Extra Exercises for Week 6

MH2500

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Most questions are taken from the textbook: A First Course in Probability (9th edition) by Sheldon Ross.

Problem 1. If X is $\mathrm{Bin}(n, p)$, then prove that as k goes from 0 to n, $P(X = k)$ increases monotonically, reaching its largest value when $k = \lfloor (n+1)p \rfloor$. You may start by considering the ratio $P(X = k)/P(X = k-1)$.

Problem 2. Let $X \sim \mathrm{Po}(\lambda)$. What value of λ maximizes $P(X = k), k \geq 0$?

Problem 3. A casino patron will continue to make \$5 bets on red in roulette until she has won 4 of these bets. On each bet, she will either win \$5 with probability 18/38 or lose \$5 with probability 20/38.

(a) What is the probability that she places a total of 9 bets?

(b) Let X be the number of bets made until she stops. Let W be the total winnings until she stops. Compute $\mathbb{E}(X)$ and $\mathbb{E}(W)$. Here, X is a negative binomial random variable.

Problem 4. People enter a casino at a rate of 1 every 2 minutes.

(a) What is the probability that no one enters between 12:00 and 12:05?

(b) What is the probability that at least 4 people enter the casino during that time?

Problem 5. Let $X \sim \mathrm{Po}(\lambda)$. Show that

$$\mathbb{E}(X^n) = \lambda \mathbb{E}((X+1)^{n-1}).$$

Problem 6. Let $X \sim \mathrm{Po}(\lambda)$. Let $a > 0$ be a constant. Is aX a Poisson random variable?

Problem 7. A 0-truncated Poisson(λ) random variable X_T has the probability mass function

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)}, \qquad x = 1, 2, 3, \ldots,$$

where $X \sim \mathrm{Po}(\lambda)$. Find $\mathbb{E}(X_T)$.

Problem 8. The probability of being dealt a full house in a hand of poker is approximately 0.0014. Find the exact probability that in 1000 hands of poker, you will be dealt at least 2 full houses. Use a Poisson approximation to find the probability.

Problem 9. Let X and Y be two independent geometric random variables with parameter p, i.e. they have the probability mass function

$$P(X = k) = (1-p)^{k-1}p, \qquad k = 1, 2, \ldots.$$

(a) For $n \geq 1$, compute, $P(X \geq n)$.

(b) Let $Z = \min\{X, Y\}$. Compute $P(Z \geq n)$, and work out the probability mass function of Z.

(c) Compute $P(Y = 2 | X + Y = 4)$.

Problem 10. Let X be a Poisson(λ) random variable. Show that

$$P(X \text{ is even}) = \frac{1}{2}(1 + e^{-2\lambda}).$$

It may be useful to consider the Taylor series expansion of e^{λ} and $e^{-\lambda}$.

Problem 11. An urn has n white and m black balls. Balls are randomly withdrawn one at a time, without replacement, until a total of k white balls have been withdrawn, $k \leq n$. The random variable X is equal to the total number of balls that are withdrawn. We say that X follows a negative hypergeometric random variable. Find $P(X = r)$, the probability mass function of X.

Answers (Let me know if there are any discrepancies):

2. k

3(a). $\binom{8}{3}\left(\frac{20}{38}\right)^5\left(\frac{18}{38}\right)^4$

3(b). $\frac{76}{9}, -\frac{20}{9}$

4(a). $e^{-2.5}$

4(b). $1 - e^{-2.5}\left(1 + 2.5 + \frac{2.5^2}{2!} + \frac{2.5^3}{3!}\right)$

6. No, if $a \neq 1$.

7. $\frac{\lambda}{1-e^{-\lambda}}$

8. $0.4083264, 0.408167$

9(a). $(1-p)^{n-1}$

9(b). $(1-p)^{2n-2}$, $P(Z = k) = p(2-p)(1-p)^{2k-2}, k = 1, 2, \ldots$

9(c). $\frac{1}{3}$

11. $\frac{\binom{n}{k-1}\binom{m}{r-k}}{\binom{n+m}{r-1}} \cdot \frac{n-k+1}{n+m-r+1}$, where $r = k, k+1, \ldots, k+m$.