Problem 8.1

Let $X_1, \dots, X_n$ be an i.i.d. sample drawn from a normal distribution $N(\mu, \sigma^2)$. Assume that σ^2 is a known constant and that μ is an unknown parameter.

a) Compute the Fisher Information $I_{X_1,\dots,X_n}(\mu)$ using the definition of Fisher Information.
b) Show that condition (*) is satisfied for the population PDF $f(x|\mu)$.
c) Compute $I_{X_1,\dots,X_n}(\mu)$ using the alternative formula for Fisher information and additivity.

$$I_{X_1,\dots,X_n}(\mu) = E\left[\left(\frac{\partial}{\partial\mu}\log L(\mu)\right)^2\right]$$

(a) $$L(X_1,\dots,X_n|\mu) = \prod_{i=1}^n f(x_i|\mu) = \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

$$= \sigma^{-n}(2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2\right)$$

$$\log L(\mu) = -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^n (X_i-\mu)^2$$

$$\frac{\partial}{\partial\mu}\log L(\mu) = -\frac{1}{2\sigma^2}\sum_{i=1}^n 2(X_i-\mu)(-1) = \frac{1}{\sigma^2}\sum_{i=1}^n (X_i-\mu)$$

$$(a_1+a_2+a_3+\dots+a_n)^2 = a_1^2+a_2^2+\dots+a_n^2 + 2\sum_{i<j} a_i a_j$$

$$\left(\frac{\partial}{\partial\mu}\log L(\mu)\right)^2 = \frac{1}{\sigma^4}\left(\sum_{i=1}^n (X_i-\mu)\right)^2$$

$$= \frac{1}{\sigma^4}\left[\sum_{i=1}^n (X_i-\mu)^2 + 2\sum_{i<j}(X_i-\mu)(X_j-\mu)\right]$$

$$E\left[\left(\frac{\partial}{\partial\mu}\log L(\mu)\right)^2\right] = \frac{1}{\sigma^4}\left[\sum_{i=1}^n E(X_i-\mu)^2 + 2\sum_{i<j} E\left[(X_i-\mu)(X_j-\mu)\right]\right]$$

$$= \frac{1}{\sigma^4}\left[\sum_{i=1}^n Var(X_i) + 2\sum_{i<j} \underbrace{Cov((X_i-\mu)(X_j-\mu))}_{0 \text{ (due to } X_i, X_j \text{ indp } i\neq j)}\right]$$

$$= \frac{1}{\sigma^4}(n\cdot\sigma^2) = \frac{n}{\sigma^2}$$

$$Cov(XY) = E(XY) - E(X)E(Y)$$

(b) Condition $(*)$ is satisfied by $f(x|\mu)$.

Check : $\int_{-\infty}^{\infty} \frac{\partial}{\partial\mu} f(x|\mu)\, dx = 0$

$$\frac{\partial}{\partial\mu}\left[\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(X-\mu)^2}{2\sigma^2}\right)\right] = \underbrace{\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(X-\mu)^2}{2\sigma^2}\right)}_{f(x|\mu)} \cdot \left(\frac{X-\mu}{\sigma^2}\right)$$

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial\mu} f(X|\mu)\, dx = \int_{-\infty}^{\infty} \left(\frac{X-\mu}{\sigma^2}\right) f(x|\mu)\, dx = \mathbb{E}\left(\frac{X-\mu}{\sigma^2}\right)$$

$$= \frac{1}{\sigma^2} E(X-\mu) = 0$$

(c) Compute $I_{X_1,\dots,X_n}(\mu)$ using alternative formula and additivity.

If Condition $(*)$, then $I_{X_1,\dots,X_n}(\mu) = n\, I_{X_1}(\mu)$ (additivity)

$I_{X_1}(\mu) = E\left[\left[\frac{\partial}{\partial\mu} \log L(X_1|\mu)\right]^2\right]$ (by definition)

If $\int_{-\infty}^{\infty} \frac{\partial^2}{\partial\mu^2} f(X|\mu)\, dx = 0 \Rightarrow I_{X_1}(\mu) = -E\left[\frac{\partial^2}{\partial\mu^2} \log L(X_1|\mu)\right]$ (alternative)

$$f(X|\mu) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(X-\mu)^2}{2\sigma^2}\right)$$

$$\frac{\partial}{\partial\mu} f(X|\mu) = f(X|\mu)\left(\frac{X-\mu}{\sigma^2}\right)$$

product rule

$$\frac{\partial^2}{\partial\mu^2} f(x|\mu) = \left(\frac{X-\mu}{\sigma^2}\right)\frac{\partial}{\partial\mu} f(x|\mu) + f(x|\mu)\left(-\frac{1}{\sigma^2}\right)$$

$$= \left(\frac{X-\mu}{\sigma^2}\right)^2 f(x|\mu) - \frac{1}{\sigma^2} f(x|\mu) = \left[\left(\frac{X-\mu}{\sigma^2}\right)^2 - \frac{1}{\sigma^2}\right] f(x|\mu)$$

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial\mu^2} f(x|\mu)\, dx = \mathbb{E}\left[\left(\frac{X-\mu}{\delta^2}\right)^2 - \frac{1}{\delta^2} \right] = \frac{1}{\delta^4} E\left[(X-\mu)^2\right] - \frac{1}{\delta^2} = \frac{\delta^2}{\delta^4} - \frac{1}{\delta^2} = 0$$

$$\therefore I_{X_1}(\mu) = -E\left[\frac{\partial^2}{\partial\mu^2} \log L(X_1|\mu)\right]$$

$$L(X_1|\mu) = \frac{1}{\delta\sqrt{2\pi}} \exp\left(-\frac{(X_1-\mu)^2}{2\delta^2}\right) \rightarrow \log L(\mu) = \log\left(\frac{1}{\delta\sqrt{2\pi}}\right) - \frac{(X_1-\mu)^2}{2\delta^2}$$

$$\rightarrow \frac{\partial}{\partial\mu} \log L(\mu) = \frac{X_1-\mu}{\delta^2}$$

$$\rightarrow \frac{\partial^2}{\partial\mu^2} \log L(\mu) = -\frac{1}{\delta^2}$$

$$I_{X_1}(\mu) = -E\left[-\frac{1}{\delta^2}\right] = \frac{1}{\delta^2}.$$

Due to additivity

$$I_{X_1,\dots,X_n}(\mu) = n I_{X_1}(\mu) = \frac{n}{\delta^2}$$

Problem 8.3

Let $X_1, \dots, X_n$ be an i.i.d. sample drawn from the distribution with PDF $f(x|\alpha) = \alpha/x^2$ for $x \geq \alpha$, and zero otherwise. Assume that $\alpha \in (0, \infty)$ is an unknown parameter.

a) Compute the Fisher Information $I_{X_1,\dots,X_n}(\alpha)$ using the definition of Fisher Information.
b) Show that condition (*) is NOT satisfied for the population PDF ~~PMF~~ $f(x|\alpha)$.
c) Show that using the alternative formula for Fisher information and additivity produces an incorrect result for $I_{X_1,\dots,X_n}(\alpha)$.

$$f(x|\alpha) = \frac{\alpha}{x^2} \quad \text{for } x \geq \alpha$$

$$f(x|\alpha) = 0 \quad \text{for } x < \alpha$$

(b) $$\int_{-\infty}^{\infty} \frac{\partial}{\partial\alpha} f(x|\alpha)\, dx = \int_{\alpha}^{\infty} \frac{\partial}{\partial\alpha} f(x|\alpha)\, dx$$

$$= \int_{\alpha}^{\infty} \frac{\partial}{\partial\alpha}\left(\frac{\alpha}{x^2}\right) dx = \int_{\alpha}^{\infty} \frac{dx}{x^2} = \lim_{t\to\infty} \int_{\alpha}^{t} \frac{1}{x^2}\, dx$$

$$= \frac{1}{\alpha} \neq 0$$

Not necessarily true that $I_{X_1,\dots,X_n}(\alpha) = n I_{X_1}(\alpha)$

Problem 8.2

Let $X_1, \dots, X_n$ be an i.i.d. sample drawn from a Poisson distribution $Po(\lambda)$ with parameter $\lambda > 0$.

a) Compute the Fisher Information $I_{X_1,\dots,X_n}(\lambda)$ using the definition of Fisher Information.
b) Show that condition (*) is satisfied for the population PMF $f(x|\lambda)$.
c) Compute $I_{X_1,\dots,X_n}(\lambda)$ using the alternative formula for Fisher information and additivity.

Problem 8.4

Let $X_1, \dots, X_n$ be an i.i.d. sample drawn from a normal distribution $N(\mu, \sigma^2)$ with both parameters μ and σ^2 unknown.

a) Show that the MLE for is σ^2 is

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

where $\bar{X}$ denotes the sample mean.

MLE of $\mu = \bar{X}$

b) Let S^2 denote the sample variance. Show that neither $\hat{\theta}$ nor S^2 is an efficient estimator for σ^2.

c) What information on an unbiased estimator for σ^2 do we get from the Cramer-Rao bound? (We may use without proof that the Cramer-Rao regularity conditions are satisfied for the normal distribution.)

(b) $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

Efficient estimator $\hat{\theta}$: $\hat{\theta}$ is unbiased

$Var(\hat{\theta}) = \frac{1}{n I_{X_1}(\theta)}$

S^2 is unbiased $\Rightarrow E(S^2) = \sigma^2 \Rightarrow$ may be efficient

$E(\hat{\theta}) = E\left[\frac{n-1}{n} S^2\right] = \frac{n-1}{n}\sigma^2 \neq \sigma^2 \Rightarrow$ biased $\Rightarrow$ not efficient.

Let $\theta = \sigma^2$. check $Var(S^2) = \frac{1}{n I_{X_1}(\theta)}$

From Chapter 1: $\frac{(n-1)S^2}{\theta} \sim \chi^2_{n-1} \sim \text{Gamma}\left(\frac{n-1}{2}, 2\right)$ if $X_1, \dots, X_n \sim N(\mu, \sigma^2)$

$Var\left(\frac{(n-1)S^2}{\theta}\right) = \frac{n-1}{2} \cdot 2 \cdot 2 \Rightarrow \frac{(n-1)^2}{\theta^2} Var(S^2) = 2(n-1)$

$\Rightarrow Var(S^2) = \frac{2\theta^2}{n-1}$

$I_{X_1}(\theta) = E\left[\left(\frac{\partial}{\partial\theta}\log L(\theta)\right)^2\right]$

$L(X_1|\theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X_1-\mu)^2}{2\theta}\right)$

$\log L(\theta) = -\frac{1}{2}\log(2\pi\theta) - \frac{(X_1-\mu)^2}{2\theta}$

$(A+B)^2 = A^2 + 2AB + B^2$

$\frac{\partial}{\partial\theta}\log L(\theta) = -\frac{1}{2}\cdot\frac{2\pi}{2\pi\theta} + \frac{(X_1-\mu)^2}{2\theta^2} = \underbrace{-\frac{1}{2\theta}}_{A} + \underbrace{\frac{(X_1-\mu)^2}{2\theta^2}}_{B}$

$$\left(\frac{\partial}{\partial\theta}\log L(\theta)\right)^2 = \frac{1}{4\theta^2} - \frac{(X_1-\mu)^2}{2\theta^3} + \frac{(X_1-\mu)^4}{4\theta^4}$$

$$E\left[\left(\frac{\partial}{\partial\theta}\log L(\theta)\right)^2\right] = \frac{1}{4\theta^2} - \frac{1}{2\theta^3}\underbrace{E\left[(X_1-\mu)^2\right]}_{Var(X_1)} + \frac{1}{4\theta^4}E\left[(X_1-\mu)^4\right]$$

$$= \frac{1}{4\theta^2} - \frac{1}{2\theta^3}\cdot\theta + \frac{1}{4\theta^4}E\left[(X_1-\mu)^4\right]$$

$$= -\frac{1}{4\theta^2} + \frac{1}{4\theta^4}E\left[(X_1-\mu)^4\right]$$

kurtosis

$Y = X_1 - \mu \sim N(0, \sigma^2)$

$$E\left[(X_1-\mu)^4\right] = E\left[Y^4\right] = \int_{-\infty}^{\infty} y^4 \cdot \overbrace{\frac{1}{\sqrt{2\pi\theta}}\exp\left(-\frac{y^2}{2\theta}\right)}^{\text{PDF}} dy = 3\theta^2$$

$$E\left[\left(\frac{\partial}{\partial\theta}\log L(\theta)\right)^2\right] = -\frac{1}{4\theta^2} + \frac{3\theta^2}{4\theta^4} = \frac{1}{2\theta^2} = I_{X_1}(\theta)$$

$$CRLB = \frac{1}{nI_{X_1}(\theta)} = \frac{2\theta^2}{n} < \frac{2\theta^2}{n-1} = Var(S^2)$$

$\therefore$ S^2 is not efficient.

github.com/y-x-y-x/documents.

MH3500 Statistics

Tutorial 8 AY2022/23 Semester 2

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