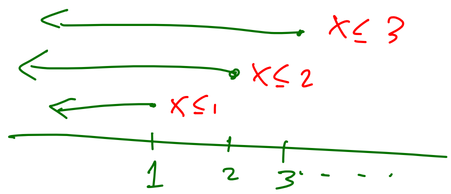
Tutorial Question

(2) An urn initially contains one red and one blue ball. At each stage, a ball is randomly chosen and then replaced along with another of the same colour. Let X denote the number of times needed until a blue ball is taken.

- For instance, if the first ball we get is red and the second one is blue, then X has value 2.

(a) Find $\mathbb{P}r\{X > i\}$ for $i \geq 1$.
(b) Prove that the proability of the event that a blue ball is eventually chosen is 1.
(c) Find $\mathbb{E}(X)$.



draw $>i$ times to get blue ball.

(a) $P(X > i)$

$= P($ first i draws are all red $)$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{4} \cdots \frac{i}{i+1} = \frac{1}{i+1}$$

1st   2nd   ith

(b) Show $P(X < \infty) = 1$

$$\{X < \infty\} = \bigcup_{n=1}^{\infty} \{X \leq n\}$$

$$P(X<\infty) = P\left(\bigcup_{n=1}^{\infty} \{X \leq n\}\right)$$

$$= \lim_{n \to \infty} P(X \leq n)$$

$$= \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

(c) $$\mathbb{E}(X) = \sum_{n=1}^{\infty} n \, P(X = n)$$

$P(X=n) = P($ first $n-1$ draws are red, nth draw is blue $)$

$$= \underbrace{\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdots \frac{\cancel{n-1}}{n}}_{\text{red}} \cdot \underbrace{\frac{1}{n+1}}_{\text{blue}} = \frac{1}{n(n+1)}$$

$\boxed{n \geq 1}$

$$= \sum_{n=1}^{\infty} n \cdot \frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n} = +\infty$$

By p-series test

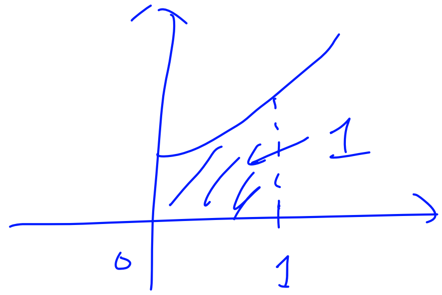
$\sum \frac{1}{n^p}$ diverges if $p \leq 1$.

(4) The probability density function of X is given by

$$f(x) = \begin{cases} a + bx^2, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

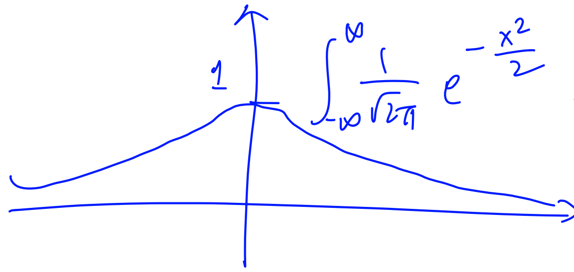
If $\mathbb{E}(X) = 3/5$, find a and b.

$$\int_0^1 f(x)\,dx = 1$$



$$\int_0^1 (a + bx^2)\,dx = \left[ax + \frac{b}{3}x^3\right]_0^1$$

$$= \boxed{a + \frac{b}{3} = 1}.$$



$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = 1$$

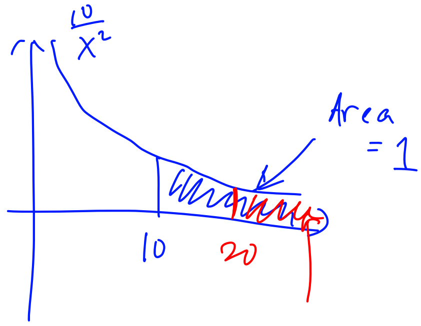
$$\mathbb{E}(X) = \int_0^1 x f(x)\,dx = \int_0^1 ax + bx^3\,dx = \left[\frac{a}{2}x^2 + \frac{b}{4}x^4\right]_0^1 = \boxed{\frac{a}{2} + \frac{b}{4} = \frac{3}{5}}$$

$$\Rightarrow a = \frac{3}{5}, \quad b = \frac{6}{5}$$

(5) The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

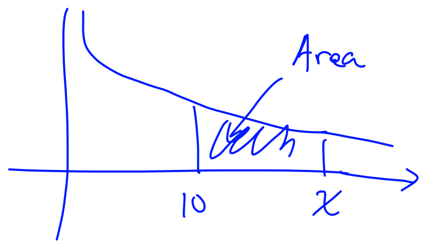
$$f(x) = \begin{cases} \frac{10}{x^2}, & \text{if } x > 10; \\ 0, & \text{if } x \le 10. \end{cases}$$

(a) Find $\mathbb{P}r\{X > 20\}$.
(b) What is the cumulative distribution function of X?
(c) What is the probability that of 6 such types of devices at least 3 will function for at least 15 hours? We assume the independence of the devices.



(a) $P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2}\,dx = \left[-\frac{10}{x}\right]_{20}^{\infty} = \frac{1}{2}$

(b) CDF of X, $F_X(x) = P(X \le x)$



$= \int_{10}^{x} \frac{10}{t^2}\,dt = \left[-\frac{10}{t}\right]_{10}^{x}$

$= 1 - \frac{10}{x}$

$$F_X(x) = \begin{cases} 1 - \frac{10}{x} & x \ge 10 \\ 0 & x < 10 \end{cases}$$

(c) $P(X \ge 15) = \int_{15}^{\infty} \frac{10}{x^2}\,dx$ or $1 - P(X \le 15)$

$= 1 - F_X(15)$

$= 1 - \left(1 - \frac{10}{15}\right) = \frac{2}{3}$

6 devices, ≥ 3 to function ≥ 15 hours.

Let Y denote the num of devices that can function ≥ 15 hours.

$Y \sim Bin\left(6, \frac{2}{3}\right)$, $P(Y \ge 3) = \underline{\sum_{k=3}^{6} \binom{6}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{6-k}}$

Practice Question

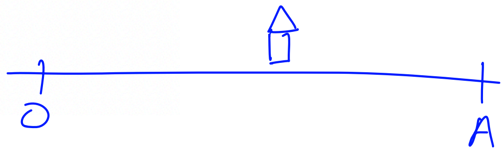
(1) A fire station is to be located along a road of length A, $A < \infty$. If fires occur at points uniformly chosen on $(0, A)$, where should the station be located so as to minimize the expected distance from the fire? That is, choose a so as to

$$\text{minimize } \mathbb{E}[|X - a|]$$

when X is uniformly distributed over $(0, A)$.

fire

fire station



$X \sim \text{Unif}(0, A)$, density function of X : $f_X(x) = \begin{cases} \frac{1}{A} & x \in (0, A) \\ 0 & \text{otherwise} \end{cases}$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$|x - a| = \begin{cases} x - a & x \geq a \\ a - x & x < a \end{cases}$$

$$\mathbb{E}[|X - a|] = \int_{-\infty}^{\infty} |x - a| f_X(x)\, dx = \int_0^A |x - a| \frac{1}{A}\, dx$$

$$= \int_a^A (x - a)\frac{1}{A}\, dx + \int_0^a (a - x)\frac{1}{A}\, dx$$

$$= -\int_A^a (x - a)\frac{1}{A}\, dx + \int_0^a (a - x)\frac{1}{A}\, dx$$

$$= -\frac{1}{A}\left[\frac{1}{2}x^2 - ax\right]_A^a + \frac{1}{A}\left[ax - \frac{1}{2}x^2\right]_0^a$$

$$= -\frac{1}{A}\left(-\frac{1}{2}a^2 - \frac{1}{2}A^2 + aA\right) + \frac{1}{A}\left(a^2 - \frac{1}{2}a^2\right)$$

$$= \frac{1}{2A} \cdot A^2 - a + \frac{a^2}{A}$$

$$\frac{d}{da}\mathbb{E}[|X - a|] = -1 + \frac{2a}{A} = 0 \Rightarrow a = \frac{A}{2}$$

(4) One thousand independent rolls of a fair die will be made. Compute an approximation to the probability that the number 6 will appear between 150 and 200 times inclusively. If the number 6 appears exactly 200 times, find the probability that the number 5 will appear less than 150 times.

n p

Let X be the num of times that '6' appears. $X \sim Bin(1000, \frac{1}{6})$

$np(1-p) = 1000 \cdot \frac{1}{6} \cdot \frac{5}{6} \geq 10$ (use normal approximation)

$$P(150 \leq X \leq 200) = \sum_{i=150}^{200} \binom{1000}{i} \left(\frac{1}{6}\right)^i \left(\frac{5}{6}\right)^{1000-i}$$

X can be approximated using a normal distribution $N(np, np(1-p))$

binom normal

$$P(150 \leq X \leq 200) \approx P(149.5 \leq X \leq 200.5) \quad N\Big(\underbrace{\tfrac{1000}{6}}_{\mu}, \underbrace{\tfrac{1000}{6} \cdot \tfrac{5}{6}}_{\sigma^2}\Big)$$

continuity correction.

$$= P\left(\frac{149.5 - \mu}{\sigma} \leq Z \leq \frac{200.5 - \mu}{\sigma}\right) \quad Z \sim N(0,1)$$

$$= P\left(Z \leq \frac{200.5 - \mu}{\sigma}\right) - P\left(Z \leq \frac{149.5 - \mu}{\sigma}\right)$$

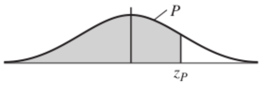
$=$ ________ use the $N(0,1)$ table.

Given that '6' appears 200 times, let Y denote the num. of times '5' appears.

$Y \sim Bin(800, \frac{1}{5})$. Y can be approximated by a normal distribution $N(800 \cdot \frac{1}{5}, 800 \cdot \frac{1}{5} \cdot \frac{4}{5})$

$$P(Y < 150) \approx P(Y \leq 149.5) = P\left(Z \leq \frac{149.5 - 800(\frac{1}{5})}{\sqrt{800(\frac{1}{5})(\frac{4}{5})}}\right) = \_\_\_\_\_\_\_\_$$

TABLE Cumulative Normal Distribution—Values of *P* Corresponding to z_p for the Normal Curve



***z* is the standard normal variable.**

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998