Problem 2.1

Let X be a random variable with the following cumulative distribution function (CDF).

$$F(x) = \begin{cases} 1 - e^{-(x/\alpha)^\beta} & for\ x > 0 \\ 0 & otherwise \end{cases}$$

where α and $\beta > 0$ are constants. The distribution of X is called a **Weibull distribution**.

a) Find the PDF of the random variable X.
b) Let $Y = (X/\alpha)^\beta$. Identify the distribution of Y.

a) For $x>0$, $f(x) = \frac{d}{dx}\left(1 - e^{-(\frac{x}{\alpha})^\beta}\right)$

$= -e^{-(\frac{x}{\alpha})^\beta} \cdot \frac{d}{dx}\left(-\frac{x^\beta}{\alpha^\beta}\right)$

$= \dfrac{\beta e^{-(\frac{x}{\alpha})^\beta}}{\alpha^\beta x^{1-\beta}}$

For $x \le 0$, $f(x) = 0$.

b) $F_Y(y) = P(Y \le y) = P\left(\frac{X^\beta}{\alpha^\beta} \le y\right) = P\left(X \le \alpha y^{\frac{1}{\beta}}\right) = 1 - e^{-\left(\frac{\alpha y^{\frac{1}{\beta}}}{\alpha}\right)^\beta}$

$= 1 - e^{-y}$

$f_Y(y) = e^{-y}$ for $y > 0$.

$= 0$ otherwise.

$\Rightarrow Y \sim Exp(1)$

Problem 2.2

Let $X \sim Exp(\lambda)$. Determine the distributions of $Y_1 = \lambda X$ and $Y_2 = 2\lambda X$.

$M_{Y_1}(t) = M_X(\lambda t) = \frac{\lambda}{\lambda - \lambda t} = \frac{1}{1-t} \Rightarrow Gamma(1, 1)$, $t < 1$. $\sim Exp(1)$

$M_{Y_2}(t) = M_X(2\lambda t) = \frac{\lambda}{\lambda - 2\lambda t} = \frac{1}{1-2t} \Rightarrow Gamma(1, 2)$, $t < \frac{1}{2}$.

$Gamma(1, \frac{1}{\lambda}) \equiv Exp(\lambda)$

Problem 2.3

Let $X_1, \ldots, X_5$ be i.i.d. $\sim N(0,1)$ and let $\bar{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$ be the sample mean. Let $Y \sim N(0,1)$ be another random variable which is independent from $X_1, \ldots, X_5$. Find the distributions of the following random variables and justify your answers.

a) Random variable $W = \sum_{i=1}^{5} X_i^2$
b) Random variable $U = \sum_{i=1}^{5} (X_i - \bar{X})^2$
c) Random variable $2Y/\sqrt{U}$
d) Random variable $2(5\bar{X}^2 + Y^2)/U$

$T = \dfrac{C}{\sqrt{\frac{D}{k}}}$ $\quad C \sim N(0,1)$ $\quad D \sim \chi_k^2$.

(a) $W \sim \chi_5^2$ $\qquad \bar{X} \sim N(0, \frac{1}{5})$

(b) $S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$ $\qquad \mu = 0,\ \sigma^2 = 1$

$U = 4S^2 \sim \chi_4^2$

(c) $\dfrac{2Y}{\sqrt{U}} \sim \dfrac{N(0,1)}{\sqrt{\frac{\chi_4^2}{4}}} = t_4$ $\qquad \dfrac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0,1)$

(d) $\bar{X} \sim N(0, \frac{1}{5}) \Rightarrow \sqrt{5}\bar{X} \sim N(0,1) \Rightarrow 5\bar{X}^2 \sim \text{Gamma}(\frac{1}{2}, 2)$.
$Y \sim N(0,1) \Rightarrow Y^2 \sim \text{Gamma}(\frac{1}{2}, 2)$

Let $D = 5\bar{X}^2 + Y^2$ $\quad$ ($\because \bar{X}, Y$ independent).

$M_D(t) = (1-2t)^{-\frac{1}{2}} (1-2t)^{-\frac{1}{2}} = (1-2t)^{-1}$

$D \sim \text{Gamma}(1, 2) \sim \chi_2^2$

$\dfrac{2(5\bar{X}^2 + Y^2)}{U} \sim \dfrac{2\chi_2^2}{\chi_4^2} = \dfrac{\frac{\chi_2^2}{2}}{\frac{\chi_4^2}{4}} \sim F(2,4)$

Problem 2.4

Let $X_1, \dots, X_{16}$ be i.i.d. $\sim N(0,1)$ and let $\bar{X}$ be the sample mean.

a) Find the constant c such that $P(|\bar{X}| < c) = 0.5$.
b) Find the mean and the variance of the sample variance S^2.

(a) $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1) \Rightarrow 4\bar{X} \sim N(0,1)$

$P(|\bar{X}|<c) = P(-4c < 4\bar{X} < 4c)$

$= 2\Phi(4c) - 1 = 0.5$

$\Phi(4c) = 0.75 \Rightarrow 4c = 0.674 \Rightarrow c = 0.169.$

(b) $\frac{15S^2}{\sigma^2} \sim \chi_{15}^2 \Rightarrow 15S^2 \sim \text{Gamma}\left(\frac{15}{2}, 2\right)$

$E(15S^2) = 15 \Rightarrow E(S^2) = 1.$

$Var(15S^2) = \frac{15}{2}(2)(2) = 30 \Rightarrow Var(S^2) = \frac{2}{15}.$

Problem 2.5

Let $X_1, \ldots, X_n \sim \text{Poisson}(1)$ be an i.i.d. random sample. For each $n \in \{10, 100, 1000, 10000\}$, compute the following.

a) The exact probability $\Pr(\sum_{i=1}^{n} X_i \leq n)$ for each n.
b) The approximation of $\Pr(\sum_{i=1}^{n} X_i \leq n)$ obtained from the Central Limit Theorem (CLT).

Hint: note that $\sqrt{n}(\bar{X} - \mu)/\sigma$ converges in distribution to $N(0, 1)$ for $n \to \infty$ where $\mu = E[X_1]$ and $\sigma = Var[X_1]$.

(a) $X_1 + \ldots + X_n \sim Po(n)$.

$\Pr(\sum_{i=1}^{10} X_i \leq 10) = P(Y \leq 10) = 0.583$

$\Pr(\sum X_i \leq 100) = 0.527$

$\Pr(\sum X_i \leq 1000) = 0.50841$

$\Pr(\sum X_i \leq 10^4) = 0.50266.$

(b) $\mu = E(X_1) = 1 \quad \sigma = \sqrt{Var(X_1)} = 1$

$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \sim N(1, 1)$

$P(\sum X_i \leq n) = P(n\bar{X} \leq n)$

$= P(\bar{X} \leq 1)$

$= P(Z \leq 0) = 0.5$ for large $n \geq 30$.

MH3500 Statistics

Tutorial 2 AY2022/23 Semester 2

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