Tutorial Question

(2) Suppose that the distribution function of X is given by

$$F(b) = \begin{cases} 0, & b < 0 \\ \frac{b}{4}, & 0 \le b < 1 \\ \frac{1}{2} + \frac{b-1}{4}, & 1 \le b < 2 \\ \frac{11}{12}, & 2 \le b < 3 \\ 1, & 3 \le b \end{cases}$$

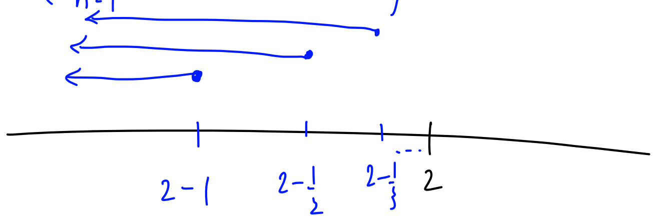
(a) Find $\mathbb{P}r\{X = i\}$, $i = 1, 2, 3$.
(b) Find $\mathbb{P}r\{\frac{1}{2} < X < \frac{3}{2}\}$.

$$F(b) = P(\underset{\substack{\downarrow \\ \text{random variable}}}{X} \le b)$$

cumulative distribution func.

(a) $P(X=2) = P(X \le 2) - P(X < 2)$

$$= F(2) - P\left(\bigcup_{n=1}^{\infty} \{X \le 2 - \tfrac{1}{n}\}\right)$$



If $A_1, A_2, \ldots, A_n, \ldots$ are events

s.t. $A_k \subseteq A_{k+1}$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_i)$$

$$= F(2) - \lim_{n\to\infty} P(X \le 2 - \tfrac{1}{n})$$

$$= F(2) - \lim_{n\to\infty} F(2 - \tfrac{1}{n}) = \frac{11}{12} - \lim_{n\to\infty}\left(\frac{1}{2} + \frac{2-\frac{1}{n}-1}{4}\right)$$

$$= \frac{1}{6}$$

$$F(b) = \begin{cases} 0, & b < 0 \\ \frac{b}{4}, & 0 \le b < 1 \\ \frac{1}{2} + \frac{b-1}{4}, & 1 \le b < 2 \\ \frac{11}{12}, & 2 \le b < 3 \\ 1, & 3 \le b \end{cases}$$

$$P\left(\tfrac{1}{2} < X < \tfrac{3}{2}\right) = P\left(X < \tfrac{3}{2}\right) - P\left(X \le \tfrac{1}{2}\right)$$

$$= \lim_{n\to\infty} P\left(\tfrac{3}{2} - \tfrac{1}{n}\right) - F\left(\tfrac{1}{2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{2} + \frac{\frac{3}{2} - \frac{1}{n} - 1}{4}\right) - \frac{\frac{1}{2}}{4} = \frac{1}{2}$$

(3) Suppose that two teams play a series of games that ends when one of them has won i games. Suppose that each game played is, independently, won by team A with probability p.

lose with probability $1-p$.

Find the expected number of games that are played when

(a) $i = 2$ and

(b) $i = 3$.

Also, show that in both cases that this number is maximized when $p = 1/2$.

(b) Let X be the number of games played. Find $E(X)$

$X \in \{3, 4, 5\}$

$$P(X=3) = \underset{AAA}{p^3} + \underset{BBB}{(1-p)^3}$$

$$P(X=4) = \binom{3}{2}(1-p)p^3 + \binom{3}{2}p(1-p)^3$$

BAAA (can swap) ABBB (can swap)

$$P(X=5) = \frac{4!}{2!\,2!}p^3(1-p)^2 + \frac{4!}{2!\,2!}(1-p)^3p^2$$

AABBA (can swap) BBAAB (can swap)

$$E(X) = \sum_x x P(X=x) = 3\cdot P(X=3) + 4\cdot P(X=4) + 5\cdot P(X=5)$$

$$= 3(1+p+p^2-4p^3+2p^4)$$ function in p.

$$\frac{d}{dp}E(X) = 0$$ show that $p = \frac{1}{2}$.

$$\frac{d}{dp}E(X) = 3(1+2p-12p^2+8p^3) = 0$$

$$8p^3 - 12p^2 + 2p + 1 = 0.$$

$$0 \le p \le 1$$

Practice Question

(3) Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on.

Let X denote the number of times player 1 is a winner. Find $\mathbb{P}r\{X = i\}$, for $i = 0, 1, 2, 3, 4$.

Five distinct numbers $= \{1, 2, 3, 4, 5\}$.

1 2 3 4 5

$$P(X=4) = \frac{4!}{5!}$$

5 _ _ _ _

$$P(X=3) = \frac{3!}{5!}$$

4 _ _ _ 5

$\underbrace{\quad\quad}_{\text{win}}$ lose

$$P(X=2) = \frac{3!}{5!} + \frac{2! \times 2!}{5!}$$

4 _ _ 5 _

3 _ _ _ _

1 or 2 — 4 or 5

$$P(X=1) = \frac{3!}{5!} + \frac{2^3}{5!} + \frac{3!}{5!}$$

4 _ 5 _ _

1 or 2 — 4 or 5

3 _ _ _ _

2 × 2 × 1 × 1

2 1 _ _ _

$$P(X=0) =$$

$\frac{4!}{5!} \leftarrow$ 1 _ _ _ _

3 3 2 1

$\frac{18}{5!} \leftarrow$ 2 _ _ _ _

3,4 or 5

2 3 2 1

$\frac{12}{5!} \swarrow$ 3 _ _ _ _

4 or 5

4 5 _ _ _ $\rightarrow \frac{3!}{5!}$

(6) You have \$1000, and a certain commodity presently sells for \$2 per ounce. Suppose that after one week the commodity will sell for either \$1 or \$4 an ounce, with these two possibilities being equally likely.

(a) If your objective is to maximize the expected amount of money that you possess at the end of the week, what strategy should you employ?

(b) If your objective is to maximize the expected amount of the commodity that you possess at the end of the week, what strategy should you employ?

(a) X : amount of money at the end of week.

k : amount of commodity to start with

With probability $\frac{1}{2}$ $\Rightarrow$ comm will sell at \$1.

$\Rightarrow$ amount of money at the end of week $= k + (1000-2k)$

sell everything at the end of week

amount of money remaining from the start

w.p. $\frac{1}{2}$ $\Rightarrow$ comm will sell at \$4 $\Rightarrow$ $4k + (1000-2k)$

$$E(X) = \frac{1}{2}\left[k + (1000-2k)\right] + \frac{1}{2}\left[4k + (1000-2k)\right] = 1000 + \frac{k}{2}$$

Max $E(X)$ $\Leftrightarrow$ max k. $\Rightarrow$ use all money to buy comm. at present.

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(b) If your objective is to maximize the expected amount of the commodity that you possess at the end of the week, what strategy should you employ?

(b) Let Y be the amount of commodity at the end of week.
k be the amount of commodity to start with.

With probability $\frac{1}{2}$: comm selling at $1

$$Y = \underbrace{k}_{\text{amount that you have from the start}} + \underbrace{(1000 - 2k)}_{\text{remaining money from the start use it to buy comm at \$1}}$$

With probability $\frac{1}{2}$: comm selling at $4

$$Y = k + \left\lfloor \frac{1000 - 2k}{4} \right\rfloor$$

remaining money from the start use all to buy comm at $4.
Take floor function because you can only buy in integer amount.

$$\therefore E(Y) = \frac{1}{2}(k + 1000 - 2k) + \frac{1}{2}\left(k + \left\lfloor \frac{1000-2k}{4} \right\rfloor\right)$$

$$= 500 + \frac{1}{2}\left\lfloor \frac{1000 - 2k}{4} \right\rfloor$$

$\max E(Y) \iff \min k \Rightarrow$ buy nothing at the beginning.