MH3500 Statistics

Tutorial 4 AY2022/23 Semester 2

Problem 4.1

Let $X_1, \ldots, X_n$ be an i.i.d. random sample drawn from a distribution with PDF

$$f(x) = \frac{\Gamma(2\theta)}{\Gamma(\theta)^2} x^{\theta-1}(1-x)^{\theta-1} \quad \text{for} \quad 0 \le x \le 1$$

and $f(x) = 0$ otherwise, with $\theta > 0$ an unknown parameter and $\Gamma(\theta)$ the gamma function.

Find the method of moments estimator for θ.

Hint: you can make use of the following identity.

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{for} \quad \alpha, \beta > 0$$

$\alpha = \theta + 1$

$\beta = \theta$

$$\mu_1 = E(X_1) = \int_0^1 \frac{\Gamma(2\theta)}{\Gamma(\theta)^2} x \cdot x^{\theta-1} (1-x)^{\theta-1}\,dx$$

$$= \frac{\Gamma(2\theta)}{\Gamma(\theta)^2} \int_0^1 x^{\theta} (1-x)^{\theta-1}\,dx = \frac{\Gamma(2\theta)}{\Gamma(\theta)^2} \cdot \frac{\Gamma(\theta+1)\cancel{\Gamma(\theta)}}{\Gamma(2\theta+1)}$$

$$= \frac{\Gamma(2\theta)\Gamma(\theta+1)}{2\theta\,\Gamma(2\theta)\Gamma(\theta)} = \frac{\theta\,\Gamma(\theta)}{2\theta\,\Gamma(\theta)} = \frac{1}{2}$$

$$\mu_2 = E(X_1^2) = \frac{\Gamma(2\theta)}{\Gamma(\theta)^2} \int_0^1 x^2 x^{\theta-1} (1-x)^{\theta-1}\,dx = \frac{\Gamma(2\theta)}{\Gamma(\theta)^2} \cdot \frac{\Gamma(\theta+2)\Gamma(\theta)}{\Gamma(2\theta+2)}$$

$$= \frac{\cancel{\Gamma(2\theta)}(\theta+1)\cancel{\theta}\cancel{\Gamma(\theta)}}{(2\theta+1)(2\cancel{\theta})\cancel{\Gamma(2\theta)}\cancel{\Gamma(\theta)}} = \frac{\theta+1}{2(2\theta+1)} = \frac{1}{4(2\theta+1)} + \frac{1}{4}$$

By LLN, $S_2 = \frac{1}{n}\sum X_i^2$ converges to μ_2 in probability

$$\therefore \quad S_2 = \frac{1}{4(2\hat{\theta}+1)} + \frac{1}{4} \Rightarrow \frac{1}{2\hat{\theta}+1} = 4S_2 - 1 \Rightarrow \hat{\theta} = \frac{1}{2}\left[\frac{1}{4S_2-1} - 1\right]$$

Problem 4.2

The Geometric distribution $Geo(p)$ has a PMF $f(x|p) = p(1-p)^{x-1}$ for $x = 1, 2, \ldots$ and zero otherwise. Let $X_1, \ldots, X_n$ be i.i.d. $\sim Geo(p)$, where $p \in [0, 1]$ is an unknown parameter.

a) Find the method of moments estimator (MME) for p based on this random sample.
b) Find the maximum likelihood estimator (MLE) for p .

(a) By MME.

$$\mu_1 = E(X_1) = \frac{1}{p}$$

By LLN, $S_1 = \frac{1}{n}\sum X_i$ conv in probability to $\frac{1}{p}$

$$\therefore \hat{p} = \frac{1}{S_1} = \frac{1}{\bar{X}}$$

(b) $$L(x_1, x_2, \ldots, x_n | p) = \prod_{i=1}^{n} f(x_i|p) = \prod_{i=1}^{n} p(1-p)^{x_i - 1}$$

$$= p^n (1-p)^{(\sum x_i) - n}$$

If $n\bar{x} - n = 0 \Rightarrow L = p^n \Rightarrow p^n$ increases for $p \in [0,1] \Rightarrow \max_{p \in [0,1]} L = 1$, $\hat{p} = 1 = \frac{1}{\bar{X}}$

When $n\bar{X} - n > 0$, $L(p)$ satisfies the standard conditions.

(i) $L(p) > 0$

(ii) $\frac{dL}{dp}$ exists for all $p \in (0,1)$ $\Rightarrow$ global max exists for $p \in (0,1)$.

(iii) $\lim_{p \to 0^+} L(p) = 0$ and $\lim_{p \to 1^-} L(p) = 0$.

maximize $L \iff$ maximize $\ln L$

$$\ln L = n(\ln p) + (\sum x_i - n)[\ln(1-p)]$$

$$\frac{d(\ln L)}{dp} = \frac{n}{p} + (\sum x_i - n)\left(\frac{1}{p-1}\right) = 0$$

$$n(1-p) = p(\sum x_i - n) \Rightarrow n = p\sum x_i \Rightarrow p = \frac{1}{\frac{\sum x_i}{n}} = \frac{1}{\bar{X}}$$

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