Recap for Tutorial 3

MH1101

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1 Improper Integral

Definition 1 (Type 1 Improper Integral)**.**

(i) If $\int_a^t f(x)\ \mathrm{d}x$ exists for all values $t \geq a$, then

$$\int_a^\infty f(x)\ \mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\ \mathrm{d}x.$$

(ii) If $\int_t^b f(x)\ \mathrm{d}x$ exists for all values $t \leq b$, then

$$\int_{-\infty}^b f(x)\ \mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\ \mathrm{d}x.$$

(iii) If both $\int_a^\infty f(x)\ \mathrm{d}x$ and $\int_{-\infty}^a f(x)\ \mathrm{d}x$ exist, then

$$\int_{-\infty}^\infty f(x)\ \mathrm{d}x = \int_a^\infty f(x)\ \mathrm{d}x + \int_{-\infty}^a f(x)\ \mathrm{d}x.$$

Definition 2 (Type 2 Improper Integral)**.**

(i) If $f(x)$ is continuous on $[a, b)$ and discontinuous on $x = b$, then

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{t\to b^-} \int_a^t f(x)\ \mathrm{d}x.$$

(ii) If $f(x)$ is continuous on $(a, b]$ and discontinuous on $x = a$, then

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{t\to a^+} \int_t^b f(x)\ \mathrm{d}x.$$

(iii) If $f(x)$ is discontinuous at $x = c$ where $a < c < b$, then

$$\int_a^b f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x,$$

provided both integrals on the right converges.

2 Area between Curves

Theorem 1. The area, A, of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$ is

$$\int_a^b (f(x) - g(x))\ \mathrm{d}x.$$

3 Volume of a Solid

Theorem 2. Let S be a solid bounded between $x = a$ and $x = b$. If the cross-sectional area of the plane at x, perpendicular to x-axis is $A(x)$, where A is a continuous function in $[a, b]$, then the volume of the solid is

$$V = \int_a^b A(x)\ \mathrm{d}x.$$

To calculate the volume of a solid of revolution (about an axis), we generally have two methods.

Disk-Washer Method: Slicing perpendicularly to the axis of rotation. Each slice corresponds to a cross-sectional area of a circle or an annulus (area between a pair of concentric circles). Hence,

$$A(x) = \pi r^2 \qquad \text{or} \qquad A(x) = \pi(r_{out}^2 - r_{in}^2)$$

Cylindrical shells: Slicing parallel to the axis of rotation. Each slice corresponds to a cross-sectional area of the curve surface of a cylinder. Hence,

$$A(x) = 2\pi r h.$$

r represents the perpendicular distance of the cross-sectional plane to the axis of rotation. h is the height of the cylindrical shell.

4 Extra Exercises

Problem 1. Evaluate each integral, if it is convergent.

(a) $\displaystyle\int_1^\infty \frac{\ln x}{x}\ \mathrm{d}x$ (Hint: Let $u = \ln x$)

(b) $\displaystyle\int_{-\infty}^\infty x e^{-x^2}\ \mathrm{d}x$ (Hint: Let $u = x^2$)

Problem 2. Explain why the following equality is false.

$$\int_{-\infty}^{\infty} x\ \mathrm{d}x = \lim_{t\to\infty} \int_{-t}^{t} x\ \mathrm{d}x.$$

Problem 3. Find the value of c such that the area bounded by the parabolas $y = c^2 - x^2$ and $y = x^2 - c^2$ is 576.