Problem 1.1

Let $X \sim \mathrm{Geo}(p)$ with $0 < p < 1$. Determine the distributions the mean and variance of X .

$$P(X=x) = (1-p)^{x-1} p$$

$$E(X) = \sum_{x=1}^{\infty} x (1-p)^{x-1} p$$

$$= p \cdot \frac{d}{dp}\left(-\sum_{x=1}^{\infty} (1-p)^{x}\right) = p \cdot \frac{d}{dp}\left(\frac{-1}{1-(1-p)} - 1\right) = \frac{p}{p^2} = \frac{1}{p}$$

$$E(X^2) = \sum_{x=1}^{\infty} x^2 (1-p)^{x-1} p$$

$$= p \cdot \frac{d}{dp}\left(-\sum_{x=1}^{\infty} x(1-p)^{x}\right)$$

$$= -p \cdot \frac{d}{dp}\left((1-p)\sum_{x=1}^{\infty} x(1-p)^{x-1}\right)$$

$$= -p \cdot \frac{d}{dp}\left(\frac{1-p}{p^2}\right) = -p \cdot \frac{-p^2-(1-p)2p}{p^4} = \frac{p^2+(1-p)2p}{p^3} = \frac{2-p}{p^2}$$

$$\mathrm{Var}(X) = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}$$

Problem 1.2

a) Let $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$ for $\alpha > 0$. Show that $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

b) Let $X \sim \text{Gamma}(\alpha, \beta)$ with $\alpha, \beta > 0$. Determine the mean and the variance of X

(a) $$\Gamma(\alpha+1) = \int_0^\infty x^{\alpha} e^{-x}\,dx$$

$$= \left[-e^{-x}x^{\alpha}\right]_0^\infty + \int_0^\infty e^{-x}\cdot \alpha x^{\alpha-1}\,dx$$

$$= -\lim_{t\to\infty}\frac{t^\alpha}{e^t} + \alpha\Gamma(\alpha) = \alpha\Gamma(\alpha)$$

(b) $$f_X(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha\,\Gamma(\alpha)} \quad x>0 \quad , f_X(x) = 0 \quad \text{for } x \le 0$$

$$E(X) = \int_0^\infty \frac{x^\alpha e^{-\frac{x}{\beta}}}{\beta^\alpha\,\Gamma(\alpha)}\,dx = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty x^\alpha e^{-\frac{x}{\beta}}\,dx$$

Let $\frac{x}{\beta} = t \Rightarrow dx = \beta\,dt$

$$E(X) = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty \beta^\alpha t^\alpha e^{-t}\cdot\beta\,dt = \frac{\beta}{\Gamma(\alpha)}\int_0^\infty t^\alpha e^{-t}\,dt = \frac{\beta\,\Gamma(\alpha+1)}{\Gamma(\alpha)}$$

$$= \alpha\beta.$$

$$E(X^2) = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty x^{\alpha+1}e^{-\frac{x}{\beta}}\,dx$$

Let $\frac{x}{\beta} = t \Rightarrow dx = \beta\,dt$

$$E(X^2) = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty t^{\alpha+1}\beta^{\alpha+1}e^{-t}\beta\,dt = \frac{\beta^2}{\Gamma(\alpha)}\int_0^\infty t^{\alpha+1}e^{-t}\,dt = \frac{\beta^2}{\Gamma(\alpha)}\cdot\Gamma(\alpha+2)$$

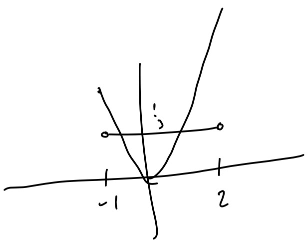
$$\mathrm{Var}(X) = \beta^2\alpha(\alpha+1) - (\alpha\beta)^2 = \alpha\beta^2.$$

Problem 1.3

Let X be a random variable with the following probability density function (PDF).

$$f(x) = \begin{cases} 1/3 & for\ -1 < x < 2 \\ 0 & otherwise \end{cases}$$

a) Find the median, lower and upper quartiles of the distribution of X.
b) Find the CDF and PDF of X^2.



(a) $F(x) = \int_{-1}^{x} \frac{1}{3}\, dt = \frac{1}{3}(x+1)$

Median : $F(x) = \frac{1}{2} \Rightarrow x = \frac{1}{2}$

Lower Q: $F(x) = \frac{1}{4} \Rightarrow x = -\frac{1}{4}$

Upper Q: $F(x) = \frac{3}{4} \Rightarrow x = \frac{5}{4}$

(b) Let $Y = X^2$

For $0 \le y \le 1$,

$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3}\, dx = \frac{2\sqrt{y}}{3}$

For $1 \le y \le 4$,

$F_Y(y) = \frac{2}{3} + P(1 \le X \le \sqrt{y}) = \frac{2}{3} + \int_{1}^{\sqrt{y}} \frac{1}{3}\, dx = \frac{1}{3}(\sqrt{y} - 1) + \frac{2}{3} = \frac{1}{3}(\sqrt{y} - 1)$

$$f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & 0 \le y < 1 \\ \frac{1}{6\sqrt{y}} & 1 \le y < 4 \\ 0 & otherwise \end{cases}$$

Problem 1.4

Let $X_1, \ldots, X_n$ be independent random variable with $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, \ldots, n$. Let $Y = \sum_{i=1}^n a_i X_i$ where $a_i \in \mathbb{R}$ are constants.

a) Use MGFs to show that Y has a normal distribution with $E[Y] = \sum_{i=1}^n a_i \mu_i$ and $Var[Y] = \sum_{i=1}^n a_i^2 \sigma_i^2$.

b) Write down the PDF of Y?

$$M_{X_i}(t) = e^{\mu_i t} e^{\frac{(\sigma_i t)^2}{2}}$$

(a) $$M_Y(t) = \prod_{i=1}^n M_{X_i}(a_i t) = \prod_{i=1}^n e^{\mu_i a_i t} e^{\frac{(\sigma_i a_i t)^2}{2}}$$

$$= e^{\sum \mu_i a_i t} \cdot e^{\sum \frac{(\sigma_i a_i t)^2}{2}} \Rightarrow \text{Normal distribution}.$$

$$E(Y) = M_Y'(0) \Rightarrow M_Y'(t) = \left(\sum \mu_i a_i + t \sum (\sigma_i a_i)^2\right) M_Y(t)$$

$$E(Y) = \sum \mu_i a_i$$

$$E(Y^2) = M_Y''(0) \Rightarrow M_Y''(t) = \sum (\sigma_i a_i)^2 \cdot M_Y(t) + \left(\sum \mu_i a_i + t \sum (\sigma_i a_i)^2\right)^2 M_Y(t)$$

$$M_Y''(0) = \sum (\sigma_i a_i)^2 + \left(\sum \mu_i a_i\right)^2$$

$$Var(Y) = E(Y)^2 - [E(Y)]^2 = \sum (\sigma_i a_i)^2$$

(b) $$f_Y(y) = \frac{1}{\sqrt{2\pi \sum (\sigma_i a_i)^2}} e^{-\frac{(y - \sum \mu_i a_i)^2}{2 \sum (\sigma_i a_i)^2}} \qquad -\infty < y < \infty$$

Problem 1.5

Let $X_1, \ldots, X_n$ be independent random variable with $X_i \sim Gamma(\alpha_i, \beta)$ where $\alpha_1, \ldots, \alpha_n, \beta > 0$. Let $Y = \sum_{i=1}^n X_i$. Use MGFs to show that $Y \sim Gamma(\sum_{i=1}^n \alpha_i, \beta)$.

$$M_{X_i}(t) = (1 - \beta t)^{-\alpha_i} \qquad t < \frac{1}{\beta}$$

$$M_Y(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n (1 - \beta t)^{-\alpha_i} = (1 - \beta t)^{-\sum \alpha_i}$$

$$\Rightarrow Y \sim \text{Gamma}\left(\sum \alpha_i, \beta\right).$$

MH3500 Statistics

Tutorial 1 AY2022/23 Semester 2

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