Extra Exercises for Week 5

MH2500

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Most questions are taken from the textbook: A First Course in Probability (9th edition) by Sheldon Ross.

Problem 1. Let X be a random variable such that

$$P(X = 1) = p = 1 - P(X = -1).$$

Find $c \neq 1$ such that $\mathbb{E}(c^X) = 1$.

Problem 2. Suppose the CDF of X is given by

$$F(x) = \begin{cases} 0, & x < 1; \\ 1/3, & 1 \leq x < 2; \\ 5/6, & 2 \leq x < 4; \\ 1, & x \geq 4. \end{cases}$$

Find $P(1 < X < 2)$ and $P(2 \leq X < 3)$.

Problem 3. Let X be a random variable with a CDF F. What is the CDF for e^X in terms of F?

Problem 4. Two coins are to be flipped. The first coin will land on heads with probability 0.6, the second with probability 0.7. Assume that the flips are independent, what is the expected number of heads appearing?

Problem 5. How many people in a group are needed so that the probability that at least one of them has the same birthday as you is greater than 0.99?

Problem 6. How many people in a group are needed so that the probability of at least two people sharing the same birthday is more than 0.99? You can't compute the answer by hand but should arrive at the inequality

$$\frac{P(365, n)}{365^n} \leq 0.01.$$

Problem 7. How many people in a group are needed to ensure there will be at least two people sharing the same birthday?

Problem 8. Let $X \sim \mathrm{Bin}(n, p)$. Show that

$$\mathbb{E}\left(\frac{1}{X+1}\right) = \frac{1 - (1-p)^{n+1}}{(n+1)p}.$$

Problem 9. Suppose that $P(X = 0) = 1 - P(X = 1)$. If $\mathbb{E}(X) = 3\mathrm{Var}(X)$, find $P(X = 0)$.

Problem 10. Suppose that

$$P(X = a) = p, \qquad P(X = b) = 1 - p.$$

Show that $\frac{X-b}{a-b}$ is a Bernoulli random variable.

Problem 11. We throw a coin until a head turns up for the k-th time, where p is the probability that a throw results in a head and we assume that the outcome of each throw is independent. Let X be the number of times we have thrown the coin. Write $P(X = n)$ in terms of k and p, for $n \geq k$.

Answers (Let me know if there are any discrepancies):

1. $\frac{1-p}{p}$
2. $0, \frac{1}{2}$
3. $F \circ \ln$
4. 1.3
5. 1679
6. Largest n that satisfies $\frac{P(365,n)}{365^n} \leq 0.01$. Using a computer, $n = 57$.
7. 366
9. 1 or $\frac{1}{3}$
11. $\binom{n-1}{k-1} p^k (1-p)^{n-k}$

Below is the solution to Problem 8.

$$
\begin{aligned}
\mathbb{E}\left(\frac{1}{X+1}\right) &= \sum_{i=0}^{n} \frac{1}{i+1} P(X=i) \\
&= \sum_{i=0}^{n} \frac{1}{i+1} \cdot \binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=0}^{n} \frac{1}{i+1} \cdot \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} \\
&= \sum_{i=0}^{n} \frac{n!}{(i+1)!(n-i)!} p^i (1-p)^{n-i} \\
&= \sum_{i=0}^{n} \frac{\frac{(n+1)!}{n+1}}{(i+1)!(n-i)!} \frac{p^{i+1}}{p} (1-p)^{n-i} \\
&= \frac{1}{(n+1)p} \sum_{i=0}^{n} \binom{n+1}{i+1} p^{i+1} (1-p)^{n-i} \\
&= \frac{1}{(n+1)p} \sum_{j=1}^{n+1} \binom{n+1}{j} p^j (1-p)^{n+1-j} \\
&= \frac{1}{(n+1)p} \left(1 - \binom{n+1}{0} p^0 (1-p)^{n+1}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}.
\end{aligned}
$$

At the last row, we use the fact that

$$\sum_{j=0}^{n+1} \binom{n+1}{j} p^j (1-p)^{n+1-j} = 1.$$