Extra Exercises Week 8

Problem 1. If X is a standard normal random variable, then the random variable $Y = e^X$ is said to be a standard lognormal random variable. Derive $\mathbb{E}(Y)$.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \qquad -\infty < x < \infty$$

CDF of Y: $F_Y(y) = P(Y \le y) = P(e^X \le y)$

$$= P(X \le \ln y)$$

$$= \int_{-\infty}^{\ln y} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

Use Fundamental Thm of Calculus.

$$\frac{d}{dy}\int_a^{g(y)} f(x)\,dx = f(g(y))\,g'(y)$$

PDF of Y: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\ln y)^2}{2}\right) \cdot \frac{1}{y} \qquad 0 < y < \infty$

$$\mathbb{E}(Y) = \int_0^\infty y\, f_Y(y)\, dy = \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\ln y)^2}{2}\right) dy$$

Let $\frac{\ln y}{\sqrt{2}} = u \Rightarrow \frac{1}{y}\,dy = \sqrt{2}\,du$

As $y \to \infty$, $u \to \infty$; $y \to 0$, $u \to -\infty$, $y = e^{\sqrt{2}u} \Rightarrow dy = \sqrt{2}e^{\sqrt{2}u}\,du$

$$\therefore \mathbb{E}(Y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-u^2) \cdot \sqrt{2}\, e^{\sqrt{2}u}\, du$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-u^2 + \sqrt{2}u\right) du$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\left(u - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right) du$$

Let $u - \frac{1}{\sqrt{2}} = t$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}} e^{-t^2}\, dt$$

$$= e^{\frac{1}{2}}$$

Since $\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$ (Gaussian Integral)

Problem 3. The median of a continuous random variable having cumulative distribution function F is that value m such that $F(m) = \frac{1}{2}$. That is, a random variable is just as likely to be larger than its median as it is to be smaller. Find the median of X if X has the following distribution.

(a) Uniform distribution over (a, b);

(b) Standard normal distribution, $N(0, 1)$;

(c) Exponential distribution with parameter λ;

(d) Lognormal distribution with parameters 0 and 1.

(b) $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \qquad -\infty < x < \infty$

$$\frac{1}{2} = F(m) = P(X \leq m) = \int_{-\infty}^{m} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$= \underbrace{\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx}_{=\frac{1}{2}} + \int_{0}^{m} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$\therefore \int_0^m \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = 0$. The integrand $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \geq 0 \quad \forall x$.

For the integral to be zero, we need $\underline{\underline{m = 0}}$ (Area under graph $= 0$)

$$\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = P(X \leq 0) = \frac{1}{2}$$ (since $N(0,1)$ is symmetric about $x = 0$)

(c) $f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0$

$$\frac{1}{2} = F(m) = \int_0^m \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x}\right]_0^m = 1 - e^{-\lambda m}$$

$$\Rightarrow e^{-\lambda m} = \frac{1}{2} \Rightarrow -\lambda m = -\ln 2 \Rightarrow m = \frac{\ln 2}{\lambda}$$

(d) $f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\ln y)^2}{2}\right) \cdot \frac{1}{y}$ $\quad 0 < y < \infty$

$$\frac{1}{2} = F(m) = \int_0^m \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\ln y)^2}{2}\right) \cdot \frac{1}{y}\, dy$$

Let $\ln y = u$

$\frac{1}{y}\, dy = du$

$$= \int_{-\infty}^{\ln m} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

$$= \underbrace{\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du}_{=\frac{1}{2}} + \int_0^{\ln m} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

$\therefore \int_0^{\ln m} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du = 0 \Rightarrow$ Same as (b), we need $\ln m = 0 \Rightarrow m = 1$

Problem 4. A standard Cauchy random variable has density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Show that if X is a standard Cauchy random variable, then $1/X$ is also a standard Cauchy random variable.

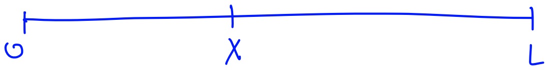
Let $Y = \frac{1}{X}$

$$F_Y(y) = P(Y \le y) = P\left(\frac{1}{X} \le y\right) = P\left(X \ge \frac{1}{y}\right) = \int_{\frac{1}{y}}^{\infty} \frac{1}{\pi(1+x^2)}\, dx = -\int_{\infty}^{\frac{1}{y}} \frac{1}{\pi(1+x^2)}\, dx$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = -\frac{1}{\pi\left(1+\left(\frac{1}{y}\right)^2\right)} \cdot \left(\frac{1}{y}\right)' = \frac{1}{\pi\left(1+\left(\frac{1}{y}\right)^2\right)} \cdot \frac{1}{y^2} = \frac{1}{\pi(1+y^2)}$$

Problem 6. A point is chosen at random on a line segment of length L. Find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

Let $X \sim \text{Unif}(0, L)$ denoting the position of the point on line segment.



Let Y be the ratio of the shorter to the longer segment.

$$Y = \min\left\{ \frac{X}{L-X}, \frac{L-X}{X} \right\}$$

$$P\left(Y < \tfrac{1}{4}\right) = 1 - P\left(\min\left\{ \frac{X}{L-X}, \frac{L-X}{X} \right\} > \frac{1}{4} \right)$$

$$= 1 - P\left(\frac{X}{L-X} > \frac{1}{4} \text{ and } \frac{L-X}{X} > \frac{1}{4} \right)$$

$$= 1 - P\left(4X > L - X \text{ and } 4L - 4X > X \right)$$

$$= 1 - P\left(\tfrac{1}{5}L < X < \tfrac{4}{5}L \right) = 1 - \int_{\frac{1}{5}L}^{\frac{4}{5}L} \frac{1}{L}\, dx$$

$$= 1 - \frac{1}{L}\left(\frac{4L}{5} - \frac{L}{5} \right) = \frac{2}{5}$$

Problem 8. Let X be a random variable with the following density function

$$f_X(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the CDF and PDF of $Y = X^2$.

Let $Y = X^2$

For $0 \leq y \leq 1$,

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3}\, dx = \frac{2\sqrt{y}}{3}$$

For $1 \leq y \leq 4$,

$$F_Y(y) = P(Y \leq y) = P(Y \leq 1) + P(Y \leq y) = \frac{2\sqrt{1}}{3} + P(X^2 \leq y)$$

$$= \frac{2}{3} + P(1 \leq x \leq \sqrt{y}) = \frac{2}{3} + \int_1^{\sqrt{y}} \frac{1}{3}\, dx = \frac{1}{3}(\sqrt{y} - 1) + \frac{2}{3} = \frac{1}{3}(\sqrt{y} - 1)$$

$$f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & 0 \leq y < 1 \\ \frac{1}{6\sqrt{y}} & 1 \leq y < 4 \\ 0 & \text{otherwise} \end{cases}$$