(3) Let $X_1, \dots, X_n$ be independent Poisson random variables with mean 1, and $X = \sum_{i=1}^{n} X_i$.

(a) Use the Markov's inequality to obtain a bound on

$$\mathbb{P}r \left\{ \sum_{i=1}^{n} X_i \geq \frac{3}{2} n \right\}.$$

(b) Use Chebyshev's inequality to obtain a bound on the same probability in part (a).

Markov's Inequality

X nonnegative RV.

$k > 0$.

$$P(X > k) \leq \frac{E(X)}{k}$$

Chebyshev's

X is RV

$Var(X) < \infty$

$k > 0$

$$P(|X - E(X)| > k) \leq \frac{Var(X)}{k^2}$$

(a) $X_i \sim Po(1)$

$X_1 \sim Po(\lambda_1)$

$X_2 \sim Po(\lambda_2)$ indp $\Rightarrow$ $X_1 + X_2 \sim Po(\lambda_1 + \lambda_2)$

Let $X = \sum X_i \sim Po(n)$

$$P\left(\sum X_i \geq \frac{3}{2} n\right) \leq \frac{E(\sum X_i)}{3/2\, n} = \frac{n}{3/2\, n} = 2/3$$

(b) $$P\left(\sum X_i \geq \frac{3}{2} n\right) = P\left(X - n \geq \frac{n}{2}\right)$$

$$\leq P\left(|X - n| \geq \frac{n}{2}\right)$$

$$\leq \frac{Var(X)}{\left(\frac{n}{2}\right)^2} = \frac{n}{\left(\frac{n}{2}\right)^2} = \frac{4}{n}$$

if $A \subseteq B$

$\Downarrow$

$P(A) \leq P(B)$

(4) Flip a fair coin 100 times. By using Central Limit Theorem, estimate the probability of more than 55 heads.

← identically & independently distributed.

CLT: If $X_1, \dots, X_n$ i.i.d. RV

and $\mu = \mathbb{E}(X_1)$, $\sigma^2 = Var(X_1)$

Then for $n > 30$ (recommended)

$$\sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2) \quad \text{approximation}$$

Let $X_i \sim Bin(1, \frac{1}{2})$ be the RV, where $X_i = 1$ if ith flip is a head.
$X_i = 0$ otherwise.

$\mu = \mathbb{E}(X_1) = \frac{1}{2}$

$\sigma^2 = Var(X_1) = 1/4$

$\sum X_i \sim N(\frac{1}{2}n, \frac{1}{4}n)$ $n = 100$

Find $P(\sum X_i > 55)$

$\approx P(\sum X_i \geq 55.5)$

$= P\left(Z \geq \frac{55.5 - 50}{\sqrt{25}}\right) =$ ______

(5) If X is a gamma random variable with parameters $(n, 1)$, approximately how large need n be so that

$$\mathbb{P}r\left\{\left|\frac{X}{n} - 1\right| > 0.01\right\} \leq 0.01?$$

Fact : $X_1 \sim \text{Gamma}(\alpha_1, \beta)$, $X_2 \sim \text{Gamma}(\alpha_2, \beta)$ indp $\Leftrightarrow$ $X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$

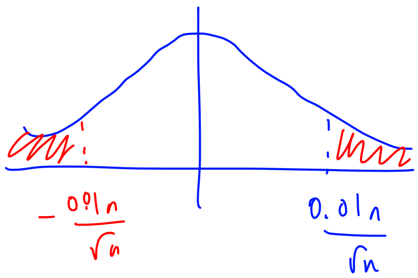
Let $X_i \sim \text{Gamma}(1, 1)$ for $i = 1, \ldots, n$ be independent RV.

$E(X_i) = 1$, $Var(X_i) = 1$

$\Rightarrow \sum X_i \sim \text{Gamma}(n, 1)$ can be approx by $N(\overset{\mu}{n \cdot 1}, \overset{\sigma^2}{n \cdot 1})$

$$0.01 \geq P\left(\left|\frac{X}{n} - 1\right| > 0.01\right) = P(|X - n| > 0.01n)$$

$$= P(|\sum X_i - n| > 0.01n)$$



$$\approx P\left(|Z| > \frac{0.01n}{\sqrt{n}}\right)$$

$$= 2\left[1 - \Phi\left(\frac{0.01n}{\sqrt{n}}\right)\right]$$

$$0.01 \geq 2\left[1 - \Phi(0.01\sqrt{n})\right]$$

$$0.005 \geq 1 - \Phi(0.01\sqrt{n})$$

$$\Phi(0.01\sqrt{n}) \geq 0.995$$

$$0.01\sqrt{n} \geq \Phi^{-1}(0.995)$$

$$n \geq \frac{1}{0.01}\left[\Phi^{-1}(0.995)\right]^2 = \underline{\qquad\qquad}$$

(6) Civil engineers believe that W, the amount of weight (in units of 1000 pounds) that a certain span of a bridge can withstand without structural damage resulting, is normally distributed with mean 400 and standard deviation 40. Suppose that the weight (again, in units of 1000 pounds) of a car is a random variable with mean 3 and standard deviation 0.3. Approximately how many cars would have to be on the bridge span for the probability of structural damage to exceed 0.1?

$W \sim N(400, 40^2)$

Let X_i be the RV for the weight of car i.

$\mathbb{E}(X_i) = 3$, $Var(X_i) = 0.09$

$X_1 \sim N(\mu_1, \sigma_1^2)$
$X_2 \sim N(\mu_2, \sigma_2^2)$

$X_1 \pm X_2$
$\sim N(\mu_1 \pm \mu_2, \sigma_1^2 + \sigma_2^2)$

Find n, such that $P\left(\sum_{i=1}^{n} X_i > W\right) \cancel{\geq} 0.1$

By CLT, $X = \sum X_i \sim N(3n, 0.09n)$

$0.1 < P(X - W > 0)$

$X - W \sim N(400 - 3n, 40^2 + 0.09n)$

$$\approx P\left(Z > \frac{0 - (400 - 3n)}{\sqrt{40^2 + 0.09n}}\right)$$

$$\Phi\left(\frac{400 - 3n}{\sqrt{40^2 + 0.09n}}\right) > 0.1$$

$$\frac{400 - 3n}{\sqrt{40^2 + 0.09n}} > \Phi^{-1}(0.1) \Rightarrow \text{solve for } n.$$

Summary

① Probability : Bayes Rule

Law of Total Probability

② Discrete RV : Bernoulli, Binomial, Geometric, Poisson.

③ Continuous RV : Normal, Uniform, Exponential, Gamma.

$E(X)$, $Var(X)$

④ Joint RV :

$$f_X(x) = \int f(x,y)\,dy$$
$$f_Y(y) = \int f(x,y)\,dx$$
$$\iint_D f(x,y)\,dy\,dx = 1$$

$f(x,y) = f_X \cdot f_Y$

$\Updownarrow$

independence.

$$Cov(X,Y) = \mathbb{E}(XY) - E(X)E(Y).$$

If X, Y indp $\Rightarrow$ $E(XY) = E(X)E(Y)$ $\Rightarrow$ $Cov(X,Y) = 0$

Converse is FALSE.

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} \qquad -1 < \rho < 1$$

If X_i indp $\Rightarrow$ $Var(\sum X_i) = \sum Var(X_i)$ (False in general)

$\mathbb{E}(\sum X_i) = \sum \mathbb{E}(X_i)$ (always true)

⑤ Law of Total Expectation.

$$E(X) = \sum_i E(X \mid A_i) P(A_i)$$

$$\mathbb{E}(X) = E_Y(\underbrace{E_X(X \mid Y)}_{\text{in terms of } Y.})$$

Conditional Variance.

$$Var(Y \mid X=x) = \mathbb{E}(Y^2 \mid X=x) - [E(Y \mid X=x)]^2$$

Law of Total Variance

$$Var(Y) = Var(\mathbb{E}(Y \mid X)) + \mathbb{E}[Var(Y \mid X)]$$

⑥ Markov Ineq, Chebyshev Ineq, CLT

Normal Approximation, Poisson Approx to Binomial