Practice Week 6 Question 3

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Here, we will go through Practice Week 6 Question 3(a) and 3(b) in details and I will explain a few different methods to approach 3(b).

For $0 < p < 1$, we say X follows a negative binomial distribution if for $n = r, r+1, \ldots$, if

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}.$$

In short, we can write $X \sim \text{NegBin}(r, p)$. The generalized binomial theorem states that for any real numbers x, y and any real number r, we have

$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k y^{r-k}.$$

This in fact generalises to $r \in \mathbb{C}$. If we consider r as a non-negative integer and use the convention that $\binom{r}{k} = 0$, if $r < k$, then we have the simpler version of the binomial theorem, i.e.,

$$(x+y)^r = \sum_{k=0}^{r} \binom{r}{k} x^k y^{r-k}.$$

For both, we define $\displaystyle\binom{r}{k} = \frac{r \cdot (r-1) \cdot \ldots \cdot (r-k+1)}{k!}$.

For Question 3(a), we have

$$\begin{aligned}
\sum_{n=r}^{\infty} P(X = n) &= \sum_{n=r}^{\infty} \binom{n-1}{r-1} p^r (1-p)^{n-r} \\
&= \sum_{n=r}^{\infty} \binom{n-1}{n-r} p^r (1-p)^{n-r} && \text{by symmetry, } \binom{n}{k} = \binom{n}{n-k} \\
&= \sum_{j=0}^{\infty} \binom{n-1}{n-r} p^r (1-p)^{n-r} && \text{let } j = n-r \\
&= p^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} (1-p)^j \\
&= p^r \sum_{j=0}^{\infty} \frac{(r+j-1) \cdot (r+j-2) \cdot \ldots \cdot (r+1) \cdot r}{j!} (1-p)^j \\
&= p^r \sum_{j=0}^{\infty} (-1)^j \frac{(-r-j+1) \cdot (-r-j+2) \cdot \ldots \cdot (-r-1) \cdot (-r)}{j!} (1-p)^j \\
&= p^r \sum_{j=0}^{\infty} (-1)^j \binom{-r}{j} (1-p)^j && \text{by definition of } \binom{-r}{j} \\
&= p^r \sum_{j=0}^{\infty} \binom{-r}{j} (p-1)^j \cdot 1^{-r-j} \\
&= p^r (p-1+1)^{-r} && \text{by generalised binomial theorem} \\
&= 1.
\end{aligned}$$

For Question 3(b), I will provide two different methods to approach it.

Method 1. We need to compute $\mathbb{E}(X)$ and $Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$. For any $k \in \mathbb{Z}$, $k \geq 1$, we can compute $\mathbb{E}(X^k)$ as follows:

$$
\begin{aligned}
\mathbb{E}(X^k) &= \sum_x x^k P(X = x) \\
&= \sum_{n=r}^{\infty} n^k \binom{n-1}{r-1} p^r (1-p)^{n-r} \\
&= r \sum_{n=r}^{\infty} n^{k-1} \binom{n}{r} p^r (1-p)^{n-r} && \text{use } r\binom{n}{r} = n\binom{n-1}{r-1} \\
&= \frac{r}{p} \sum_{n=r}^{\infty} n^{k-1} \binom{n}{r} p^{r+1} (1-p)^{n-r} \\
&= \frac{r}{p} \sum_{m=r+1}^{\infty} (m-1)^{k-1} \binom{m-1}{r} p^{r+1} (1-p)^{m-r-1} && \text{let } m = n+1 \\
&= \frac{r}{p} \cdot \mathbb{E}\left[(Y-1)^{k-1}\right] && \text{based on pmf of the distribution where } Y \sim \text{NegBin}(r+1, p).
\end{aligned}
$$

Hence, we have

$$\mathbb{E}(X) = \frac{r}{p} \cdot \mathbb{E}(Y-1)^0 = \frac{r}{p}.$$

We can also compute

$$\mathbb{E}(X^2) = \frac{r}{p}\mathbb{E}(Y-1) = \frac{r}{p}(\mathbb{E}(Y) - 1) = \frac{r}{p}\left(\frac{r+1}{p} - 1\right).$$

Hence,

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{r(1-p)}{p^2}.$$

This method is a more combinatorial approach and allows us to calculate $\mathbb{E}(X^k)$ for all k in a recursive manner. $\mathbb{E}(X^k)$ is the k-th moment of the negative binomial distribution (r, p). The moments of a distribution essentially define the distribution itself.

Method 2. This will be a calculus-based approach where we need to analyse the convergence of infinite series under differentiation.

$$
\begin{aligned}
\mathbb{E}(X) &= \sum_{n=r}^{\infty} n\binom{n-1}{r-1}p^r(1-p)^{n-r} \\
&= \sum_{n=r}^{\infty} n\binom{n-1}{n-r}p^r(1-p)^{n-r} && \text{by symmetry} \\
&= \sum_{j=0}^{\infty}(j+r)\binom{r+j-1}{j}p^r(1-p)^j && \text{let } j=n-r \\
&= rp^r\sum_{j=0}^{\infty}\binom{j+r-1}{j}(1-p)^j + p^r\sum_{j=0}^{\infty}\binom{r+j-1}{j}j(1-p)^j \\
&= r + p^r\sum_{j=0}^{\infty}\binom{r+j-1}{j}j(1-p)^j && \text{from (a): } p^r\sum_{j=0}^{\infty}\binom{j+r-1}{j}(1-p)^j = 1 \\
&= r + p^r\sum_{j=0}^{\infty}\binom{-r}{j}j(p-1)^j && \text{from (a)} \\
&= r + p^r(p-1)\sum_{j=0}^{\infty}\binom{-r}{j}j(p-1)^{j-1} \\
&= r + p^r(p-1)\sum_{j=0}^{\infty}\binom{-r}{j}\frac{d}{dp}(p-1)^j \\
&= r + p^r(p-1)\frac{d}{dp}\left[\sum_{j=0}^{\infty}\binom{-r}{j}(p-1)^j\right] && \text{interchanging summation and differentiation} \\
&= r + p^r(p-1)\frac{d}{dp}(p^{-r}) && \text{from (a)} \\
&= r - \frac{rp^r(p-1)}{p^{r+1}} = \frac{r}{p}.
\end{aligned}
$$

We can compute $\mathbb{E}(X^2)$ in a similar manner.

$$
\begin{aligned}
\mathbb{E}(X^2) &= \sum_{n=r}^{\infty} n^2\binom{n-1}{r-1}p^r(1-p)^{n-r} \\
&= \sum_{j=0}^{\infty}(j+r)^2\binom{r+j-1}{j}p^r(1-p)^j \\
&= r^2\underbrace{p^r\sum_{j=0}^{\infty}\binom{r+j-1}{j}(1-p)^j}_{=1,\text{ from (a)}} + 2r\underbrace{p^r\sum_{j=0}^{\infty}j\binom{r+j-1}{j}(1-p)^j}_{=\frac{r}{p}-r,\text{ from }\mathbb{E}(X)} + p^r\sum_{j=0}^{\infty}j^2\binom{r+j-1}{j}(1-p)^j \\
&= r^2 + 2r\left(\frac{r}{p}-r\right) + p^r\sum_{j=0}^{\infty}j(j-1)\binom{r+j-1}{j}(1-p)^j + \underbrace{p^r\sum_{j=0}^{\infty}j\binom{r+j-1}{j}(1-p)^j}_{=\frac{r}{p}-r} && \text{use } j^2 = j(j-1)+j \\
&= r^2 + (2r+1)\left(\frac{r}{p}-r\right) + p^r(1-p)^2\sum_{j=0}^{\infty}\binom{-r}{j}j(j-1)(p-1)^{j-2} \\
&= r^2 + (2r+1)\left(\frac{r}{p}-r\right) + p^r(1-p)^2\sum_{j=0}^{\infty}\binom{-r}{j}\frac{d^2}{dp^2}(p-1)^j \\
&= r^2 + (2r+1)\left(\frac{r}{p}-r\right) + p^r(1-p)^2\frac{d^2}{dp^2}\left[\sum_{j=0}^{\infty}\binom{-r}{j}(p-1)^j\right]
\end{aligned}
$$

Then, we have

$$\begin{aligned}\mathbb{E}(X^2) &= r^2 + (2r+1)\left(\frac{r}{p} - r\right) + p^r(1-p)^2\frac{d^2}{dp^2}(p^{-r}) \\ &= r^2 + (2r+1)\left(\frac{r}{p} - r\right) + \frac{(1-p)^2 r(r+1)}{p^2}\end{aligned}$$

Thus,

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{r^2p^2 + pr(2r+1)(1-p) + r(r+1)(1-p)^2}{p^2} - \frac{r^2}{p^2} = \frac{r(1-p)}{p^2}.$$

Extra information on calculus: In both the computations of $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$, we interchange differentiation and summation. This always works for a finite series, since differentiation is linear. However, it is not always the case for infinite series. Given a power series centered at $x = a$,

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad \text{for } |x-a| < R.$$

R is the radius of convergence. f is differentiable on the open interval of convergence $(a-R, a+R)$ and we can interchange summation and differentiation to do term-by-term differentiation, i.e.,

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n\frac{d}{dx}(x-a)^n = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$

f' also converges on $(a-R, a+R)$. In our example in $\mathbb{E}(X)$, we have to check the interval of convergence of $\sum_{j=0}^{\infty}\binom{-r}{j}(p-1)^j$.

We can use the ratio test.

$$\begin{aligned}L &= \lim_{n\to\infty}\left|\frac{\binom{-r}{n+1}(p-1)^{n+1}}{\binom{-r}{n}(p-1)^n}\right| \\ &= |p-1|\lim_{n\to\infty}\left|\frac{(-r)(-r-1)\dots(-r-n)}{(-r)(-r-1)\dots(-r-n+1)}\cdot\frac{n!}{(n+1)!}\right| \\ &= |p-1|\lim_{n\to\infty}\frac{n+r}{n+1} = |p-1|.\end{aligned}$$

By ratio test, the power series converges if and only if $|p-1| < 1$. From our assumption of the negative binomial distribution that $0 < p < 1$, this falls within the interval of convergence, hence the power series is always differentiable for $0 < p < 1$ and we are able to interchange differentiation and summation.

Method 3. Suppose you already know $\mathbb{E}(X) = r/p$, this gives an easier way to compute $\mathbb{E}(X^2)$. The same method has been applied to find the variance of geometric distribution.

$$\begin{aligned}
\mathbb{E}(X(X+1)) &= \sum_{n=r}^{\infty} n(n+1)\binom{n-1}{r-1}p^r(1-p)^{n-r} \\
&= \sum_{n=r}^{\infty} n(n+1)\frac{(n-1)!}{(r-1)!(n-r)!}p^r(1-p)^{n-r} \\
&= \sum_{n=r}^{\infty} \frac{(n+1)!}{(r-1)!(n-r)!}p^r(1-p)^{n-r} \\
&= \sum_{n=r}^{\infty} \frac{(n+1)!}{(r+1)!(n-r)!} \cdot r(r+1) \cdot p^r(1-p)^{n-r} \\
&= r(r+1)\sum_{n=r}^{\infty} \binom{n+1}{r+1}p^r(1-p)^{n-r} \\
&= \frac{r(r+1)}{p^2}\sum_{n=r}^{\infty} \binom{n+1}{r+1}p^{r+2}(1-p)^{n-r} \\
&= \frac{r(r+1)}{p^2}\sum_{n=r+2}^{\infty} \binom{n-1}{r+1}p^{r+2}(1-p)^{n-r-2} \\
&= \frac{r(r+1)}{p^2}\underbrace{\sum_{n=r+2}^{\infty} P(Y=n)}_{=1} && \text{where } Y \sim \text{NegBin}(r+2,p) \\
&= \frac{r(r+1)}{p^2}
\end{aligned}$$

. Hence,

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = (\mathbb{E}(X(X+1)) - \mathbb{E}(X)) - \mathbb{E}(X)^2 = \frac{r(1-p)}{p^2}.$$