Let $f_0(x) = \begin{cases} 1, & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$ and $f_1(x) = \begin{cases} 12\left(x - \frac{1}{2}\right)^2, & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

be two probability density functions.

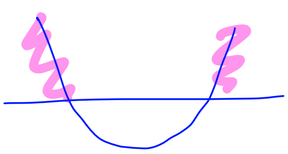
Let X be a random variable whose PDF is either $f_0(\cdot)$ or $f_1(\cdot)$. Based on a single observation of X, construct the MP test of the hypothesis $H_0: f(x) = f_0(x)$ against $H_1: f(x) = f_1(x)$ with size $\alpha = 0.05$. What is the power of this test?

$$\Lambda(X) = \frac{L(X|H_0)}{L(X|H_1)} = \frac{L(X|f_0)}{L(X|f_1)} = \frac{1}{12\left(x-\frac{1}{2}\right)^2}$$

Neyman Pearson Lemma

: The MP Test rejects H_0 when $\Lambda(X) \le t$ for some t.

$$\frac{1}{12\left(x-\frac{1}{2}\right)^2} \le t \Rightarrow \left(x-\frac{1}{2}\right)^2 \ge \frac{1}{12t} \Rightarrow \left|x-\frac{1}{2}\right| \ge \sqrt{\frac{1}{12t}}$$



$$\Rightarrow x \ge \frac{1}{2} + \sqrt{\frac{1}{12t}} \quad \text{or} \quad x \le \frac{1}{2} - \sqrt{\frac{1}{12t}}$$

Reject H_0 if $x \ge \frac{1}{2} + \sqrt{\frac{1}{12t}}$ or $x \le \frac{1}{2} - \sqrt{\frac{1}{12t}}$

Find t, use $\alpha = 0.05$

$$0.05 = P(\text{Rej } H_0 \mid H_0) = P\left(X \ge \frac{1}{2} + \sqrt{\frac{1}{12t}} \text{ or } X \le \frac{1}{2} - \sqrt{\frac{1}{12t}} \;\middle|\; f_0\right)$$

$$= 1 - P\left(\frac{1}{2} - \sqrt{\frac{1}{12t}} \le X \le \frac{1}{2} + \sqrt{\frac{1}{12t}} \;\middle|\; f_0\right)$$

$$= 1 - \int_{\frac{1}{2}-\sqrt{\frac{1}{12t}}}^{\frac{1}{2}+\sqrt{\frac{1}{12t}}} 1 \, dx \quad \leftarrow \text{pdf } (f_0)$$

$$= 1 - 2\sqrt{\frac{1}{12t}}$$

$$0.05 = 1 - 2\sqrt{\frac{1}{12t}}$$

$$\sqrt{\frac{1}{12t}} = \frac{0.95}{2}$$

The MP Test rejects H_0 when $x \geq \frac{1}{2} + \sqrt{\frac{1}{12t}}$ or $x \leq \frac{1}{2} - \sqrt{\frac{1}{12t}}$

$$\Leftrightarrow \quad X \geq \frac{1}{2} + \frac{0.95}{2} \quad \text{or} \quad X \leq \frac{1}{2} - \frac{0.95}{2}$$

$$\Leftrightarrow \quad X \geq 0.975 \quad \text{or} \quad X \leq 0.025 .$$

$$\text{Power of Test} = P(\text{Reject } H_0 \mid H_1)$$

$$= P(X \geq 0.975 \text{ or } X \leq 0.025 \mid f_1)$$

$$= 1 - P(0.025 \leq X \leq 0.975 \mid f_1)$$

$$= 1 - \int_{0.025}^{0.975} 12\left(x - \tfrac{1}{2}\right)^2 dx$$

pdf (f_1)

$$= 1 - 4\left[0.475^3 + 0.475^3\right] = 0.142625.$$

Let $X_1, \ldots, X_n$ be an i.i.d. sample drawn from a normal distribution $N(\mu, \sigma^2)$ where μ is known. Consider the test statistic $T = \sum_{i=1}^{n}(X_i - \mu)^2$.

a) For testing $H_0\colon \sigma^2 = \sigma_0^2$ against $H_1\colon \sigma^2 = \sigma_1^2$, with $\sigma_1^2 > \sigma_0^2$, find a MP test with size α.
b) Find a size α, UMP test for testing $H_0\colon \sigma^2 = \sigma_0^2$ against $H_1\colon \sigma^2 > \sigma_0^2$.
c) Is there a UMP for testing $H_0\colon \sigma^2 \le \sigma_0^2$ against $H_1\colon \sigma^2 > \sigma_0^2$, with size α?

(a)
$$\Lambda(X_1,\ldots,X_n) = \frac{L(X_1,\ldots,X_n \mid \overset{\sigma_0^2}{H_0})}{L(X_1,\ldots,X_n \mid \underset{\sigma_1^2}{H_1})} = \frac{\prod_{i=1}^{n} \frac{1}{\cancel{\sqrt{2\pi}}\,\sigma_0} \exp\left(-\frac{(X_i-\mu)^2}{2\sigma_0^2}\right)}{\prod_{i=1}^{n} \frac{1}{\cancel{\sqrt{2\pi}}\,\sigma_1} \exp\left(-\frac{(X_i-\mu)^2}{2\sigma_1^2}\right)}$$

$$= \left(\frac{\sigma_1}{\sigma_0}\right)^n \frac{\exp\left(-\frac{1}{2\sigma_0^2}\sum(X_i-\mu)^2\right)}{\exp\left(-\frac{1}{2\sigma_1^2}\sum(X_i-\mu)^2\right)} = \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left(\frac{T}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\right)$$

NP Lemma: MP Test rejects H_0 when $\Lambda(X_1,\ldots,X_n) \le t$ for some t.

Given that $\sigma_1^2 > \sigma_0^2 \Rightarrow \frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2} < 0$

$\Rightarrow \Lambda$ is a decreasing function in T (MLR)

If $\Lambda \le t$ for some t $\Rightarrow$ $\boxed{T \ge t' \text{ for some } t'}$ rejection rule.

Find t', use α.

$T = \sum(X_i-\mu)^2$

$$\alpha = P(\text{Reject } H_0 \mid H_0) = P(T \ge t' \mid \sigma^2 = \sigma_0^2)$$

$$= P\Big(\underbrace{\boxed{\frac{T}{\sigma_0^2}}}_{\chi^2_n} \ge \frac{t'}{\sigma_0^2}\Big)$$

Know that $X_i \sim N(\mu, \sigma^2)$

$\frac{X_i-\mu}{\sigma} \sim N(0,1)$

$\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_1$

$\sum\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_n$

$\parallel$

$\frac{T}{\sigma^2}$

Let $\chi^2_{n,\alpha}$ be the number that $P\left(\frac{T}{\sigma_0^2} \ge \chi^2_{n,\alpha}\right) = \alpha$.

$$\Rightarrow \frac{t'}{\sigma_0^2} = \chi^2_{n,\alpha} \Rightarrow t' = \sigma_0^2 \chi^2_{n,\alpha}$$

MP Test : Reject H_0 when $T \geq \sigma_0^2 \chi^2_{n,\alpha}$ $\qquad T = \sum (X_i - \mu)^2$

(b) $H_0 : \sigma^2 = \sigma_0^2 \qquad H_1 : \sigma^2 > \sigma_0^2$

The rejection rule in (a) does not depend on $\sigma_1^2 > \sigma_0^2$.

So, the MP Test in (a) is the UMP test in (b).

$T = \sum (X_i - \mu)^2$

$\frac{T}{\sigma^2} \sim \chi^2_n$

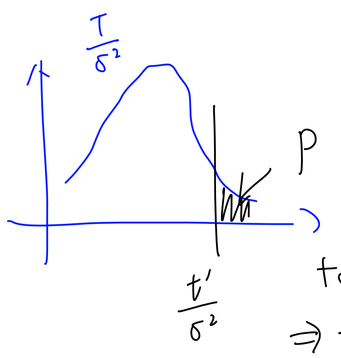
(c) $H_0 : \sigma^2 \leq \sigma_0^2 \qquad H_1 : \sigma^2 > \sigma_0^2$

For a test of size α,

$$\alpha = \sup_{\sigma^2 \leq \sigma_0^2} P(\text{Reject } H_0) = \sup_{\sigma^2 \leq \sigma_0^2} P(T \geq t') \text{ for some } t'$$

$$= \sup_{\sigma^2 \leq \sigma_0^2} P\left(Y \geq \frac{t'}{\sigma^2}\right) \qquad Y = \frac{T}{\sigma^2} \sim \chi^2_n$$

$$= P\left(Y \geq \frac{t'}{\sigma_0^2}\right)$$



to max P

$\Rightarrow$ to min $\frac{t'}{\sigma^2}$

$\Rightarrow$ to max σ^2

Similar as (a), $t' = \sigma_0^2 \chi^2_{n,\alpha}$

Reject H_0 when $T \geq \sigma_0^2 \chi^2_{n,\alpha}$

The MP Test in (a) is the MP Test in (c).

Common Mistakes

(a) $X_1, \ldots, X_n$ i.i.d. , $E(X_1) = \mu$ $Var(X_1) = \sigma^2$

$\bar{X} = \frac{1}{n} \sum X_i$

then $\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0,1)$ wrong!!

reason: distribution of X_i is not known.

By CLT, $\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \to N(0,1)$ convergence in distribution correct!

as n increases

(b) When $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$

$\Rightarrow \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0,1)$ correct!

(c) $X_1, \ldots, X_n$ i.i.d.

$\sum X_i = n X_1$ wrong!!

$\sum E(X_i) = n E(X_1)$ correct!!

If X_1, X_2 are indp $\Rightarrow Var(X_1 + X_2) = Var(X_1) + Var(X_2)$. correct ✓

If X_1, X_2 not indp $\Rightarrow Var(X_1 + X_2) =$ or $\neq Var(X_1) + Var(X_2)$.

correct ✓. $Var(X_1+X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$

$X_1, \dots, X_n$ not indp $\quad Var(\sum X_i) = \sum Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$