Tutorial Week 4 Question 4

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Here, we will go through Tutorial Week 4 Question 4 in details and I will provide a few alternative methods to approach this question.

Method 1. Same as the solution provided in the tutorial. We define events $E_i, i = 1, 2, 3, 4$ as follows

- $E_1 = \{\text{the ace of spades is in any one of the piles}\}$;
- $E_2 = \{\text{the ace of spades and the ace of hearts are in different piles}\}$;
- $E_3 = \{\text{the aces of spades, hearts, and diamonds are all in different piles}\}$;
- $E_4 = \{\text{all 4 aces are in different piles}\}$.

We want to compute $P(E_4)$. By conditioning on the event $E_1E_2E_3$ and using De Morgan's Law, we have

$$\begin{aligned} P(E_4) &= P(E_4 \cap E_1E_2E_3) + P(E_4 \cap (E_1E_2E_3)') \\ &= P(E_4|E_1E_2E_3)P(E_1E_2E_3) + \underbrace{P(E_4|E_1' \cup E_2' \cup E_3')}_{=0} P(E_1' \cup E_2' \cup E_3'). \end{aligned}$$

The underbraced term is 0 because, if one of events (E_1, E_2 or E_3) do not occur, then E_4 will never occur. By the multiplication rule for conditional probability, we have

$$P(E_4) = P(E_4|E_1E_2E_3)P(E_1E_2E_3) = P(E_4|E_1E_2E_3)P(E_3|E_1E_2)P(E_2|E_1)P(E_1).$$

Clearly, $P(E_1) = 1$ from the way we define the event. For $P(E_2|E_1)$, to make sure $A\spadesuit$ and $A\heartsuit$ are in different piles, apart from these two cards, there are 50 cards, we choose 12 cards and put them with $A\spadesuit$. This forces the two cards to not be in the same pile. In total, the size of the sample space for this conditional event is $\binom{51}{12}$ and the probability will be

$$P(E_2|E_1) = \frac{\binom{50}{12}}{\binom{51}{12}} = \frac{39}{51}.$$

For $P(E_3|E_1E_2)$, we want to make sure $A\spadesuit$, $A\heartsuit$ and $A\diamondsuit$ are in different piles while $A\spadesuit$, $A\heartsuit$ are already in different piles. Apart from these 3 cards, we have 49 cards, we choose 12 to put them with $A\spadesuit$ and again choose 12 from the remaining to put them with $A\heartsuit$. This ensures that the three cards to not be in the same pile. There are $\binom{49}{12}\binom{37}{12}$ possible combinations. The size of this sample space is $\binom{50}{12}\binom{38}{12}$ where no restrictions are imposed when choosing the cards. Hence,

$$P(E_3|E_1E_2) = \frac{\binom{49}{12}\binom{37}{12}}{\binom{50}{12}\binom{38}{12}} = \frac{26}{50}.$$

For $P(E_4|E_1E_2E_3)$, the idea is similar as above and we should have

$$P(E_4|E_1E_2E_3) = \frac{\binom{48}{12}\binom{36}{12}\binom{24}{12}}{\binom{49}{12}\binom{37}{12}\binom{25}{12}} = \frac{13}{49}.$$

Therefore,

$$P(E_4) = P(E_1E_2E_3E_4) = \frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49}.$$

Method 2. We define events $E_i, i = 1,2,3,4$, where $E_i = \{i\text{-th pile has exactly 1 ace}\}$. We aim to compute $P(E_1E_2E_3E_4)$. By the multiplication rule for conditional probability, we have

$$P(E_1E_2E_3E_4) = P(E_4|E_1E_2E_3)P(E_3|E_1E_2)P(E_2|E_1)P(E_1).$$

Clearly, $P(E_4|E_1E_2E_3) = 1$, because if the first 3 piles each has exactly 1 ace, then the fourth pile must have exactly 1 ace. Next, to make sure that first pile has exactly 1 ace, we have

$$P(E_1) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}.$$

For $P(E_2|E_1)$, if the first pile has been arranged with exactly an ace, to make sure second pile also having exactly an ace, there are $\binom{3}{1}\binom{36}{12}$ outcomes. Size of sample space in this conditional event is thus $\binom{39}{13}$, since given that the first pile has been arranged. Therefore,

$$P(E_2|E_1) = \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}}.$$

For $P(E_3|E_1E_2)$, the idea is similar as above and we should have

$$P(E_3|E_1E_2) = \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}}.$$

Therefore,

$$P(E_1E_2E_3E_4) = \frac{4\cdot 3\cdot 2\cdot \binom{48}{12}\binom{36}{12}\binom{24}{12}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}} = \frac{39}{51}\cdot\frac{26}{50}\cdot\frac{13}{49}.$$

Method 3. We fix the four aces each in one pile and choose another 12 cards for each pile. Discarding four aces, we have 48 cards and to partition it into 4 piles, we have a total of $\binom{48}{12,12,12,12}$ possible combinations. However, we can still swap the four aces among different piles which contributes another 4!. Of course, the size of the sample space here is $\binom{52}{13,13,13,13}$, i.e., to distribute 52 cards into 4 piles of 13 each. Hence the required probability is

$$\frac{4!\cdot\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}} = \frac{4\cdot 3\cdot 2\cdot \binom{48}{12}\binom{36}{12}\binom{24}{12}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}} = \frac{39}{51}\cdot\frac{26}{50}\cdot\frac{13}{49}.$$

Method 4. Suppose we line up the 52 cards in a row. The first set of 13 cards form the first pile. The second set of 13 cards form the second pile, and so on. We will have four piles of card. Now, to make sure that aces appear in different pile, each ace can only take up one of the 13 entries. For example, only one ace in the first 13 cards, only one ace in the next 13 cards, and so on. So, the probability is

$$\frac{13^4}{\binom{52}{4}} = \frac{39}{51}\cdot\frac{26}{50}\cdot\frac{13}{49}.$$