# Tutorial Question

(4) An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

**Solution**: Define events $E_i$, $i = 1, 2, 3, 4$, as follows:

- $E_1 = \{$the ace of spades is in any one of the piles$\}$;
- $E_2 = \{$the ace of spades and the ace of hearts are in different piles$\}$;
- $E_3 = \{$the aces of spades, hearts, and diamonds are all in different piles$\}$;
- $E_4 = \{$all 4 aces are in different piles$\}$.

$$P(E_1 E_2 E_3 E_4)$$

$$P(E_4) = P(E_4 \cap E_1E_2E_3) + P(E_4 \cap (E_1E_2E_3)')$$

$$= P(E_4 | E_1E_2E_3)\, P(E_1E_2E_3) + P(E_4 | (E_1E_2E_3)')\, P[(E_1E_2E_3)']$$

$$= P(E_4 | E_1E_2E_3)\, P(E_1E_2E_3) + \underbrace{P(E_4 | E_1' \cup E_2' \cup E_3')}_{0}\, P[\quad]$$

$$= P(E_4 | E_1E_2E_3)\, P(E_3 | E_1E_2)\, P(E_2 | E_1)\, P(E_1)$$ (multiplicative rule)

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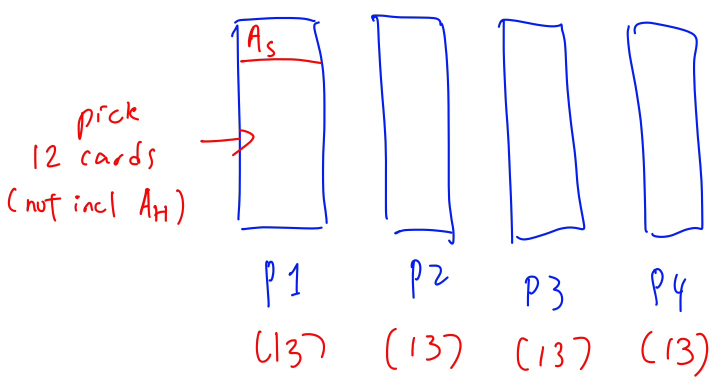
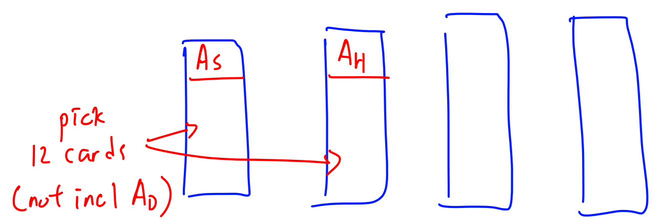
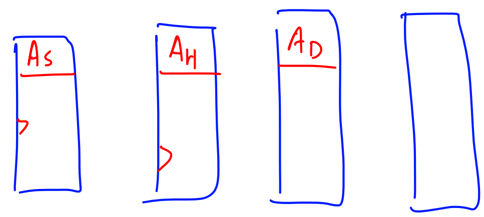



$$P(E_1) = 1$$

$$P(E_2 | E_1) = \frac{39}{51} = \frac{\binom{50}{12}}{\binom{51}{12}}$$




$$P(E_3 | E_1E_2) = \frac{26}{50} = \frac{\binom{49}{12}\binom{37}{12}}{\binom{50}{12}\binom{38}{12}}$$

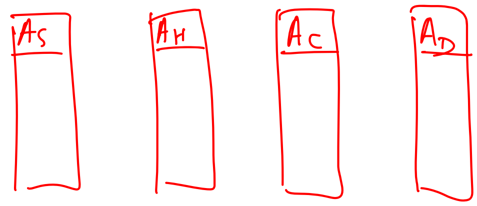



$$P(E_4 | E_1E_2E_3) = \frac{13}{49}$$

$$P(E_1 E_2 E_3 E_4) = P(E_4 \mid E_1 E_2 E_3)\, P(E_3 \mid E_1 E_2)\, P(E_2 \mid E_1)\, P(E_1)$$

$$= \frac{13}{49} \cdot \frac{26}{50} \cdot \frac{39}{51}$$

## Alternative Method




$$\frac{4! \binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}} = \frac{4! \dfrac{48!}{12!\,12!\,12!\,12!}}{\dfrac{52!}{13!\,13!\,13!\,13!}}$$

Practice Question

(1) The probability that a new car battery functions for over 10,000 miles is 0.8, the probability that it functions for over 20,000 miles is 0.4, and the probability that it functions for over 30,000 miles is 0.1. If a new car battery is still working after 10,000 miles, what is the probability that

(a) its total life will exceed 20,000 miles?
(b) its additional life will exceed 20,000 miles?

Let $X$ be the distance for a new car battery to function.

$P(X > 10000) = 0.8$

$P(X > 20000) = 0.4$

$P(X > 30000) = 0.1$

(a) $P(X > 20000 \mid X > 10000)$

$$= \frac{P(X > 20000 \cap X > 10000)}{P(X > 10000)}$$

$$= \frac{P(X > 20000)}{P(X > 10000)} = \frac{1}{2}$$

(b) $P(X > 30000 \mid X > 10000) = \frac{1}{8}$

(2) An urn $A$ contains 2 white balls and 1 black ball, whereas urn $B$ contains 1 white ball and 5 black balls. A ball is drawn at random from urn $A$ and placed in urn $B$. A ball is then drawn from urn $B$. It happens to be white.

What is the probability that the ball transferred was white?

| 2 W<br>1 B | 1 W<br>5 B |
|---|---|
| A | B |

$B$: Ball drawn from urn $B$ is white

$T$: Ball transferred is white.

$$P(T \mid B) = \frac{P(T \cap B)}{P(B)} = \frac{P(B \mid T)\,P(T)}{P(B \mid T)\,P(T) + P(B \mid T')\,P(T')}$$

$$= \frac{\frac{2}{7} \cdot \frac{2}{3}}{\frac{2}{7} \cdot \frac{2}{3} + \frac{1}{7} \cdot \frac{1}{3}} = \frac{4}{5}$$

(3) A friend randomly chooses two cards, without replacement, from an ordinary deck of 52 playing cards. In each of the following situations, determine the conditional probability that both cards are aces.

(a) You ask your friend if one of the cards is the ace of spades, and your friend answers in the affirmative.
(b) You ask your friend if the first card selected is an ace, and your friend answers in the affirmative.
(c) You ask your friend if the second card selected is an ace, and your friend answers in the affirmative.
(d) You ask your friend if either of the cards selected is an ace, and your friend answers in the affirmative.

(a) $P(\text{both are aces} \mid \text{one is ace of spade}) = \dfrac{\binom{1}{1}\binom{3}{1}/\binom{52}{2}}{\binom{1}{1}\binom{51}{1}/\binom{52}{2}} = \dfrac{1}{17}$

(b) $P(\text{both are aces} \mid \text{first card is ace}) = \dfrac{3}{51} = \dfrac{1}{17}$

$= \dfrac{P(\text{both are aces} \cap \text{first is ace})}{P(\text{first is ace})} = \dfrac{\binom{4}{2}/\binom{52}{2}}{4/52 \times \frac{51}{51}} = \dfrac{1}{17}$

(c) $P(\text{both are aces} \mid \text{second is ace}) = \dfrac{1}{17}$

$P(\text{second is ace}) = P(\text{second is ace} \mid \text{first is ace})\, P(\text{first is ace})$

$+ P(\text{second is ace} \mid \text{first is not ace})\, P(\text{first is not ace})$

$= \dfrac{3}{51} \cdot \dfrac{4}{52} + \dfrac{4}{51} \cdot \dfrac{48}{52}$

$\therefore \dfrac{\binom{4}{2}/\binom{52}{2}}{\frac{3}{51} \cdot \frac{4}{52} + \frac{4}{51} \cdot \frac{48}{52}} = \dfrac{12}{12 + 192} = \dfrac{1}{17}$

(d) $P(\text{both aces} \mid \geq 1 \text{ ace}) = \dfrac{\binom{4}{2}/\binom{52}{2}}{1 - P(\text{no ace})}$

$$= \frac{\binom{4}{2}/\binom{52}{2}}{1 - \binom{48}{2}/\binom{52}{2}}$$

$$= \frac{\cancel{12}\,3}{\underset{13}{\cancel{52}} \times 51 - \underset{12}{\cancel{48}} \times 47} = \frac{3}{663 - 564} = \frac{1}{33}$$

(4) A type $C$ battery is in working condition with probability 0.7, whereas a type $D$ battery is in working condition with probability 0.4. A battery is randomly chosen from a bin consisting of 8 type $C$ and 6 type $D$ batteries.

(a) What is the probability that the battery works?
(b) Given that the battery does not work, what is the conditional probability that it was a type $C$ battery?

$C$ : Type c battery chosen
$D$ : Type d battery chosen
$W$ : Battery works.

(a) $$P(W) = P(W|C)\,P(C) + P(W|D)\,P(D)$$

$$= 0.7 \cdot \frac{8}{14} + 0.4 \cdot \frac{6}{14}$$

$$= \frac{4}{7}$$

(b) $$P(C|W') = \frac{P(W'|C)\,P(C)}{P(W')} = \frac{0.3 \cdot \frac{8}{14}}{\frac{3}{7}} = \frac{3}{10} \times \frac{8}{6} = \frac{2}{5}$$

(5) Two local factories, $A$ and $B$, produce radios. Each radio produced at factory A is defective with probability 0.05, whereas each one produced at factory $B$ is defective with probability 0.01. Suppose you purchase two radios that were produced at the same factory, which is equally likely to have been either factory $A$ or factory $B$. If the first radio that you check is defective, what is the conditional probability that the other one is also defective?

$D_1$ : first radio defective

$D_2$ : second radio defective

$A$ : radios purchased from $A$

$B$ : radios purchased from $B$.

$$P(D_2 \mid D_1) = \frac{P(D_2 D_1)}{P(D_1)} = \frac{P(D_2 D_1 \mid A) P(A) + P(D_2 D_1 \mid B) P(B)}{P(D_1 \mid A) P(A) + P(D_1 \mid B) P(B)}$$

$$= \frac{0.05^2 \cdot 0.5 + 0.01^2 \cdot 0.5}{0.05 \cdot 0.5 + 0.01 \cdot 0.5}$$

$$= \frac{0.05^2 + 0.01^2}{0.05 + 0.01} = \frac{26}{600} = \frac{13}{300}$$