Risk Parity and Budgeting

Introduction

> **Risk parity** (or **risk premia parity**) is an approach to investment portfolio management which focuses on allocation of risk, usually defined as volatility, rather than allocation of capital. The risk parity approach asserts that when asset allocations are adjusted (leveraged or deleveraged) to the same risk level, the risk parity portfolio can achieve a higher Sharpe ratio and can be more resistant to market downturns than the traditional portfolio.

Reference book: Roncalli, Thierry - Introduction to Risk Parity and Budgeting

Risk measures

Properties

- Coherency
- Convexity

Common risk measures

- Volatility (standard deviation-based risk measure)
- Value-at-risk
- Expected shortfall

Here we use volatility as risk measures.

Risk Contribution

Let $R(x)$ be the risk measure of portfolio $x = (x_1, \ldots, x_n)$. The **risk contribution (RC)** of asset i is defined as

$$RC_i = x_i \frac{\partial R(x)}{\partial x_i}$$

and

$$R(x) = \sum_{i=1}^{n} x_i \frac{\partial R(x)}{\partial x_i} = \sum_{i=1}^{n} RC_i$$

When

$$R(x) = \sigma(x) = \sqrt{x^T \Sigma x}$$

we have **marginal risk contribution (MRC)**

$$MRC_i \equiv \frac{\partial \sigma(x)}{\partial x_i} = \frac{\Sigma x}{\sqrt{x^T \Sigma x}}$$

and hence

$$RC_i = x_i \frac{(\Sigma x)_i}{\sqrt{x^T \Sigma x}}$$

For an **equally weighted risk contribution (ERC) portfolio**

$$RC_i(x_1, \ldots, x_n) = \frac{1}{n}$$

When consider a set of given risk budgets $\{B_1, \ldots, B_n\}$, the **risk budgeting (RB) portfolio** is defined by

$$RC_i(x_1, \ldots, x_n) = B_i$$

Risk Budgeting Portfolio

Specification

$$\begin{cases} RC_i(x) = b_i R(x) \\ b_i \geq 0 \\ x_i \geq 0 \\ \sum_{i=1}^{n} b_i = 1 \\ \sum_{i=1}^{n} x_i = 1 \end{cases}$$

Numerical solutions

Optimize

$$
\begin{aligned}
x^* &= argminR(x) \\
u.c. \sum_{i=1}^{n} & b_i \ln x_i \geq c
\end{aligned}
$$

with c an arbitrary constant such that $c < \sum_{i=1}^{n} b_i \ln x_i$.

Here the optimization problem is realized in MATLAB using fmincon function.

Problems

1. ERC portfolios have high Sharpe ratio (~1.4) with relatively low annual return (~4.8%), which may not meet the expectation from customers.
2. Bonds usually takes up a large portion in ERC portfolios (~70-80%), due to their low volatility and risk, which may limit the performance of the portfolio.
3. Quantitative methods to determined the risk budget for RB portfolios.

Optimality

> Maillard *et al.* (2010) show that the ERC portfolio corresponds to the tangency portfolio when the correlations are the same and when the assets have the same Sharpe ratio.

Maximize Sharpe ratio (SR) gives

$$
\frac{\partial SR}{\partial x_i} \equiv \frac{\partial}{\partial x_i}\left(\frac{\mu}{\sigma}\right) = 0
$$

then we have

$$
\frac{\partial}{\partial x_i}\left(\frac{\mu}{\sigma}\right) = \frac{1}{\sigma^2}\left(\sigma\frac{\partial \mu}{\partial x_i} - \mu\frac{\partial \sigma}{\partial x_i}\right) = 0
$$

and hence

$$
\frac{\frac{\partial \sigma}{\partial x_i}}{\frac{\partial \mu}{\partial x_i}} = \frac{\mu}{\sigma} \equiv const.
$$

As

$$
\mu = \sum_{i=1}^{n} x_i \mu_i
$$

we have

$$
\frac{\partial \mu}{\partial x_i} = \mu_i
$$

Therefore

$$
\frac{MRC_i}{\mu_i} = \frac{\frac{\partial \sigma}{\partial x_i}}{\mu_i} \equiv const. \tag{1}
$$

RB portfolio gives

$$RC_i(x) = b_i R(x)$$

which can be written as

$$\frac{RC_i}{b_i} = \frac{x_i MRC_i}{b_i} = const. \tag{2}$$

From Eq.(1) and Eq.(2) we have

$$\frac{x_i \mu_i}{b_i} = const. \tag{3}$$

When

$$R(x) = \sigma(x) = \sqrt{x^T \Sigma x}$$

we have

$$MRC_i = \frac{\partial \sigma(x)}{\partial x_i} = \frac{\sigma_i \sum_{j=1}^n \rho_{ij} \sigma_j x_j}{\sigma(x)} \tag{4}$$

Combine Eq.(1) and Eq.(4) we have

$$\frac{\sum_{j=1}^n \rho_{ij} \sigma_j x_j}{SR_i} = \frac{\sum_{j=1}^n \rho_{ij} \sigma_j x_j}{\frac{\mu_i}{\sigma_i}} = const. \tag{5}$$

Use Eq.(2) and Eq.(5) to eliminate x_i

$$\frac{\sum_{j=1}^n \rho_{ij} \sigma_j \frac{b_j}{\mu_j}}{SR_i} = const.$$

Hence

$$\sum_{j=1}^n \frac{\rho_{ij}}{SR_j} b_j \propto SR_i \tag{6}$$

which is equivalent to solving the system of linear equations

$$\begin{pmatrix} \frac{\rho_{11}}{SR_1} & \frac{\rho_{12}}{SR_2} & \cdots & \frac{\rho_{1n}}{SR_n} \\ \frac{\rho_{21}}{SR_1} & \frac{\rho_{22}}{SR_2} & \cdots & \frac{\rho_{2n}}{SR_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\rho_{n1}}{SR_1} & \frac{\rho_{n2}}{SR_2} & \cdots & \frac{\rho_{nn}}{SR_n} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} SR_1 \\ SR_2 \\ \vdots \\ SR_n \end{pmatrix} \tag{7}$$

Note that $\rho_{ii} = 1$ and $\rho_{ij} = \rho_{ji}$ for any i, j.

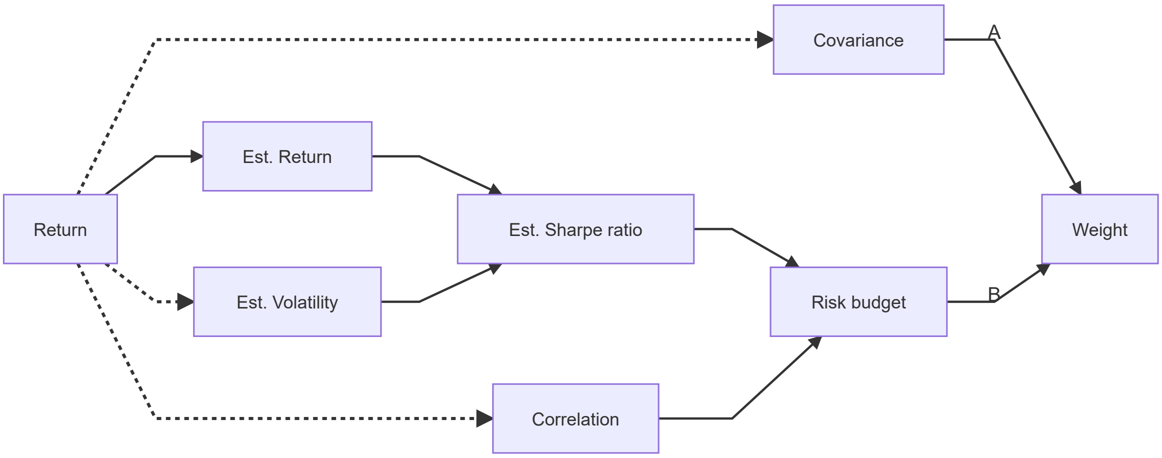
The solution $\{b_1, b_2, \ldots, b_n\}$ maximize RB portfolio's Sharpe ratio.

When $\rho_{ij} = 0$ for any $i \neq j$, from Eq.(6) we have

$$b_i \propto SR_i^2 \tag{8}$$

MATLAB Realization

Process Flow



Two parameters are needed for the Risk Budgeting portfolio. One is risk budget for each asset, the other is covariance matrix to compute weight from risk budget. We are going to test three ways of setting the risk budget:

- **Equally weighted risk contribution (ERC) portfolio**

  Risk budget for each asset is equal

- **Squared Sharpe ratio (SRS) portfolio**

  Risk budget is proportional to squared Sharpe ratio as from Eqs.(8).

- **Max Sharpe ratio (maxSR) portfolio**

  Risk budget is the solution of Eqs.(7) , which maximize portfolio's Sharpe ratio

We have known that the portfolio is insensitive to covariance, which is a main property of ERC portfolio. Estimated Sharpe ratio of each asset is needed for both SRS and maxSR portfolio. In addition, correlation matrix is required for maxSR portfolio.

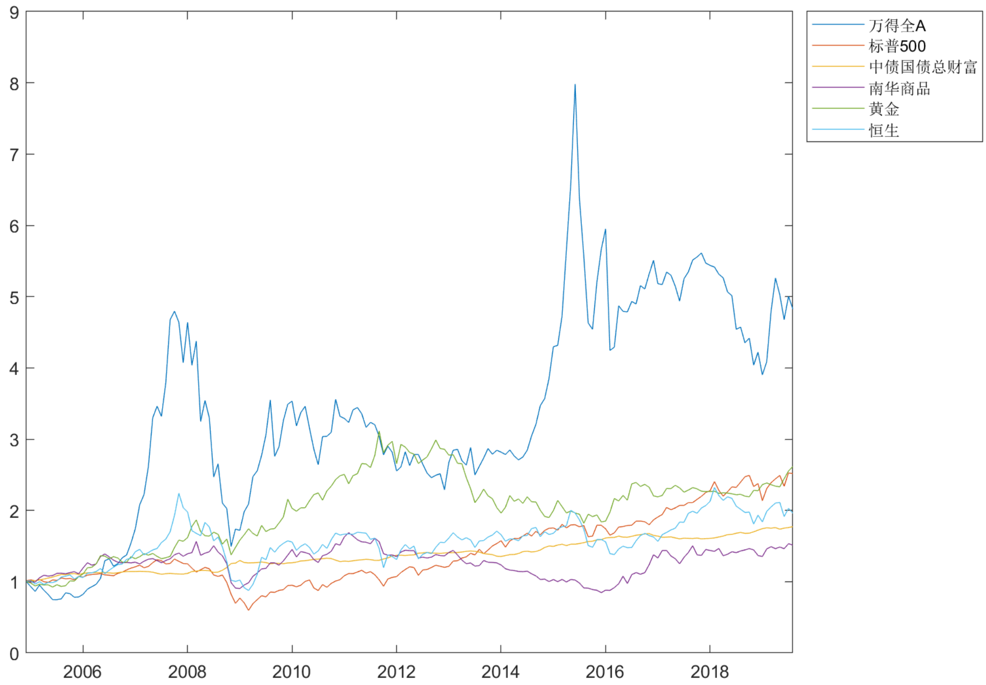
As mentioned, the portfolio is insensitive to covariance. We use covariance of daily return from the past N months to compute covariance. We tested with $N=1,2,4,6,8,10,12$, and it shows not much difference, which verified the insensitivity, and we use $N=4$ in the following backtesting.

Data

We get daily return of Wind A-Shares, S&P 500, China Treasury Bond Index, Nanhua Commodity Index, AU9999 and Hang Seng Index during the period of 12/01/2004 to 08/05/2019 from [Wind Information](#).

[*[Original data](#)*]

The following figure shows the net value of the 6 assets mentioned above from 12/01/2004 to 08/01/2019.



Backtestings are performed during the time period from 12/01/2009 to 08/05/2009. **Positions are changed monthly. In all backtesting, we account for 0.3% transaction cost.**

SRS & maxSR

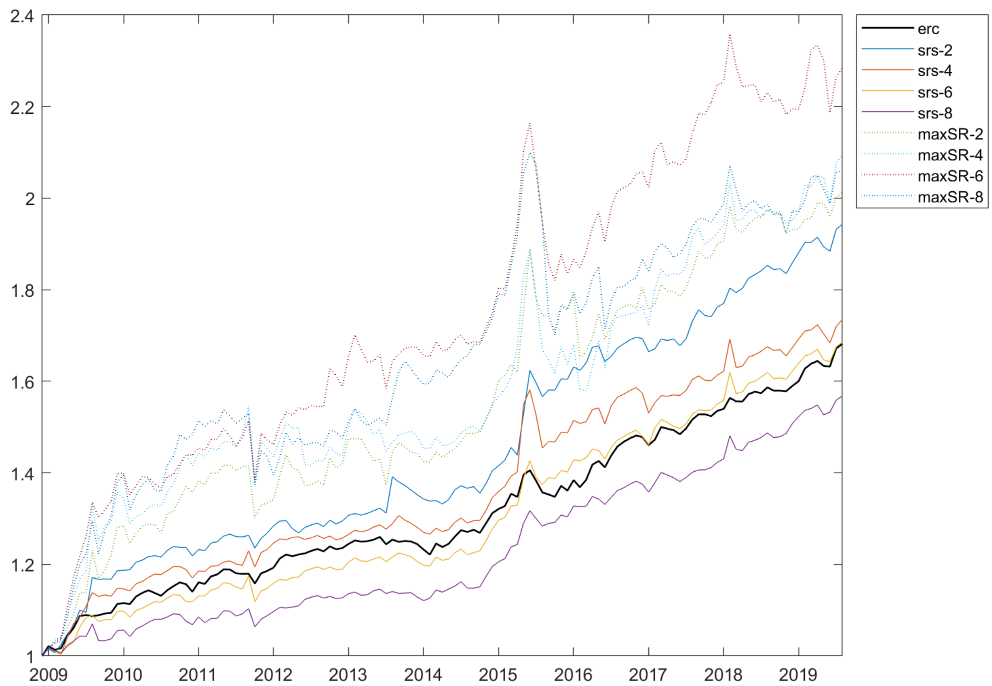
- Covariance - covariance of daily return from the past 4 months
- Est. Return - mean of daily return from the past N months
- Est. Volatility - standard deviation of daily return from the past N months
- Correlation (for maxSR) - correlation of daily return from the past N months
- $N = 1, 2, 4, 6, 8, 10, 12$

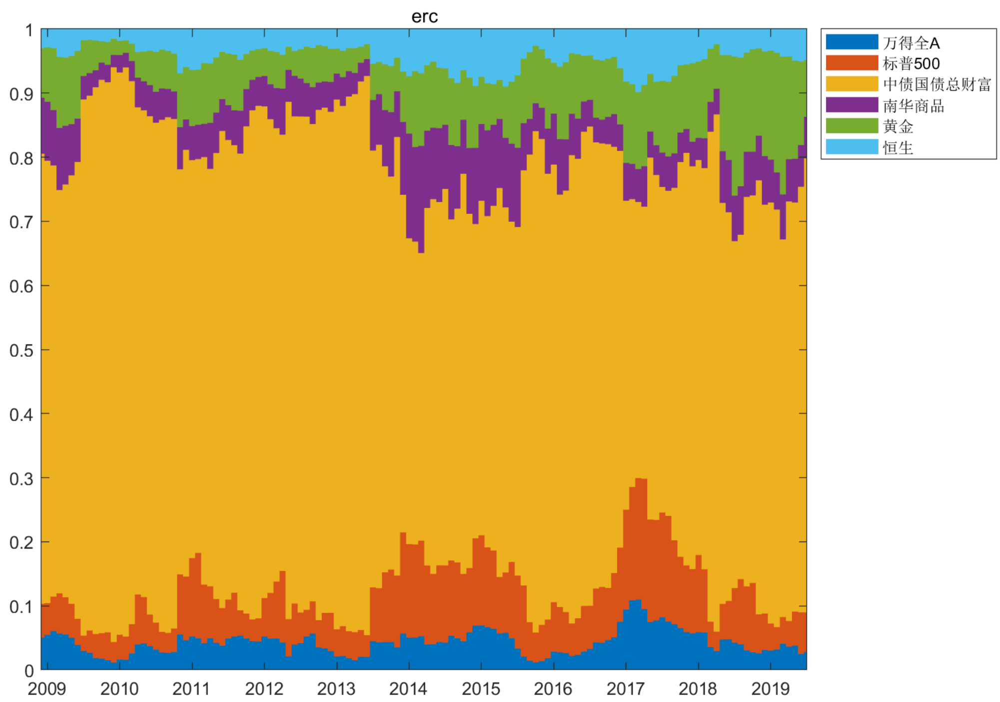
	'erc'	'srs_1'	'srs_2'	'srs_4'	'srs_6'	'srs_8'	'srs_10'	'srs_12'
Annual Ret (%)	4.89	4.90	6.43	5.36	5.05	4.29	4.30	4.98
Annual Vot (%)	3.38	4.43	5.03	5.48	4.15	3.75	3.94	3.73
Sharpe Ratio	1.45	1.11	1.28	0.98	1.22	1.14	1.09	1.33
Max DD (%)	4.13	5.68	4.28	8.05	4.78	3.61	5.45	4.04

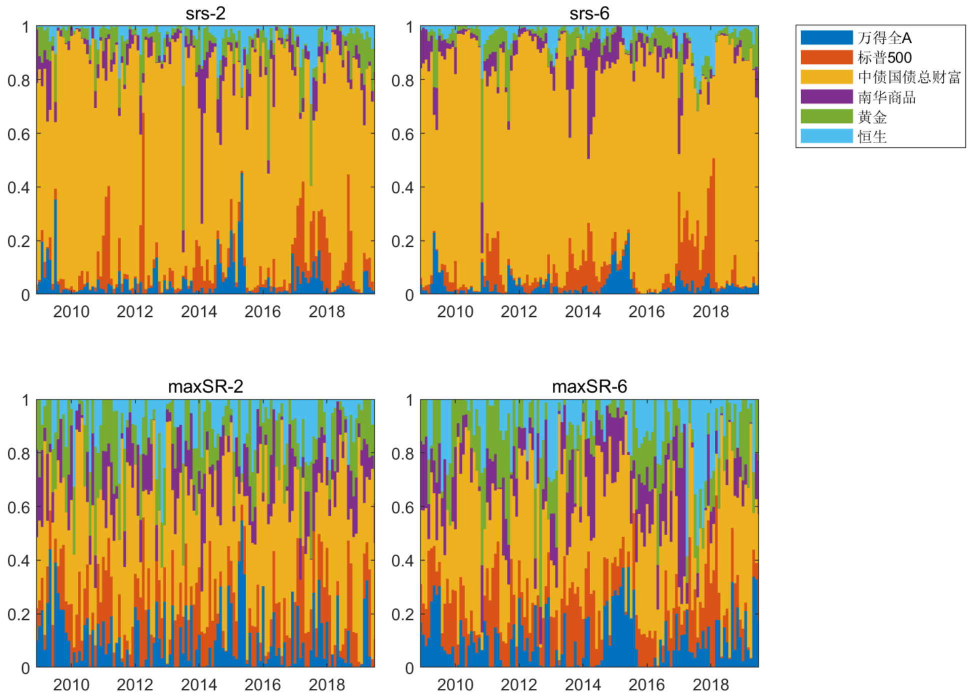
[srs_nm.m]

	'erc'	'mSR1'	'mSR2'	'mSR4'	'mSR6'	'mSR8'	'mSR10'	'mSR12'
Annual Ret (%)	4.89	4.69	6.83	7.12	8.48	7.13	6.09	8.91
Annual Vot (%)	3.38	8.44	8.42	9.89	8.69	9.43	9.25	9.18
Sharpe Ratio	1.45	0.56	0.81	0.72	0.98	0.76	0.66	0.97
Max DD (%)	4.13	15.35	12.62	16.18	15.88	18.96	14.50	13.70

[maxSR_cost.m]







As we can see from the backtesting, squared Sharpe ratio (SRS) portfolio has higher return than the ERC portfolio. When time window N is large, SRS portfolio has similar performance with the ERC portfolio. Mathematically SRS portfolio is a generalization of ERC portfolio, and SRS shows similar property of ERC - high (>1) Sharpe Ratio and low volatility, while have better return.

MaxSR portfolio, on the other hand, is a generalization of SRS portfolio, as it does not assume assets have no correlations. However, in the backtesting, it does not show similar properties with SRS and ERC portfolio. while having higher return, maxSR model also has higher Sharpe ratio and higher max drawdown.

The difference between SRS and maxSR portfolio is we use correlations between the assets to compute risk budget. Here we use correlation of daily returns as estimated the correlations, by analyzing the change of estimated correlations through time (Appendix), we can know that the estimation is constantly changing over time, and this method to estimate correlation is unreliable. Since we temporarily do not have a reliable way of estimating correlations, we do not do further analysis on maxSR portfolio.

Theoretically, maxSR is the more accurate portfolio, but it is difficult for us to realize it in reality, as the parameter it needs, correlations, is not easy to acquire.

Momentum Effect

Here we incorporate the momentum effect into the SRS portfolio. If the Est. Return of an asset is negative, then we set the weight of that asset as 0. It means that the weights are distributed among the assets who have positive Est. Return and Sharpe ratio. If all assets have negative Sharpe ratio, we set weight of China Treasury Bond Index as 1.

Th original momentum portfolio use return instead of Sharpe ratio in SRS portfolio to allocate weight among asset. By bringing volatility in, we have a more stable portfolio. To test this, in addition to SRS portfolio with momentum effect, we test portfolio of risk budget proportional to Sharpe ratio (instead of squared Sharpe ratio) with momentum effect, here we call it Sharpe ratio budget (SRB) portfolio.

- Covariance - covariance of daily return from the past 4 months
- Est. Return - mean of daily return from the past N months
- Est. Volatility - standard deviation of daily return from the past N months
- $N = 1, 2, 4, 6, 8, 10, 12$

SRB portfolio with momentum effect

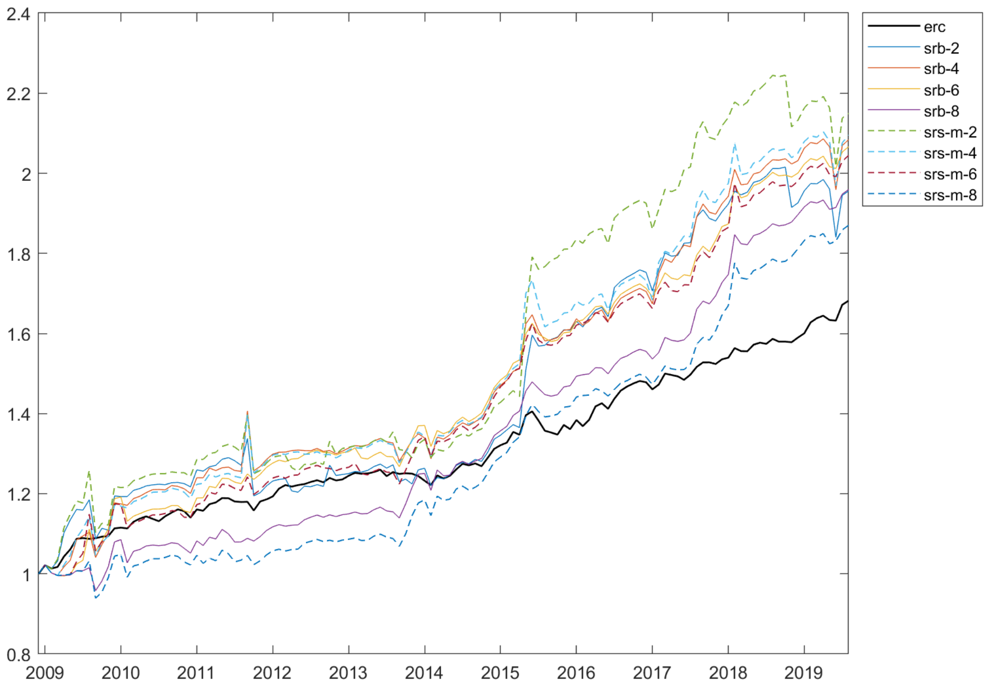
	'erc'	**'srb-1'**	**'srb-2'**	**'srb-4'**	**'srb-6'**	**'srb-8'**	**'srb-10'**	**'srb-12'**
Annual Ret (%)	4.89	8.01	6.80	7.31	7.10	6.53	6.09	5.81
Annual Vot (%)	3.38	9.37	8.11	7.72	5.87	5.58	6.23	6.50
Sharpe Ratio	1.45	0.86	0.84	0.95	1.21	1.17	0.98	0.89
Max DD (%)	4.13	10.49	10.73	10.62	5.10	6.31	10.26	10.26

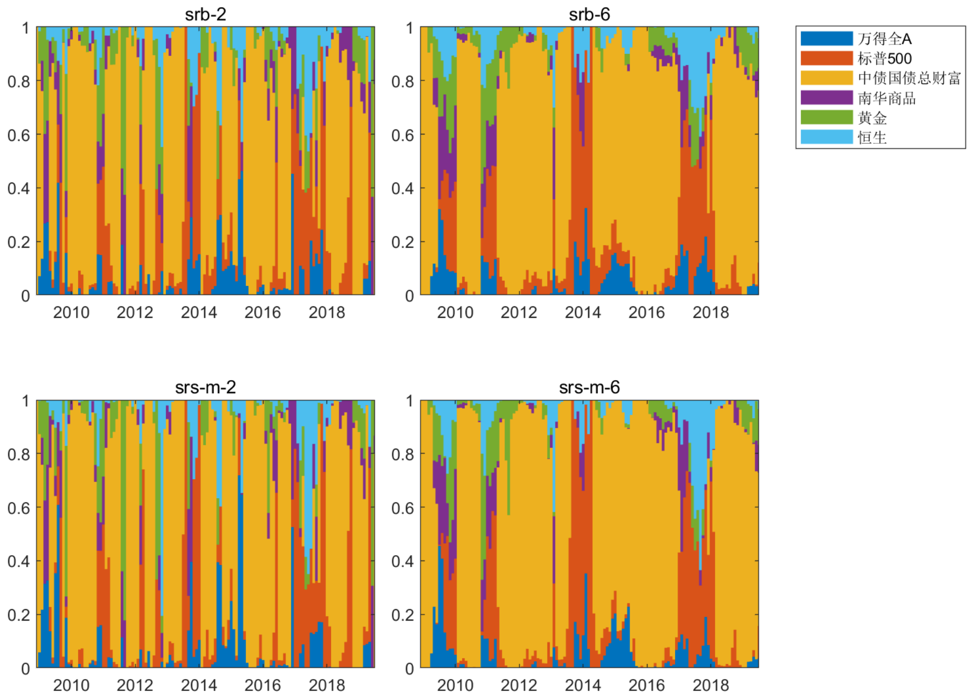
[srb.m]

SRS portfolio with momentum effect

	'erc'	'srsm1'	'srsm2'	'srsm4'	'srsm6'	'srsm8'	'srsm10'	'srsm12'
Annual Ret (%)	4.89	8.25	7.82	7.40	7.01	6.08	6.50	6.24
Annual Vot (%)	3.38	10.34	9.85	8.91	6.81	6.22	6.58	6.91
Sharpe Ratio	1.45	0.80	0.79	0.83	1.03	0.98	0.99	0.90
Max DD (%)	4.13	10.48	12.45	10.42	8.06	9.08	10.37	10.23

[srs_cost.m]





We see that the results from backtesting is consistent with the expectation. SRS and SRB with momentum have similar characteristic. By bringing volatility in, the portfolios become more stable and also have higher return. This two portfolio is like a combination of ERC portfolio and momentum portfolio. In addition, SRS has stronger performance than SRB portfolio with momentum,

Volatility Predicted by GARCH

Here we use GARCH model to improve the estimation for volatility. We choose GARCH(1,1) model after examining the data.

- Covariance - covariance of daily return from the past 4 months
- Est. Return - mean of daily return from the past N months
- Est. Volatility - standard deviation of predicted daily return future month by GARCH(1,1) model fitted using daily return of past 48 months
- $N = 1, 2, 4, 6, 8, 10, 12$

SRS model with GARCH

	'erc'	'srsv-1'	'srsv-2'	'srsv-4'	'srsv-6'	'srsv-8'	'srsv-10'	'srsv-12'
Annual Ret (%)	4.89	5.38	6.66	5.99	5.09	4.48	4.42	5.10
Annual Vot (%)	3.38	4.96	5.60	5.97	4.34	3.85	4.04	3.83
Sharpe Ratio	1.45	1.08	1.19	1.00	1.17	1.16	1.10	1.33

Max DD (%)	4.13	6.15	3.55	6.73	5.18	4.15	5.66	4.02
	'erc'	**'srsv-1'**	**'srsv-2'**	**'srsv-4'**	**'srsv-6'**	**'srsv-8'**	**'srsv-10'**	**'srsv-12'**

[srs_volt_nm.m]

SRB model with momentum effect and GARCH

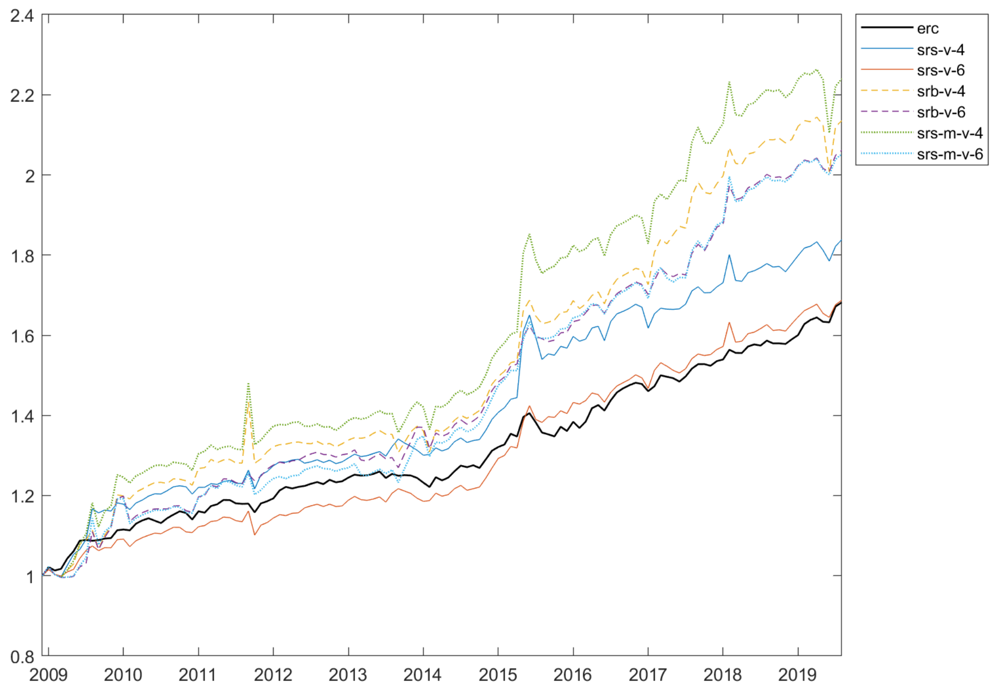
	'erc'	'srbv1'	'srbv2'	'srbv4'	'srbv6'	'srbv8'	'srbv10'	'srbv12'
Annual Ret (%)	4.89	8.20	7.23	7.60	7.10	6.66	6.03	5.87
Annual Vot (%)	3.38	9.34	8.03	7.68	5.81	5.44	6.23	6.51
Sharpe Ratio	1.45	0.88	0.90	0.99	1.22	1.23	0.97	0.90
Max DD (%)	4.13	10.48	10.74	10.66	5.23	5.43	10.41	10.40

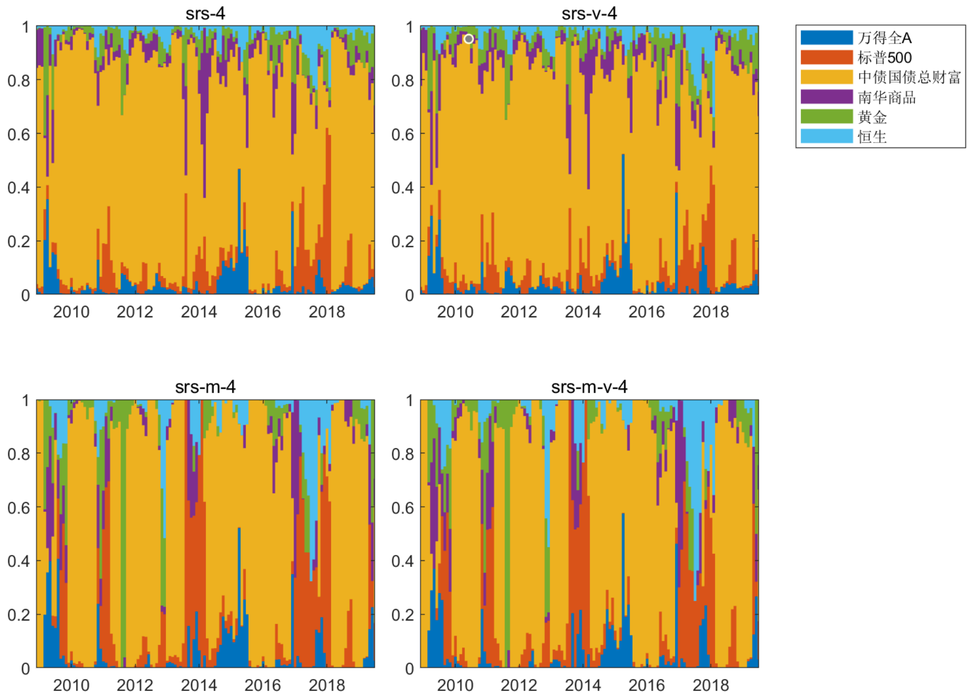
[srb_volt.m]

SRS with momentum effect and GARCH

	'erc'	'srsmv1'	'srsmv2'	'srsmv4'	'srsmv6'	'srsmv8'	'srsmv10'	'srsmv12'
Annual Ret (%)	4.89	8.16	8.60	8.16	7.09	6.48	6.44	6.35
Annual Vot (%)	3.38	10.38	9.70	8.90	6.70	5.92	6.59	6.93
Sharpe Ratio	1.45	0.79	0.89	0.92	1.06	1.10	0.98	0.92
Max DD (%)	4.13	10.46	10.83	10.45	5.81	6.58	10.52	10.56

[srs_volt.m]





With volatility predicted by GARCH(1,1) model, all three portfolios have better performance in return and Sharpe ratio. By comparison we can see that SRS with momentum portfolio has best performance, followed by SRB with momentum portfolio, and then SRS portfolio.

Conclusion

We explore two types of portfolios. One is based on maximize Sharpe ratio under ERC frame (SRS and maxSR), the other is modify momentum effect model by bringing in volatility (SRS and SRB with momentum). We also improve these portfolios by using GARCH model to get better prediction of volatility.

As mentioned, SRS is a strengthened version of ERC, maintaining most of the Sharpe ratio when getting higher return. SRS and SRB with momentum effect sacrifice some Sharpe ratio and get even higher return than original SRS portfolio, and SRS generally has better performance than SRB. MaxSR is difficult to implement in practice as correlations between assets are difficult to estimate. All models will be more stable if we take longer time window for estimation of Sharpe ratio.

It is important to notice that higher return of SRS, SRS with momentum and SRB with momentum is acquired during years when equity securities have higher return.

Appendix

Correlation