MH2500 AY17/18

Solution 1.

(a) $P(X > Y) = 0.23$.

(b) $P(X = 0|Y = 0) = 0.4$, $P(X = 1|Y = 0) = 0.2$, $P(X = 2|Y = 0) = 0.4$.

(c) $P(Y = 1) = P(Y = 0) = 0.25$, $P(X = 0, Y = 1) = 0.25 - P(X = 1, Y = 1) - P(X = 2, Y = 1) = 0.13$.

(d) $P(X = 1) = 0.21$, $P(X = 1, Y = 1) \neq P(X = 1)P(Y = 1)$. Hence, X and Y are not independent.

Solution 2.
Let F denote the event that the coin is fair. Let H_i be the event that the coin shows a head at the i-th throw.

(a)
$$P(F|H_1) = \frac{P(H_1|F)P(F)}{P(H_1|F)P(F) + P(H_1|F')P(F')} = \frac{0.5(0.5)}{0.5(0.5) + 1(0.5)} = \frac{1}{3}.$$

(b)
$$P(H_2|H_1) = \frac{P(H_2H_1)}{P(H_1)} = \frac{P(H_2H_1|F)P(F) + P(H_2H_1|F')P(F')}{P(H_1|F)P(F) + P(H_1|F')P(F')} = \frac{0.25(0.5) + 1(0.5)}{0.5(0.5) + 1(0.5)} = \frac{5}{6}.$$

(c)
$$P(F|H_2H_1) = \frac{P(H_2H_1|F)P(F)}{P(H_2H_1|F)P(F) + P(H_2H_1|F')P(F')} = \frac{0.25(0.5)}{0.25(0.5) + 1(0.5)} = \frac{1}{5}.$$

Solution 3.

(a) The conditional distribution of H given $X = x$ is $H|X = x \sim$Unif$(0, x)$.
$$\mathbb{E}(H|X = x) = \int_0^x h f_{H|X}(h|x)dh = \int_0^x \frac{h}{x}dh = \frac{1}{2}x.$$

(b)
$$\mathbb{E}(A) = \mathbb{E}(HX) = \mathbb{E}_X(\mathbb{E}(HX|X)) = \mathbb{E}_X(X\mathbb{E}(H|X)) = \mathbb{E}\left(X \cdot \frac{1}{2}X\right) = \frac{1}{2}\mathbb{E}(X^2) = \frac{1}{2}\int_0^1 x^2 f_X(x)dx = \frac{1}{6}.$$

(c)
$$\mathbb{E}(H^2|X = x) = \int_0^x h^2 f_{H|X}(h|x)dx = \int_0^x \frac{h^2}{x}dh = \frac{1}{3}x^2.$$
$$\mathbb{E}(A^2) = \mathbb{E}(H^2X^2) = \mathbb{E}_X(\mathbb{E}(H^2X^2|X)) = \mathbb{E}_X(X^2\mathbb{E}(H^2|X)) = \mathbb{E}\left(X^2 \cdot \frac{1}{3}X^2\right) = \frac{1}{3}\mathbb{E}(X^4) = \frac{1}{3}\int_0^1 x^4 dx = \frac{1}{15}.$$
$$Var(A) = \mathbb{E}(A^2) - \mathbb{E}(A)^2 = \frac{7}{180}.$$

(d)
$$\mathbb{E}(X) = \int_0^1 x dx = \frac{1}{2}, \qquad Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$
$$\mathbb{E}(XA) = \mathbb{E}(X^2H) = \mathbb{E}_X(\mathbb{E}(X^2H|X)) = \mathbb{E}_X(X^2\mathbb{E}(H|X)) = \frac{1}{2}\mathbb{E}(X^3) = \frac{1}{8}.$$

$$\rho = \frac{Cov(X, A)}{\sqrt{Var(X)Var(A)}} = \frac{\mathbb{E}(XA) - \mathbb{E}(X)\mathbb{E}(A)}{\sqrt{\frac{1}{12} \cdot \frac{7}{180}}} = 0.7319.$$

Solution 4.

(a)

$$1 = \int_1^\infty Cx^{-\alpha-1}dx = \frac{C}{\alpha} \implies C = \alpha.$$

(b)

$$F_{Y_i}(y) = P(Y_i \le y) = P(\ln X_i \le y) = P(X_i \le e^y) = \int_1^{e^y} \alpha x^{-\alpha-1}dx = 1 - e^{-y\alpha}.$$

$$f_{Y_i}(y) = \frac{d}{dy}F_{Y_i}(y) = \alpha e^{-y\alpha}, \qquad \text{for } y \ge 0.$$

Y_i follows an exponential distribution with parameter α.

(c)

$$Z_n = (X_1X_2\dots X_n)^{1/n} \implies \ln Z_n = \frac{1}{n}(\ln X_1 + \dots + \ln X_n) = \frac{1}{n}\sum_{i=1}^n Y_i$$

By the strong law of large numbers, with probability 1,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n Y_i = \mathbb{E}(Y_i) = \frac{1}{\alpha}. \qquad \text{(mean of exponential distribution)}$$

Therefore,

$$\lim_{n\to\infty} Z_n = \lim_{n\to\infty} e^{\ln Z_n} = e^{1/\alpha}.$$

(d)

$$\begin{aligned}
P(Z_n \le e^{1/\alpha}L) &= P\left(\ln Z_n \le \frac{1}{\alpha} + \ln L\right) \\
&= P\left(\ln Z_n - \mathbb{E}(\ln Z_n) \le \frac{1}{\alpha}\right) && \mathbb{E}(\ln Z_n) = \frac{1}{n}\sum \mathbb{E}(Y_i) = \frac{1}{\alpha} \\
&\ge P\left(|\ln Z_n - \mathbb{E}(\ln Z_n)| \le \frac{1}{\alpha}\right) \\
&\ge 1 - \frac{Var(\ln Z_n)}{\frac{1}{\alpha^2}} && \text{Chebyshev's inequality} \\
&= 1 - \frac{\frac{1}{n\alpha^2}}{\frac{1}{\alpha^2}} = 1 - \frac{1}{n}. && Var(\ln Z_n) = \frac{1}{n^2}\sum Var(Y_i) = \frac{1}{n\alpha^2}
\end{aligned}$$