Tutorial Question

(4) Suppose that 10 balls are put into 5 boxes, with each ball independently being put in box i with probability p_i, $\sum_{i=1}^{5} p_i = 1$.

(a) Find the expected number of boxes that do not have any balls.
(b) Find the expected number of boxes that have exactly 1 ball.

(a) $\mathbb{E}(X) = \sum_{x} x P(X=x)$, where X is num of boxes with no balls.

$P(X=1) = (1-p_1)^{10} + (1-p_2)^{10} + (1-p_3)^{10} + \dots$

$P(X=2) =$

$P(X=3) =$

not encouraged.

X_i = ith box does not contain any balls.

X_i is an indicator variable. $\Rightarrow$ $X_i = \begin{cases} 1, & \text{when ith box doesn't contain any balls.} \\ 0, & \text{otherwise.} \end{cases}$

$i = 1,2,3,4,5$

X = number of boxes that do not contain any balls.

$X = X_1 + X_2 + X_3 + X_4 + X_5$

$$\mathbb{E}(X) = \mathbb{E}(X_1 + \dots + X_5) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \dots + \mathbb{E}(X_5)$$
$$= \sum_{i=1}^{5} (1-p_i)^{10}$$

$\mathbb{E}(X_i) = \cancel{0 \cdot P(}\quad)$
$+ 1 \cdot P(\text{ith box doesn't contain any balls}),$
$= P(\text{ith box doesn't contain any balls})$
$= (1-p_i)^{10}$

(4) Suppose that 10 balls are put into 5 boxes, with each ball independently being put in box i with probability p_i, $\sum_{i=1}^{5} p_i = 1$.

(a) Find the expected number of boxes that do not have any balls.
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(b) Y_i : box i has exactly 1 ball (indicator variable) $\in \{0, 1\}$

$\mathbb{E}(Y_i) = P(\text{box } i \text{ has exactly 1 ball}) = \binom{10}{1} p_i (1-p_i)^9$

Y = num of boxes that have exactly 1 ball.

$\Rightarrow Y = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 \quad \Rightarrow \mathbb{E}(Y) = \sum \mathbb{E}(Y_i) = \sum_{i=1}^{5} \binom{10}{1} p_i (1-p_i)^9$

(5) There are k types of coupons. Independently of the types of previously collected coupons, each new coupon collected is of type i with probability p_i, $\sum_{i=1}^{k} p_i = 1$.

If n coupons are collected, find the expected number of distinct types that appear in this set. (That is, find the expected number of types of coupons that appear at least once in the set of n coupons.)

$\in \{0,1\}$

Indicator variable : X_i = ith coupon has been collected (among n)

X = number of distinct types of coupon that has been collected.

$$X = X_1 + X_2 + \dots + X_k \Rightarrow \mathbb{E}(X) = \sum_{i=1}^{k} \mathbb{E}(X_i)$$

$$= \sum_{i=1}^{k} \mathbb{P}(\text{ith coupon has been collected})$$

To collect ith coupon : p_i

among n

$$= \sum_{i=1}^{k} [1 - P(\text{ith coupon has not been collected})]$$

$$= \sum_{i=1}^{k} [1 - (1-p_i)^n]$$

Practice Question

(2) An urn contains N white and M black balls. Balls are randomly selected, one at a time, until a black one is obtained.

If we assume that each ball selected is replaced before the next one is drawn, what is the probability that

(a) exactly n draws are needed?
(b) at least k draws are needed?

X = number of draws needed. $X \sim Geo\left(\frac{M}{N+M}\right)$

(a) $P(X=n) = P($ first $(n-1)$ are all white, n^{th} ball is black$)$

$$= \left(\frac{N}{N+M}\right)^{n-1}\left(\frac{M}{N+M}\right)$$

$$\boxed{\sum_{n\geq 1} r^n = \frac{r}{1-r} \qquad |r|<1}$$

(b) $P(X\geq k) = P(X=k) + P(X=k+1) + \ldots$

$$= \sum_{i=k}^{\infty} P(X=i) = \sum_{i=k}^{\infty}\left(\frac{N}{N+M}\right)^{i-1}\left(\frac{M}{N+M}\right)$$

$$= \frac{M}{N+M}\sum_{i=k}^{\infty}\left(\frac{N}{N+M}\right)^{i-1} = \frac{M}{\cancel{N+M}} \cdot \frac{\left(\frac{N}{N+M}\right)^{k-1}}{\cancel{\left(1-\frac{N}{N+M}\right)}}$$

$$= \left(\frac{N}{N+M}\right)^{k-1}$$

$P(\geq k \text{ draws}) = P($ first $(k-1)$ draws are all white$)$

(3) Suppose that independent trials, each having probability p, $0 < p < 1$, of being a success are performed until a total of r successes is accumulated. Let X equal the number of trials required, then

$$\mathbb{P}r\{X = n\} = \binom{n-1}{r-1} p^r (1-p)^{n-r},$$

for $n = r, r+1, \cdots$.

$$\underbrace{\_\_ \ \ \_\_ \ \ \_\_ \ \ \cdots \ \ \_\_}_{r-1 \text{ successes}} \quad \overset{r\text{th success}}{\_\_}$$

$$1 \ \cdots \qquad n-1 \quad n$$

- A random variable X whose probability mass function is given as above is said to be a *negative binomial* random variable with parameters (r, p).
- Note that a geometric random variable is just a negative binomial with parameter $(1, p)$.

(a) Check that $\sum_{n=r}^{\infty} \mathbb{P}r\{X = n\} = 1$.

$$\sum_{n=r}^{\infty} \binom{n-1}{r-1} p^r (1-p)^{n-r} = \sum_{n=r}^{\infty} \binom{n-1}{n-r} p^r (1-p)^{n-r} \qquad \binom{n}{k} = \binom{n}{n-k}$$

$$= p^r \sum_{j=0}^{\infty} \binom{r+j-1}{j} (1-p)^j \qquad j = n-r$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\overbrace{n(n-1)\cdots(n-k+1)}^{k \text{ terms}}}{k!}$$

$$= p^r \sum_{j=0}^{\infty} \frac{\overbrace{(r+j-1)(r+j-2)\cdots(r+1)r}^{j \text{ terms}}}{j!} (1-p)^j$$

$$= p^r \sum_{j=0}^{\infty} \frac{(-r)(-r-1)\cdots(-r-j+2)(-r-j+1)}{j!} (-1)^j (1-p)^j$$

$$= p^r \sum_{j=0}^{\infty} \binom{-r}{j} (p-1)^j$$

$$= p^r \sum_{j=0}^{\infty} \binom{-r}{j} (p-1)^j \, 1^{-r-j} = p^r (p-1+1)^{-r}$$

$$= 1$$

Generalised Binomial Theorem. $\forall a, b \in \mathbb{R}$, $r \in \mathbb{R}$.

$$(a+b)^r = \sum_{j=0}^{\infty} \binom{r}{j} a^j b^{r-j} \qquad \binom{r}{j} = \frac{r(r-1)\cdots(r-j+1)}{j!}$$

(b) Prove that $\mathbb{E}(X) = r/p$ and $Var(X) = r(1-p)/p^2$.

$\mathbb{E}(X^2) - \mathbb{E}(X)^2$

$$\mathbb{E}(X^k) = \sum_n n^k P(X=n)$$

$$n\binom{n-1}{r-1} = r\binom{n}{r}$$

$$= \sum_{n=r}^{\infty} n^k \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

$$= \sum_{n=r}^{\infty} n^{k-1} \cdot r \binom{n}{r} p^r (1-p)^{n-r}$$

$$= \frac{r}{p} \sum_{n=r}^{\infty} n^{k-1} \binom{n}{r} p^{r+1} (1-p)^{n-r}$$

Let $m = n+1$

$$= \frac{r}{p} \sum_{m=r+1}^{\infty} (m-1)^{k-1} \underbrace{\binom{m-1}{r} p^{r+1} (1-p)^{m-r-1}}_{\text{p.m.f of } Y \sim NB(r+1,p)}$$

$$P(X=n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

$$X \sim NB(r,p)$$

$$= \frac{r}{p} \mathbb{E}\left[(Y-1)^{k-1}\right]$$

$$\mathbb{E}(X) = \frac{r}{p} \mathbb{E}\left[(Y-1)^0\right] = \frac{r}{p}$$

$$\mathbb{E}(X^2) = \frac{r}{p}\left[\mathbb{E}(Y-1)\right] = \frac{r}{p}\left[\mathbb{E}(Y) - 1\right] = \frac{r}{p}\left[\frac{r+1}{p} - 1\right]$$

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{r}{p}\left[\frac{(r+1)}{p} - 1\right] - \frac{r^2}{p^2} = \frac{r(1-p)}{p^2}$$