Tutorial Question

(5) Show that if $\mathbb{P}(A_i) = 1$ for all $i \geq 1$, then $\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1$.

De Morgan's Law : $\bigcup_{i=1}^{n} A_i' = \left(\bigcap_{i=1}^{n} A_i\right)'$ as sets.

To show $A = B$.

$A \subseteq B$

and $B \subseteq A$

We claim that : $\bigcup_{i=1}^{\infty} A_i' = \left(\bigcap_{i=1}^{\infty} A_i\right)'$

Proof

Let $x \in \bigcup_{i=1}^{\infty} A_i' \iff x \in A_i'$ for some i

$\iff x \notin A_i$ for some i

$\iff x \notin \bigcap_{i=1}^{\infty} A_i \iff x \in \left(\bigcap_{i=1}^{\infty} A_i\right)'$

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - P\left[\left(\bigcap_{i=1}^{\infty} A_i\right)'\right]$$

$$= 1 - P\left(\bigcup_{i=1}^{\infty} A_i'\right) \quad \text{(by claim)}$$

$$\geq 1 - \sum_{i=1}^{\infty} \underbrace{P(A_i')}_{0}$$

$$= 1$$

Boole's Ineq : $P\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \sum_{i=1}^{\infty} P(B_i)$

So, $P\left(\bigcap_{i=1}^{\infty} A_i\right) \geq 1$, since $P(A) \leq 1$ for any event A.

then $P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1$

Practice Question

(1) Three students, A, B, and C, take turns flipping a coin. That is, A flips first, then B, then C, then A, and so on. The sample space of this experiment is given by

$$S = \begin{cases} H, TH, TTH, TTTH, \ldots, \\ TTTT\cdots \end{cases}$$

(a) Interpret the sample space.
(b) Say that the first one to get a head wins. Let X be the event that A wins, and Y be the event that B wins. Find $(X \cup Y)'$.

(a) The experiment stops when a "head" is flipped.

(b) $(X \cup Y)' = X' \cap Y' =$ "A does not win and B does not win"

$(X \cup Y)' \subseteq S$, $(X \cup Y)' = \{TTH, TTTTTH, \ldots, TTTT\ldots\}$.

(2) Two symmetric dice have both had two of their sides painted red, two painted black, one painted yellow, and the other painted white. When this pair of dice is rolled, what is the probability that both dice land with the same colour facing up?

R, R, B, B, Y, W

$P(RR) + P(BB) + P(YY) + P(WW)$

$$= \underset{\text{first}}{\frac{2}{6}} \cdot \underset{\text{second}}{\frac{2}{6}} + \frac{2}{6} \cdot \frac{2}{6} + \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} = \frac{5}{18}$$

(3) Two dice are thrown n times in succession. Compute the probability that double 6 appears at least once. How large need n be to make this probability at least 1/2?

P(double 6 appears at least once for n times)

$n = \left\lceil \dfrac{\ln \frac{1}{2}}{\ln \frac{35}{36}} \right\rceil$

= 1 − P(double 6 does not appear for n times)

$= 1 - \prod_{i=1}^{n}$ P(double 6 does not appear once) (independence of events)

$$= 1 - \prod_{i=1}^{n} \left(\frac{35}{36}\right) = 1 - \left(\frac{35}{36}\right)^n \geq \frac{1}{2}$$

$$\left(\frac{35}{36}\right)^n \leq \frac{1}{2}$$

$$\Rightarrow n \geq \frac{\ln \frac{1}{2}}{\ln 35/36}$$

(4) Prove Boole's inequality:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$ (A_i's may not be mutually exclusive).

Recall that countable additivity says that when $\{A_i\}$ is a countable sequence of mutually disjoint events, we have the identity, i.e., we will have

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$ (Axiom of probability)

We just need to prove $P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$ for all n.

Proof by induction

Base case: $n=1$.

$$P\left(\bigcup_{i=1}^{1} A_i\right) = P(A_1) = \sum_{i=1}^{1} P(A_i)$$

True

Assume true up to $n=k$

Assumption: $P\left(\bigcup_{i=1}^{k} A_i\right) \le \sum_{i=1}^{k} P(A_i)$

Proceed to prove: $P\left(\bigcup_{i=1}^{k+1} A_i\right) \le \sum_{i=1}^{k+1} P(A_i)$

Principle of I-E

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P(A_1 \cup A_2 \cup \dots \cup A_k \cup A_{k+1})$$

$$= P\left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right)$$

$$= P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\bigcup_{i=1}^{k} A_i \cap A_{k+1}\right)$$

hypothesis

$$\le \sum_{i=1}^{k} P(A_i) + P(A_{k+1})$$

$P(\ \ ) \ge 0$

$$= \sum_{i=1}^{k+1} P(A_i)$$

This completes the induction and $P\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} P(A_i)$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} P(A_i)$$

2nd way

Let $B_1 = A_1$

$B_2 = A_2 \setminus A_1$

$B_3 = A_3 \setminus (A_1 \cup A_2)$

$\vdots$

$B_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i$

$B_1, B_2, \ldots$ are all mutually exclusive.

Key : Show $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$.

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i) \le \sum_{i=1}^{\infty} P(A_i)$$

mutually exclusive

since $B_i \subseteq A_i$

To show $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$

Let $x \in \bigcup_{i=1}^{\infty} B_i$, $x \in B_k$ for some k $\Rightarrow x \in A_k \setminus \bigcup_{i=1}^{k-1} A_i$ for some k

$\Rightarrow x \in A_k$ for some k

$\Rightarrow x \in \bigcup_{i=1}^{\infty} A_i$ $\quad\therefore \bigcup B_i \subseteq \bigcup A_i$

Let $x \in \bigcup_{i=1}^{\infty} A_i$, then there is a minimum k, such that $x \in A_k$ and $x \notin A_\ell$, $\ell \le k-1$

$\Rightarrow x \in A_k \setminus \bigcup_{i=1}^{k-1} A_i$ for some k

$\Rightarrow x \in B_k$ for some k $\quad\therefore \bigcup A_i \subseteq \bigcup B_i$

$\Rightarrow x \in \bigcup_{i=1}^{\infty} B_i$

(5) Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. We assume that at each draw, each ball in the urn is equally likely to be chosen. What is the probability that both balls drawn are red?

$$\frac{8}{12} \times \frac{7}{11}$$

github.com/y-x-y-x/documents.