } \theta+1 < 0.92 \end{cases}$$

← PdF of Unif



21/22 Q4

$X_1, \dots, X_n \sim N(\mu, 1)$

$H_0 : \mu = 0 \qquad H_1 : \mu > 0.$

$$\Lambda(X_1, \dots, X_n) = \frac{L(X_1, \dots, X_n \mid H_0)}{L(X_1, \dots, X_n \mid H_1)} \leq t$$

NP-Lemma: MP Test rejects H_0 when $\Lambda \leq t$ for some t.

Test statistic: $\bar{X}$, $\Lambda(X_1, \dots, X_n)$ is increasing in $\bar{X}$

MP Test rejects H_0 if $\Lambda(\bar{X}) \leq t$

$$\Downarrow$$

$$\bar{X} \leq \Lambda^{-1}(t) = t^*$$

$\Lambda(X_1, \dots, X_n)$ is dec in $\bar{X}$

MP Test rejects H_0 if $\Lambda(\bar{X}) \leq t \Rightarrow \bar{X} \geq \Lambda^{-1}(t) = t^*$

MH3500 Statistics

Tutorial 11 AY2022/23 Semester 2

Problem 11.1

Let $X_1, \ldots, X_n$ be an i.i.d. sample drawn from a normal distribution $N(\mu, 1)$ where $\mu \in \mathbb{R}$ is an unknown parameter.

Consider a test for $H_0: \mu \leq 1$ against $H_1: \mu > 1$ which rejects H_0 if and only if $\bar{X} > 2$.

a) Compute the size of this test.
b) Determine the power of this test as a function of μ, for $\mu > 1$.

Problem 11.2

Let X_1, X_2 be i.i.d. $U(\theta, \theta + 1)$. For testing $H_0: \theta = 0$ against $H_1: \theta > 0$, we have two competing tests:

$$\phi_1(X_1): \text{ Reject } H_0 \text{ if } X_1 > 0.92$$

$$\phi_2(X_1, X_2): \text{ Reject } H_0 \text{ if } X_1 + X_2 > C$$

a) Find the value of C so that ϕ_2 has the same size as ϕ_1.
b) Calculate the power function of the test each test ϕ_1. Draw a graph of this power function.

Hint: Notice that the joint distribution of X_1 and X_2 has a uniform distribution over the $(\theta, \theta + 1) \times (\theta, \theta + 1)$ square. Therefore, letting $Y = X_1 + X_2$,

$$F_Y(y) = \begin{cases} 0, & if\ y < 2\theta \\ \frac{1}{2}(y - 2\theta)^2, & if\ 2\theta \leq y < 2\theta + 1 \\ 1 - \frac{1}{2}(2\theta + 2 - y)^2, & if\ 2\theta + 1 \leq y < 2\theta + 2 \\ 1, & if\ y \geq 2\theta + 2 \end{cases}$$

Problem 11.3

Let $f_0(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$ and $f_1(x) = \begin{cases} 12\left(x - \frac{1}{2}\right)^2, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

be two probability density functions.

Let X be a random variable whose PDF is either $f_0(\cdot)$ or $f_1(\cdot)$. Based on a single observation of X, construct the MP test of the hypothesis $H_0: f(x) = f_0(x)$ against $H_1: f(x) = f_1(x)$ with size $\alpha = 0.05$. What is the power of this test?

Problem 11.4

Let $X_1, \dots, X_n$ be an i.i.d. sample drawn from a normal distribution $N(\mu, \sigma^2)$ where μ is known. Consider the test statistic $T = \sum_{i=1}^{n}(X_i - \mu)^2$.

a) For testing $H_0 \colon \sigma^2 = \sigma_0^2$ against $H_1 \colon \sigma^2 = \sigma_1^2$, with $\sigma_1^2 > \sigma_0^2$, find a MP test with size α.
b) Find a size α, UMP test for testing $H_0 \colon \sigma^2 = \sigma_0^2$ against $H_1 \colon \sigma^2 > \sigma_0^2$.
c) Is there a UMP for testing $H_0 \colon \sigma^2 \leq \sigma_0^2$ against $H_1 \colon \sigma^2 > \sigma_0^2$, with size α?