MH3500 Statistics

Tutorial 9 AY2022/23 Semester 2

Problem 9.1

Let $X_1, \ldots, X_n$ be an i.i.d. sample drawn from a normal distribution $N(\mu, \sigma^2)$ with both parameters μ and σ^2 unknown. Determine c such that $(-\infty, \bar{X} + c)$ is a 95% confidence interval for μ.

Hint: use the pivotal quantity $\sqrt{n}(\bar{X} - \mu)/S$ to construct the confidence interval. Your result should be an expression for c involving n, the sample variance S^2, and a percentage point for a t-distribution.

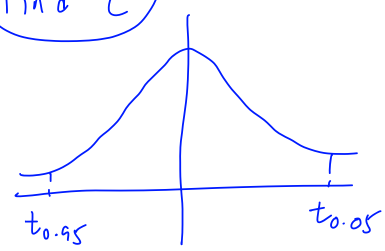
If $X_1, \ldots, X_n \sim N(\mu, \sigma^2) \Rightarrow T = \dfrac{\sqrt{n}(\bar{X} - \mu)}{S} \sim t_{n-1}$

$$P(\mu \le \bar{X} + c) = 0.95$$

Find c

$$P(\bar{X} - \mu \ge -c) = 0.95$$

$$P\left(\frac{\sqrt{n}(\bar{X} - \mu)}{S} \ge -\frac{c\sqrt{n}}{S}\right) = 0.95$$



$t_{0.95} = -t_{0.05}$

$t_\alpha = -t_{1-\alpha}$

$P\left(T \ge -\dfrac{c\sqrt{n}}{S}\right) = 0.95$, where $T \sim t_{n-1}$

Let t_α be the number, s.t. $P(T > t_\alpha) = \alpha$

$$\Rightarrow -\frac{c\sqrt{n}}{S} = t_{0.95} \Rightarrow c = -\frac{S t_{0.95}}{\sqrt{n}} = \frac{S t_{0.05}}{\sqrt{n}}$$

Problem 9.2

Let $X_1, \dots, X_n \sim N(\mu_X, \sigma_X^2)$ and $Y_1, \dots, Y_m \sim N(\mu_Y, \sigma_Y^2)$ be i.i.d. samples which are independent for each other. Using a pivotal quantity approach, construct a 95% confidence interval for the variance ratio σ_X^2/σ_Y^2.

Hint: show that $\left(\frac{S_X^2}{\sigma_X^2}\right) / \left(\frac{S_Y^2}{\sigma_Y^2}\right)$ follows an F-distribution.

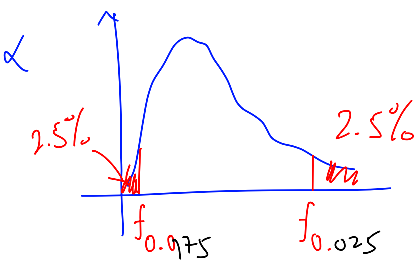
$$S_X^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$$

$$S_Y^2 = \frac{1}{m-1} \sum (Y_i - \bar{Y})^2$$

$$U = \frac{(n-1) S_X^2}{\sigma_X^2} \sim \chi^2_{n-1} \qquad V = \frac{(m-1) S_Y^2}{\sigma_Y^2} \sim \chi^2_{m-1}$$

$$\frac{\frac{U}{n-1}}{\frac{V}{m-1}} \sim F(n-1, m-1) \Rightarrow \underset{\substack{\uparrow \\ \text{pivotal quantity}}}{F} = \frac{\left(\frac{S_X^2}{\sigma_X^2}\right)}{\left(\frac{S_Y^2}{\sigma_Y^2}\right)} \sim \overset{\text{distribution}}{\overset{\downarrow}{F(n-1, m-1)}}$$

Let f_α be the number, s.t. $P(F > f_\alpha) = \alpha$



$$P\left(f_{0.975} \le \frac{S_X^2}{\sigma_X^2} \cdot \frac{\sigma_Y^2}{S_Y^2} \le f_{0.025}\right) = 0.95$$

$$P\left(f_{0.975} \cdot \frac{S_Y^2}{S_X^2} \le \frac{\sigma_Y^2}{\sigma_X^2} \le f_{0.025} \cdot \frac{S_Y^2}{S_X^2}\right) = 0.95$$

$$P\left(\frac{S_X^2}{S_Y^2 f_{0.025}} \le \frac{\sigma_X^2}{\sigma_Y^2} \le \frac{S_X^2}{S_Y^2 f_{0.975}}\right) = 0.95$$

$$95\% \text{ CI}: \left(\frac{S_X^2}{S_Y^2 f_{0.025}}, \frac{S_X^2}{S_Y^2 f_{0.975}}\right), \text{ for } \frac{\sigma_X^2}{\sigma_Y^2}$$

Problem 9.3

Suppose the fat content of certain steaks follows a $N(\mu, \sigma^2)$ distribution. The following observations $X_1, \dots, X_{16}$ for the fat content are given.

$$5.33, 4.25, 3.15, 3.70, \quad 1.61, 6.39, 3.12, 6.59,$$

$$3.53, 4.74, 0.11, 1.60, \quad 5.49, 1.72, 4.15, 2.28$$

a) Suppose $\sigma^2 = 3.2$. use the pivotal quantity $\sqrt{n}(\bar{X} - \mu)/\sigma$ to find 90%, 95% and 99% confidence intervals for μ based on the observations above.

b) In Part a), if we want to cut down the length of the confidence interval to half their length, how much would we need to increase the sample size?

For the remaining pats of this problem, suppose that both parameters μ and σ^2 unknown.

c) Find 90%, 95% and 99% confidence intervals for μ.

d) Find 90%, 95% and 99% confidence intervals for σ^2.

e) Use the main result on approximate confidence intervals to construct an approximate 95% confidence interval for σ^2.
Hint: to remove unknown quantities from the interval, replace σ^2 by the MLE for σ^2 in the expressions for the interval endpoints.

(a) $Z = \dfrac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0,1)$

Let z_α be the number st.

$P(Z > z_\alpha) = \alpha$

$P(z_{0.95} \leq Z \leq z_{0.05}) = 0.9$

$z_{0.05} = -z_{0.95}$ (symmetry)

$P\left(-z_{0.05} \leq \dfrac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \leq z_{0.05}\right) = 0.9$

90% CI for μ

$\left(\bar{X} - \dfrac{\sigma z_{0.05}}{\sqrt{n}}, \bar{X} + \dfrac{\sigma z_{0.05}}{\sqrt{n}}\right) \Leftarrow P\left(\bar{X} - \dfrac{\sigma z_{0.05}}{\sqrt{n}} \leq \mu \leq \bar{X} + \dfrac{\sigma z_{0.05}}{\sqrt{n}}\right) = 0.9$

(b) For 90% CI for μ, length $= 2 \cdot \dfrac{\sigma z_{0.05}}{\sqrt{n}}$

For a shorter interval with sample size m $\Rightarrow$ $\underbrace{2 \cdot \dfrac{\sigma z_{0.05}}{\sqrt{m}}}_{\text{new length}} = \underbrace{\dfrac{1}{2} \cdot 2 \dfrac{\sigma z_{0.05}}{\sqrt{n}}}_{\frac{1}{2} \text{ of existing length.}}$

$\sqrt{m} = 2\sqrt{n} \Rightarrow m = 4n$

c) Find 90%, 95% and 99% confidence intervals for μ.

d) Find 90%, 95% and 99% confidence intervals for σ^2.

e) Use the main result on approximate confidence intervals to construct an approximate 95% confidence interval for σ^2.
Hint: to remove unknown quantities from the interval, replace σ^2 by the MLE for σ^2 in the expressions for the interval endpoints.

$X_1, \dots, X_{16} \sim N(\mu, \sigma^2)$

μ, σ^2 unknown.

(c) Problem 9.1 Consider $T = \dfrac{\sqrt{n}(\bar{X} - \mu)}{S} \sim t_{n-1}$

(d) Problem 9.2 $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

(e) $\sqrt{n I_{X_1}(\hat{\theta})}\,(\hat{\theta} - \theta_0) \to N(0,1)$ where $\hat{\theta}$ is MLE of θ_0.

MLE for σ^2 in $N(\mu, \sigma^2) = \boxed{\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \hat{\sigma}^2}$

Compute $I_{X_1}(\sigma^2) = E\left[\left(\frac{\partial}{\partial \sigma^2} \ln L(x_1 | \sigma^2)\right)^2\right] = \frac{1}{2\sigma^4}$

$n = 16$. $\sqrt{n I_{X_1}(\hat{\theta})}\,(\hat{\theta} - \theta_0) = \sqrt{\frac{8}{\hat{\sigma}^4}}\,(\hat{\sigma}^2 - \sigma^2)$ ← pivotal quantity

$$P\left(-z_{0.025} < \sqrt{\frac{8}{\hat{\sigma}^4}}\,(\hat{\sigma}^2 - \sigma^2) < z_{0.025}\right) = 0.95.$$

Problem 9.4

A theoretical model suggests that the time to breakdown of an insulating fluid between electrodes at a particular voltage has an $\exp(\lambda)$ distribution. The following observations $X_1, \ldots, X_{10}$ for the breakdown time are given.

$$41.53, 18.73, 2.99, 30.34, 12.33, \quad 117.52, 73.02, 223.63, 4.00, 26.78$$

a) Show that $2\lambda \sum X_i$ is a pivotal quantity for estimating λ.
b) Use the result from part a) to find 95% confidence intervals for λ and $1/\lambda$ based on the observations given.
c) Using the main result on approximate confidence intervals to construct approximate 95% confidence intervals for λ and $1/\lambda$ based on the observations.

(a) $X_i \sim Exp(\lambda) \sim Gamma\left(1, \frac{1}{\lambda}\right)$

$P(X_i \le x) = 1 - e^{-\lambda x}$ Let $Y_i = 2\lambda X_i$

$$P(Y_i \le y) = P(2\lambda X_i \le y) = P\left(X_i \le \frac{y}{2\lambda}\right) = 1 - e^{-\lambda \cdot \frac{y}{2\lambda}} = 1 - e^{-\frac{y}{2}}$$

$$Y_i \sim Gamma(1, 2) \qquad \lambda = \frac{1}{2}$$

$$\sim \chi^2_2$$

$$2\lambda \sum X_i = \sum Y_i \sim \chi_2^2 + \chi_2^2 + \ldots + \chi_2^2 \quad (n \text{ times})$$

$$\sim \chi^2_{2n}$$

MH3500 Statistics

Tutorial 9 AY2022/23 Semester 2

Problem 9.1

Let $X_1, \ldots, X_n$ be an i.i.d. sample drawn from a normal distribution $N(\mu, \sigma^2)$ with both parameters μ and σ^2 unknown. Determine c such that $(-\infty, \bar{X} + c)$ is a 95% confidence interval for μ.

Hint: use the pivotal quantity $\sqrt{n}(\bar{X} - \mu)/S$ to construct the confidence interval. Your result should be an expression for c involving n, the sample variance S^2, and a percentage point for a *t*-distribution.

Problem 9.2

Let $X_1, \ldots, X_n \sim N(\mu_X, \sigma_X^2)$ and $Y_1, \ldots, Y_m \sim N(\mu_Y, \sigma_Y^2)$ be i.i.d. samples which are independent for each other. Using a pivotal quantity approach, construct a 95% confidence interval for the variance ratio σ_X^2/σ_Y^2.

Hint: show that $\left(\frac{S_X^2}{\sigma_X^2}\right) / \left(\frac{S_Y^2}{\sigma_Y^2}\right)$ follows an *F*-distribution.

Problem 9.3

Suppose the fat content of certain steaks follows a $N(\mu, \sigma^2)$ distribution. The following observations $X_1, \ldots, X_{16}$ for the fat content are given.

$$5.33, 4.25, 3.15, 3.70, \quad 1.61, 6.39, 3.12, 6.59,$$

$$3.53, 4.74, 0.11, 1.60, \quad 5.49, 1.72, 4.15, 2.28$$

a) Suppose $\sigma^2 = 3.2$. use the pivotal quantity $\sqrt{n}(\bar{X} - \mu)/\sigma$ to find 90%, 95% and 99% confidence intervals for μ based on the observations above.
b) In Part a), if we want to cut down the length of the confidence interval to half their length, how much would we need to increase the sample size?

For the remaining pats of this problem, suppose that both parameters μ and σ^2 unknown.

c) Find 90%, 95% and 99% confidence intervals for μ.
d) Find 90%, 95% and 99% confidence intervals for σ^2.
e) Use the main result on approximate confidence intervals to construct an approximate 95% confidence interval for σ^2.
Hint: to remove unknown quantities from the interval, replace $\sigma^2$by the MLE for σ^2in the expressions for the interval endpoints.

Problem 9.4

A theoretical model suggests that the time to breakdown of an insulating fluid between electrodes at a particular voltage has an $\exp(\lambda)$ distribution. The following observations $X_1, \dots, X_{10}$ for the breakdown time are given.

$$41.53, 18.73, 2.99, 30.34, 12.33, \quad 117.52, 73.02, 223.63, 4.00, 26.78$$

a) Show that $2\lambda \sum X_i$ is a pivotal quantity for estimating λ.
b) Use the result from part a) to find 95% confidence intervals for λ and $1/\lambda$ based on the observations given.
c) Using the main result on approximate confidence intervals to construct approximate 95% confidence intervals for λ and $1/\lambda$ based on the observations.