Recap for Tutorial 8

MH1101

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1 Monotone and Boundedness

Definition 1 (Monotone Sequence)**.** A sequence $(a_n)_{n\geq 1}$ is **monotonically increasing** if $a_n \leq a_{n+1}$ for $n \geq 1$. A sequence $(a_n)_{n\geq 1}$ is **monotonically decreasing** if $a_n \geq a_{n+1}$ for $n \geq 1$.

Definition 2 (Boundedness)**.** A sequence $(a_n)_{n\geq 1}$ in bounded if there exists a number M, such that $|a_n| < M$ for all n.

Theorem 1 (Monotonic Sequence Theorem)**.** A bounded, monotonic sequence is convergent.

Extra Information: If a sequence $(a_n)_{n\geq 1}$ is bounded and monotonically increasing, then it converges to its least upper bound, also known as the supremum. It is written as $\sup a_n$. If a sequence $(a_n)_{n\geq 1}$ is bounded and monotonically decreasing, then it converges to its greatest lower bound, also known as the infimum. It is written as $\inf a_n$.

Example: Consider $a_n = 1 - \frac{1}{n}$. This sequence is monotonically increasing and bounded above. We have

$$\lim_{n\to\infty} a_n = 1.$$

We also note that the least upper bound, $\sup a_n = 1$.

2 Series

Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots,$$

we define the partial sum,

$$s_k = \sum_{n=1}^{k} a_n = a_1 + \dots + a_k.$$

$(s_k)_{k\geq 1}$ forms a sequence. To analyse the convergence of a series, we just need to analyse the convergence of its partial sum $(s_k)_{k\geq 1}$.

- If $(s_k)_{k\geq 1}$ converges and $\lim_{k\to\infty} s_k = L$, then

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k = L.$$

- If $(s_k)_{k\geq 1}$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 2 (Geometric Series)**.**

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r}, & \text{if } |r| < 1, \\ \text{diverges}, & \text{otherwise.} \end{cases}$$

Theorem 3. If $\sum_{n\geq 1} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. (Converse is **FALSE**)

Think about it: We can interpret the theorem above as, given a convergent series $\sum a_n$,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ such that } |a_n| < \varepsilon, \text{ whenever } n > N.$$

This is not counter-intuitive. For a convergent series, we can make the tail sequence $(a_n)_{n \geq N}$ arbitrarily small, by choosing a suitable N.

Theorem 4 (n-th term Test)**.** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n\geq 1} a_n$ diverges.

Theorem 5. If $\sum a_n$ and $\sum b_n$ are convergent series and c is a constant, then

- $\sum ca_n = c \sum a_n$ is convergent.
- $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$ is convergent.

Tip: To examine the convergence of a series, we just have to examine the convergence of the tail series. In other words, for the series $\sum_{n\geq 1} a_n$, we just need to check that for some $N \in \mathbb{N}$, such that $\sum_{n>N} a_n$ converges. This is because $a_1 + \ldots a_N$ is finite and will not affect the convergence.

3 Extra Exercises

Problem 1. Determine whether the following series converges or diverges. If it converges, then find the sum.

- $\displaystyle\sum_{n=1}^{\infty} \tan^{-1} n.$
- $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}.$

Problem 2. Given the k-th partial sum of a series $\sum_{n\geq 1} a_n$ as

$$s_k = \frac{k-1}{k+1}.$$

Find a_n and $\sum_{n\geq 1} a_n$.

Problem 3. Find the value of c, such that

$$\sum_{n=0}^{\infty} e^{nc} = 10.$$