Axion Lepton Notes

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1 Lorentz Boost Between CM Frame and Rest Frame

Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ be the 4 momentum in the rest frame (SM background frame), $\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3, \mathbf{p}'_4$ be in the CM frame.

For the scattering process $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$, under relativistic limit, the total momentum is

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = (p_1 + p_2,\ \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2)$$

The CM velocity is

$$\vec{\mathbf{v}} = \frac{\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2}{p_1 + p_2}$$

General Lorentz transformation (wikipedia)

$$(A, \vec{\mathbf{Z}}) \longrightarrow (A', \vec{\mathbf{Z}}')$$

$$A' = \gamma(A - v\hat{\mathbf{n}} \cdot \vec{\mathbf{Z}})$$

$$\vec{\mathbf{Z}}' = \vec{\mathbf{Z}} + (\gamma - 1)(\vec{\mathbf{Z}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - \gamma v A\hat{\mathbf{n}}$$

Thus,

$$p'_1 = \gamma(p_1 - v\hat{\mathbf{n}} \cdot \vec{\mathbf{p}}_1), \quad \vec{\mathbf{p}}'_1 = \vec{\mathbf{p}}_1 + (\gamma - 1)(\vec{\mathbf{p}}_1 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - \gamma v p_1\hat{\mathbf{n}}$$
$$p'_2 = \gamma(p_2 - v\hat{\mathbf{n}} \cdot \vec{\mathbf{p}}_2), \quad \vec{\mathbf{p}}'_2 = \vec{\mathbf{p}}_2 + (\gamma - 1)(\vec{\mathbf{p}}_2 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - \gamma v p_2\hat{\mathbf{n}}$$

One can check that $\mathbf{p}'_1, \mathbf{p}'_2$ are in the CM frame by finding $\vec{\mathbf{p}}'_1 + \vec{\mathbf{p}}'_2 = 0$

And the reverse transformation is

$$p_1 = \gamma(p'_1 + v\hat{\mathbf{n}} \cdot \vec{\mathbf{p}}'_1), \quad \vec{\mathbf{p}}_1 = \vec{\mathbf{p}}'_1 + (\gamma - 1)(\vec{\mathbf{p}}'_1 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \gamma v p'_1\hat{\mathbf{n}}$$

$$p_2 = \gamma(p'_2 + v\hat{\mathbf{n}} \cdot \vec{\mathbf{p}}'_2), \quad \vec{\mathbf{p}}_2 = \vec{\mathbf{p}}'_2 + (\gamma - 1)(\vec{\mathbf{p}}'_2 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \gamma v p'_2\hat{\mathbf{n}}$$

2 Production Rate

The production rate of the interaction we are interested in

$$\gamma(p_1) + \psi(p_2) \longrightarrow \phi(p_3) + \psi(p_4)$$

is

$$\Gamma = \frac{1}{n_a^{eq}} \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} f_1(p_1) f_2(p_2)(1+f_3(p_3))(1-f_4(p_4)) \\ (2\pi)^3 \delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)(2\pi)\delta(E_1+E_2-E_3-E_4)|\mathcal{M}|^2 \tag{1}$$

where the amplitude is

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 = 4\pi\alpha|\tilde{\epsilon}|^2 \frac{t^2}{(s-m_\psi^2)(m_\psi^2-u)}$$

and

$$f_2(p) = f_4(p) = \frac{1}{e^{E(p)/T}+1}$$

$$f_1(p) = f_3(p) = \frac{1}{e^{p/T}-1}$$

Adopt from Schwartz:

$$\begin{aligned} d\Pi_{\text{LIPS}} =& (2\pi)^4\delta^4(\Sigma p) \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} \qquad \text{(Schwartz 5.26)} \\ & \frac{d\Omega}{16\pi^2} \frac{p_f}{E_{\text{CM}}} \Theta(E_{\text{CM}} - m_3 - m_4) \;\; \text{in CM frame} \qquad \text{(Schwartz 5.29)} \end{aligned} \tag{2}$$

Therefore,

$$\Gamma = \frac{1}{n_a^{eq}} \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} f_1(p_1) f_2(p_2) \int \frac{d\Omega}{16\pi^2} \frac{p_f}{E_{\text{CM}}} (1+f_3(p_3))(1-f_4(p_4))|\mathcal{M}|^2 \tag{3}$$

Comparing to Dan and Ben's formula (A.16):

$$\Gamma_{2\to2} \approx \frac{1}{n_a^{eq}} \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} f_1(p_1) f_2(p_2)[1\pm f_3][1\pm f_4] 2s\sigma_{\text{cm}}(s)$$

I'll name our last integral $2s\sigma$ without scrutinizing and focus on it for now:

$$2s\sigma = \int \frac{d\Omega}{16\pi^2} \frac{p_f}{E_{\text{CM}}} (1+f_3(p_3))(1-f_4(p_4))|\mathcal{M}|^2$$

3 Rest Frame Cross Section (calculated in CM frame)

Given $\mathbf{p}_1, \mathbf{p}_2$, and assuming leptons relativistic in CM frame,

$$s = (\mathbf{p}_1 + \mathbf{p}_2)^2 \quad \text{and} \quad p_i = p_f = \frac{1}{2}E_{\text{CM}} = \frac{\sqrt{s}}{2}$$

Let $\vec{\mathbf{p}}_3' = \vec{\mathbf{p}}_f,\ \vec{\mathbf{p}}_4' = -\vec{\mathbf{p}}_f$, then the outcome momentum in the rest frame are

$$p_3 = \gamma(p_3' + v\hat{\mathbf{n}}\cdot\vec{\mathbf{p}}_3') = \gamma(p_f + v\hat{\mathbf{n}}\cdot\vec{\mathbf{p}}_f)$$

$$p_4 = \gamma(p_4' + v\hat{\mathbf{n}}\cdot\vec{\mathbf{p}}_4') = \gamma(p_f - v\hat{\mathbf{n}}\cdot\vec{\mathbf{p}}_f)$$

Let

$$\hat{\mathbf{n}}\cdot\vec{\mathbf{p}}_f = p_f \cos\alpha$$

Then

$$p_3 = \frac{\sqrt{s}}{2}\gamma(1+v\cos\alpha)$$

$$p_4 = \frac{\sqrt{s}}{2}\gamma(1 - v\text{cos}\alpha)$$

Note that this might solve our double Bose-Einstein enhancement problem since

$$\mathbf{p}_1 + \mathbf{p}_2 = (p_1 + p_2,\ \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2) = (\gamma\sqrt{s}, \gamma\sqrt{s}\vec{\mathbf{v}})$$

and when $p_1 \to 0$, $\gamma\sqrt{s} \to p_2$, thus

$$p_3 \to \frac{p_2}{2}(1 + v\text{cos}\alpha) \neq 0 \qquad \text{when } \alpha \neq \pi$$

The amplitude

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 &= 4\pi\alpha|\tilde{\epsilon}|^2 \frac{t^2}{(s - m_\psi^2)(m_\psi^2 - u)} \\
&= 4\pi\alpha|\tilde{\epsilon}|^2 \frac{(\mathbf{p}_1' - \mathbf{p}_3')^4}{(s - m_\psi^2)(m_\psi^2 - (\mathbf{p}_1' - \mathbf{p}_4')^2)} \\
&= \frac{4\pi\alpha|\tilde{\epsilon}|^2}{s - m_\psi^2} \frac{((p_i - p_f)^2 - (\vec{\mathbf{p}}_i - \vec{\mathbf{p}}_f)^2)^2}{m_\psi^2 - (p_i - p_f)^2 + (\vec{\mathbf{p}}_i + \vec{\mathbf{p}}_f)^2} \\
&= \frac{4\pi\alpha|\tilde{\epsilon}|^2}{s - m_\psi^2} \frac{(-(\vec{\mathbf{p}}_i - \vec{\mathbf{p}}_f)^2)^2}{m_\psi^2 + (\vec{\mathbf{p}}_i + \vec{\mathbf{p}}_f)^2} \\
(s \gg m_\psi^2) &= \frac{4\pi\alpha|\tilde{\epsilon}|^2}{s} \frac{4p_f^4(1 - \text{cos}\theta)^2}{m_\psi^2 + 2p_f^2(1 + \text{cos}\theta)}
\end{aligned} \tag{4}$$

Where $\vec{\mathbf{p}}_i \cdot \vec{\mathbf{p}}_f = p_i p_f \text{cos}(\theta)$, and $p_f = \sqrt{s}/2$

Thus

$$\begin{aligned}
2s\sigma(s, \vec{\mathbf{v}}, T) &= \int \frac{d\Omega}{16\pi^2} \frac{p_f}{E_{\text{CM}}} (1 + f_3(p_3))(1 - f_4(p_4))|\mathcal{M}|^2 \\
&= \frac{\alpha|\tilde{\epsilon}|^2}{16\pi} \int d\phi d\text{cos}\theta \frac{1}{1 - e^{-\frac{\gamma\sqrt{s}}{2T}(1 + v\text{cos}\alpha)}} \frac{1}{1 + e^{-\frac{\gamma\sqrt{s}}{2T}(1 - v\text{cos}\alpha)}} \frac{(1 - \text{cos}\theta)^2}{\frac{2m_\psi^2}{s} + (1 + \text{cos}\theta)} \\
&= \frac{\alpha|\tilde{\epsilon}|^2}{16\pi} \int d\phi d\text{cos}\theta \frac{1}{1 + 2e^{-\frac{\gamma\sqrt{s}}{2T}} \sinh(\frac{\gamma\sqrt{s}}{2T} v\text{cos}\alpha) - e^{-2\frac{\gamma\sqrt{s}}{2T}}} \frac{(1 - \text{cos}\theta)^2}{\frac{2m_\psi^2}{s} + (1 + \text{cos}\theta)}
\end{aligned} \tag{5}$$

Align $\vec{\mathbf{v}}$ in the x-z plane, θ' be the angle from $+\vec{\mathbf{z}}$,

$$\vec{\mathbf{v}} = v(\text{sin}\theta',\ 0,\ \text{cos}\theta')$$

$$\vec{\mathbf{p}}_f = p_f(\text{sin}\theta\text{cos}\phi,\ \text{sin}\theta\text{sin}\phi,\ \text{cos}\theta)$$

Thus,

$$\text{cos}\alpha = \text{sin}\theta'\text{sin}\theta\text{cos}\phi + \text{cos}\theta'\text{cos}\theta$$

No idea how to integrate this analytically. If integrating numerically, we'll have a pole at $v \to 1,\ \alpha \to \pi$, which is the approximation previously used $(p_3 = p_1,\ p_4 = p_2)$.

4 Parametrization (CM frame)

Attempt to write the integral using $\gamma = \frac{1}{\sqrt{1-v^2}},\ p = p_i = p_f,\ \theta'$(azimuthal angle of $\vec{\mathbf{v}}$), $\theta,\ \phi$(CM angles).

We find $\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2$ in the rest frame.

$$p_1 = \gamma(p_i + \vec{\mathbf{v}} \cdot \vec{\mathbf{p}}_i) = \gamma p + \sqrt{\gamma^2 - 1} p \mathrm{cos}\theta'$$

$$p_2 = \gamma(p_i - \vec{\mathbf{v}} \cdot \vec{\mathbf{p}}_i) = \gamma p - \sqrt{\gamma^2 - 1} p \mathrm{cos}\theta'$$

$$\vec{\mathbf{p}}_1 = p_i \hat{\mathbf{z}} + (\gamma - 1) p_i \mathrm{cos}\theta' \hat{\mathbf{v}} + \gamma v p_i \hat{\mathbf{v}} = ([(\gamma - 1) p \mathrm{cos}\theta' + \gamma v p] \mathrm{sin}\theta',\ 0,\ [(\gamma - 1) p \mathrm{cos}\theta' + \gamma v p] \mathrm{cos}\theta' + p)$$

$$\vec{\mathbf{p}}_2 = -p_i \hat{\mathbf{z}} - (\gamma - 1) p_i \mathrm{cos}\theta' \hat{\mathbf{v}} + \gamma v p_i \hat{\mathbf{v}} = ([-(\gamma - 1) p \mathrm{cos}\theta' + \gamma v p] \mathrm{sin}\theta',\ 0,\ [-(\gamma - 1) p \mathrm{cos}\theta' + \gamma v p] \mathrm{cos}\theta' - p)$$

$$\vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_2 = p^2 [\gamma^2 - 2 - (\gamma^2 - 1) \mathrm{cos}^2\theta']$$

The measure

$$\begin{aligned}
\int \frac{d^3 p_1}{(2\pi)^3 2p_1} \int \frac{d^3 p_2}{(2\pi)^3 2p_2} &= \int 4\pi \frac{p_1^2 dp_1}{(2\pi)^3 2p_1} \int 2\pi \frac{p_2^2 dp_2}{(2\pi)^3 2p_2} \int d\mathrm{cos}\theta_{12} \\
&= \frac{1}{32\pi^4} \int dp_1 \int dp_2 \int d(\vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_2) \\
&= \frac{1}{32\pi^4} \int d\gamma \int dp \int d\mathrm{cos}\theta' |\mathrm{Jacobian}| \\
&= \frac{1}{32\pi^4} \int_1^\infty d\gamma \int_0^\infty dp \int_{-1}^1 d\mathrm{cos}\theta' 8p^3 \sqrt{\gamma^2 - 1}
\end{aligned} \tag{6}$$

Thus the production rate is

$$\begin{aligned}
\Gamma =& \frac{1}{n_a^{eq}} \frac{1}{32\pi^4} \int_1^\infty d\gamma \int_0^\infty dp \int_{-1}^1 d\mathrm{cos}\theta' 8p^3 \sqrt{\gamma^2 - 1} \frac{1}{(e^{\frac{p}{T}(\gamma + \sqrt{\gamma^2 - 1}\mathrm{cos}\theta')} - 1)(e^{\frac{p}{T}(\gamma - \sqrt{\gamma^2 - 1}\mathrm{cos}\theta')} + 1)} \\
& \frac{\alpha |\tilde{\epsilon}|^2}{16\pi} \int d\phi d\mathrm{cos}\theta \frac{1}{(1 - e^{-\frac{p}{T}(\gamma + \sqrt{\gamma^2 - 1}\mathrm{cos}\alpha)})(1 + e^{-\frac{p}{T}(\gamma - \sqrt{\gamma^2 - 1}\mathrm{cos}\alpha)})} \frac{(1 - \mathrm{cos}\theta)^2}{\frac{m_\psi^2}{2p^2} + (1 + \mathrm{cos}\theta)} \\
=& \frac{1}{n_a^{eq}} \frac{\alpha |\tilde{\epsilon}|^2}{64\pi^5} \int_1^\infty \sqrt{\gamma^2 - 1} d\gamma \int_0^\infty p^3 dp \int_{-1}^1 d\mathrm{cos}\theta' \frac{1}{(e^{\frac{p}{T}(\gamma + \sqrt{\gamma^2 - 1}\mathrm{cos}\theta')} - 1)(e^{\frac{p}{T}(\gamma - \sqrt{\gamma^2 - 1}\mathrm{cos}\theta')} + 1)} \\
& \int d\phi d\mathrm{cos}\theta \frac{1}{(1 - e^{-\frac{p}{T}(\gamma + \sqrt{\gamma^2 - 1}\mathrm{cos}\alpha)})(1 + e^{-\frac{p}{T}(\gamma - \sqrt{\gamma^2 - 1}\mathrm{cos}\alpha)})} \frac{(1 - \mathrm{cos}\theta)^2}{\frac{m_\psi^2}{2p^2} + (1 + \mathrm{cos}\theta)}
\end{aligned} \tag{7}$$

5 Rest Frame Cross Section (calculated in rest frame)

Align $\vec{\mathbf{p}}_1$ with $+\vec{\mathbf{z}}$, $\theta_1 = 0$, align $\vec{\mathbf{p}}_2$ in the x-z plane, $\phi_2 = 0$.

The amplitude in rest frame:

$$\begin{aligned}
\frac{1}{4} \sum_{\mathrm{spin}} |\mathcal{M}|^2 &= 4\pi\alpha |\tilde{\epsilon}|^2 \frac{t^2}{(s - m_\psi^2)(m_\psi^2 - u)} \\
&= 4\pi\alpha |\tilde{\epsilon}|^2 \frac{(\mathbf{p}_1 - \mathbf{p}_3)^4}{(s - m_\psi^2)(m_\psi^2 - (\mathbf{p}_1 - \mathbf{p}_4)^2)} \\
(s \gg m_\psi^2) &= \frac{4\pi\alpha |\tilde{\epsilon}|^2}{s} \frac{((p_1 - p_3)^2 - (\vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_3)^2)^2}{m_\psi^2 - (p_1 - p_4)^2 + (\vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_4)^2} \\
&= \frac{4\pi\alpha |\tilde{\epsilon}|^2}{s} \frac{4p_1^2 p_3^2 (1 - \mathrm{cos}\theta_{13})^2}{m_\psi^2 + 2p_1 p_4 (1 - \mathrm{cos}\theta_{14})}
\end{aligned} \tag{8}$$

The production rate:

$$\Gamma = \frac{1}{n_a^{eq}} \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} f_1(p_1) f_2(p_2) \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} (1+f_3(p_3))(1-f_4(p_4)) \\ (2\pi)^3 \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)(2\pi)\delta(E_1+E_2-E_3-E_4)|\mathcal{M}|^2 \tag{9}$$

$$\begin{aligned} 2s\sigma(\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2) &= \int \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{1}{2E_4} (1+f_3(p_3))(1-f_4(p_4))(2\pi)\delta(E_1+E_2-E_3-E_4)|\mathcal{M}|^2 \Big|_{\vec{\mathbf{p}}_4 = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_3} \\ (p_4 \gg m_\psi) &= \int \frac{d^3p_3}{(2\pi)^3 4p_3p_4} (1+f_3(p_3))(1-f_4(p_4))(2\pi)\delta(p_1+p_2-p_3-p_4)|\mathcal{M}|^2 \Big|_{\vec{\mathbf{p}}_4 = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_3} \\ &= \frac{1}{16\pi^2} \int d\Omega_3 \frac{p_3}{p_4} (1+f_3(p_3))(1-f_4(p_4))|\mathcal{M}|^2 \Big|_{\mathbf{p}_3 + \mathbf{p}_4 = \mathbf{p}_1 + \mathbf{p}_2} \\ &= \frac{1}{16\pi^2} \frac{4\pi\alpha|\tilde{\epsilon}|^2}{s} \int d\Omega_3 \frac{p_3}{p_4} \frac{1}{(1-e^{-p_3/T})(1+e^{-p_4/T})} \frac{4p_1^2 p_3^2 (1-\cos\theta_{13})^2}{m_\psi^2 + 2p_1p_4(1-\cos\theta_{14})} \Big|_{\mathbf{p}_3 + \mathbf{p}_4 = \mathbf{p}_1 + \mathbf{p}_2} \end{aligned} \tag{10}$$

6 Parametrization (rest frame)

For arbitrary $\vec{\mathbf{p}}_1$, $\vec{\mathbf{p}}_2$, we have 6 degrees of freedom. And for $\vec{\mathbf{p}}_3$, $\vec{\mathbf{p}}_4$, we have $6-4=2$ dof due to the energy-momentum conservation.

In hope of solving momenta for angles showing simpler forms than the inverse, I used the following parameters (see fig1).

$$\begin{gathered} E = p_1 + p_2 \\ p = |\vec{\mathbf{p}}| = |\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2| \\ p_1 = |\vec{\mathbf{p}}_1| \\ p_3 = |\vec{\mathbf{p}}_3| \\ \phi, \text{ the polar angle of } \vec{\mathbf{p}}_3 \text{ and } \vec{\mathbf{p}}_4 \end{gathered}$$

Note that here's only 5 dof, while we should have a total of 8. The hidden 3 are the azimuthal angle and polar angle of $\vec{\mathbf{p}}$, and the rotation around $\vec{\mathbf{p}}$. Due to symmetry, we are free to set $\vec{\mathbf{p}}$ in the +z direction, and $\vec{\mathbf{p}}_1$, $\vec{\mathbf{p}}_2$ in the +x-z semi-plane. The hidden dof then integrates into $2 \times 2\pi \times 2\pi = 8\pi^2$.

To find Cartesian coordinates, we define θ_i as the angle between $\vec{\mathbf{p}}_i$ and $\vec{\mathbf{p}}$.

$$\begin{aligned} \cos\theta_i &= \frac{p^2 + p_i^2 - (E-p_i)^2}{2pp_i} = \frac{E}{p} - \frac{E^2 - p^2}{2pp_i} \\ \sin\theta_i &= \sqrt{1 - \cos\theta_i} = \frac{\sqrt{E^2 - p^2}}{p} \sqrt{(1 - \frac{E-p}{2p_i})(\frac{E+p}{2p_i} - 1)} \end{aligned}$$

The coordinates are then

$$\vec{\mathrm{p}}_1 = p_1(\sin\theta_1, 0, \cos\theta_1), \;\; \vec{\mathrm{p}}_2 = p_2(-\sin\theta_2, 0, \cos\theta_2),$$

$$\vec{\mathrm{p}}_3 = p_3(\sin\theta_3\cos\phi, \sin\theta_3\sin\phi, \cos\theta_3), \;\; \vec{\mathrm{p}}_4 = p_4(-\sin\theta_4\cos\phi, -\sin\theta_4\sin\phi, \cos\theta_4)$$

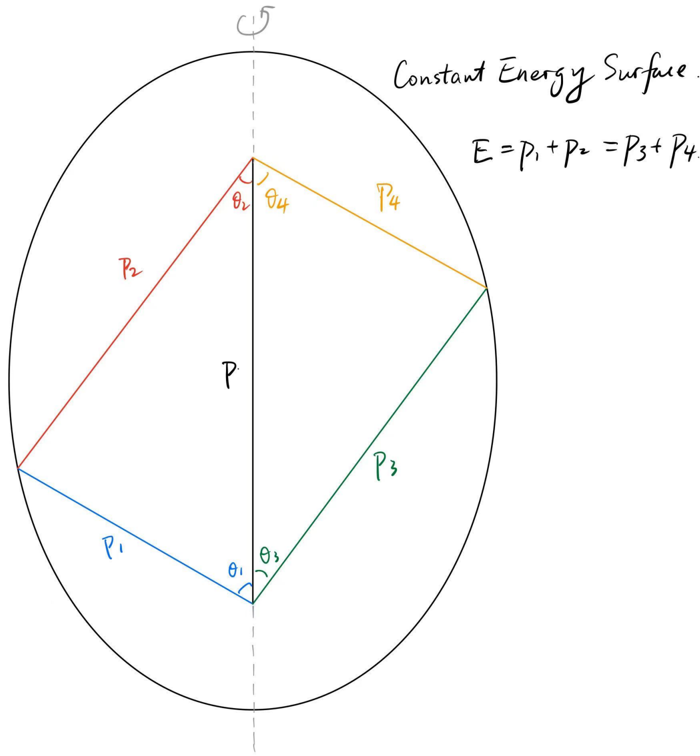

Figure 1: Parametrization of $\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 = \vec{\mathbf{p}}_3 + \vec{\mathbf{p}}_4$.

The angles are then

$$\cos\theta_{13} = \sin\theta_1 \sin\theta_3 \cos\phi + \cos\theta_1 \cos\theta_3 = a \cdot \cos\phi + b$$

$$\cos\theta_{14} = -\sin\theta_1 \sin\theta_4 \cos\phi + \cos\theta_1 \cos\theta_4 = -c \cdot \cos\phi + d$$

The measure in this parametrization is

$$\begin{aligned}
&\int \frac{d^3p_1}{(2\pi)^3 2p_1} \int \frac{d^3p_2}{(2\pi)^3 2p_2} \int \frac{d^3p_3}{(2\pi)^3 2p_3} \int \frac{d^3p_4}{(2\pi)^3 2p_4} (2\pi)^3 \delta^3(\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_3 - \vec{\mathbf{p}}_4)(2\pi)\delta(p_1 + p_2 - p_3 - p_4) \\
=& \int d^3p_1 \int d^3p_2 \int d^3p_3 \int d^3p_4 \frac{1}{(2\pi)^8 p_1 p_2 p_3 p_4} \delta^3(\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_3 - \vec{\mathbf{p}}_4)\delta(p_1 + p_2 - p_3 - p_4) \\
=& \int d^3p_1 \int d^3p_2 \int d^3p \delta(\vec{\mathbf{p}} - \vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_2) \int d^3p_3 \int d^3p_4 \frac{1}{(2\pi)^8 p_1 p_2 p_3 p_4} \delta^3(\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_3 - \vec{\mathbf{p}}_4)\delta(p_1 + p_2 - p_3 - p_4) \\
=& \int d^3p \int d^3p_1 \int d^3p_3 \int d^3p_4 \frac{1}{(2\pi)^8 p_1 p_2 p_3 p_4} \delta^3(\vec{\mathbf{p}} - \vec{\mathbf{p}}_3 - \vec{\mathbf{p}}_4)\delta(p_1 + p_2 - p_3 - p_4)\Big|_{\vec{\mathbf{p}}_2 = \vec{\mathbf{p}} - \vec{\mathbf{p}}_1} \\
=& \int d^3p \int d^3p_1 \int d^3p_3 \frac{1}{(2\pi)^8 p_1 p_2 p_3 p_4} \delta(p_1 + p_2 - p_3 - p_4)\Big|_{\vec{\mathbf{p}}_2 = \vec{\mathbf{p}} - \vec{\mathbf{p}}_1,\ \vec{\mathbf{p}}_4 = \vec{\mathbf{p}} - \vec{\mathbf{p}}_3} \\
=& \int d\Omega_p p^2 dp \int d\phi_1 d\cos\theta_1 p_1^2 dp_1 \int d\phi_3 d\cos\theta_3 p_3^2 dp_3 \frac{1}{(2\pi)^8 p_1 p_2 p_3 p_4} \delta(p_1 + p_2 - p_3 - p_4)\Big|_{\vec{\mathbf{p}}_2, \vec{\mathbf{p}}_4}
\end{aligned} \tag{11}$$

When p is fixed, the direction of $\vec{\mathrm{p}}$ does not affect the production ratio. We align $\vec{\mathrm{p}}$ in the +z direction. Similarly, there's a freedom to rotate all the $\vec{\mathrm{p}}_i$'s around $\vec{\mathrm{p}}$ (i.e. the z axis). Only the relative angle matters.

Let $\phi = \phi_3 - \phi_1$ be the polar angle between the p_1, p_2 plane and the p_3, p_4 plane. Then,

The Measure (continued)

$$
\begin{aligned}
&= \int 4\pi p^2 dp \int 2\pi d\cos\theta_1 p_1^2 dp_1 \int d\phi d\cos\theta_3 p_3^2 dp_3 \frac{1}{(2\pi)^8 p_1 p_2 p_3 p_4} \delta(p_1 + p_2 - p_3 - p_4) \Big|_{\vec{\mathbf{p}}_2, \vec{\mathbf{p}}_4} \\
&= \int 4\pi p^2 dp \int 2\pi d\cos\theta_1 p_1^2 dp_1 \int d\phi d\cos\theta_3 p_3^2 dp_3 \frac{1}{(2\pi)^8 p_1 p_2 p_3 p_4} \delta(p_1 + p_2 - p_3 - p_4) \int dE \delta(E - p_1 - p_2) \Big|_{\vec{\mathbf{p}}_2, \vec{\mathbf{p}}_4} \\
&= 8\pi^2 \int dE \int p^2 dp \int d\phi \int p_1^2 dp_1 \int d\cos\theta_1 \delta(E - p_1 - p_2) \int p_3^2 dp_3 \int d\cos\theta_3 \delta(E - p_3 - p_4) \frac{1}{(2\pi)^8 p_1 p_2 p_3 p_4} \Big|_{\vec{\mathbf{p}}_2, \vec{\mathbf{p}}_4}
\end{aligned} \tag{12}
$$

Look at the delta functions,

$$
p_2 = \sqrt{p_1^2 + p^2 - 2p_1 p \cos\theta_1}
$$

$$
\frac{dp_2}{d\cos\theta_1} = \frac{1}{2} \frac{1}{p_2} (-2p_1 p) = -\frac{p p_1}{p_2}
$$

$$
\int d\cos\theta_1 \delta(E - p_1 - p_2) = \frac{1}{|\frac{dp_2}{d\cos\theta_1}|} \Theta(E - p_1 - p_2 \text{ can be } 0) = \frac{p_2}{p p_1} \Theta(E - p) \Theta(p_1 - \frac{E-p}{2}) \Theta(\frac{E+p}{2} - p_1)
$$

Thus,

The Measure (continued)

$$
\begin{aligned}
&= 8\pi^2 \int dE \int^E p^2 dp \int d\phi \int_{\frac{E-p}{2}}^{\frac{E+p}{2}} p_1^2 dp_1 \int_{\frac{E-p}{2}}^{\frac{E+p}{2}} p_3^2 dp_3 \frac{p_2}{p p_1} \frac{p_4}{p p_3} \frac{1}{(2\pi)^8 p_1 p_2 p_3 p_4} \Big|_{\vec{\mathbf{p}}_2, \vec{\mathbf{p}}_4} \\
&= \frac{1}{32\pi^6} \int dE \int^E dp \int d\phi \int_{\frac{E-p}{2}}^{\frac{E+p}{2}} dp_1 \int_{\frac{E-p}{2}}^{\frac{E+p}{2}} dp_3 \Big|_{\vec{\mathbf{p}}_2, \vec{\mathbf{p}}_4}
\end{aligned} \tag{13}
$$

The whole integral is then,

$$
\begin{aligned}
\Gamma =& \frac{1}{n_a^{eq}} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \int \frac{d^3 p_4}{(2\pi)^3 2E_4} f_1(p_1) f_2(p_2)(1 + f_3(p_3))(1 - f_4(p_4)) \\
& (2\pi)^3 \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)(2\pi)\delta(E_1 + E_2 - E_3 - E_4)|\mathcal{M}|^2 \\
=& \frac{1}{n_a^{eq}} \frac{1}{32\pi^6} \int_0^\infty dE \int_0^E dp \int_0^{2\pi} d\phi \int_{\frac{E-p}{2}}^{\frac{E+p}{2}} dp_1 \int_{\frac{E-p}{2}}^{\frac{E+p}{2}} dp_3 f_1(p_1) f_2(p_2)(1 + f_3(p_3))(1 - f_4(p_4))|\mathcal{M}|^2 \\
=& \frac{1}{n_a^{eq}} \frac{1}{32\pi^6} \int_0^\infty dE \int_0^E dp \int_0^{2\pi} d\phi \int_{\frac{E-p}{2}}^{\frac{E+p}{2}} dp_1 \int_{\frac{E-p}{2}}^{\frac{E+p}{2}} dp_3 \\
& f_1(p_1) f_2(p_2)(1 + f_3(p_3))(1 - f_4(p_4)) \frac{4\pi\alpha|\tilde{\epsilon}|^2}{s} \frac{4p_1^2 p_3^2 (1 - \cos\theta_{13})^2}{m_\psi^2 + 2p_1 p_4 (1 - \cos\theta_{14})} \\
=& \frac{1}{n_a^{eq}} \frac{4\pi\alpha|\tilde{\epsilon}|^2}{32\pi^6} \int_0^\infty dE \int_0^E dp \frac{1}{s} \int_0^{2\pi} d\phi \int_{\frac{E-p}{2}}^{\frac{E+p}{2}} dp_1 \int_{\frac{E-p}{2}}^{\frac{E+p}{2}} dp_3 \\
& \frac{1}{(e^{\frac{p_1}{T}} - 1)(e^{\frac{p_2}{T}} + 1)(1 - e^{-\frac{p_3}{T}})(1 + e^{-\frac{p_4}{T}})} \frac{4p_1^2 p_3^2}{2p_1 p_4} \frac{(1 - \cos\theta_{13})^2}{\frac{m_\psi^2}{2p_1 p_4} + (1 - \cos\theta_{14})} \Big|_{p_2 = E - p_1, p_4 = E - p_3}
\end{aligned} \tag{14}
$$

We can first integrate ϕ out:

$$
\int_0^{2\pi} d\phi \frac{(1 - \cos\theta_{13})^2}{\epsilon + 1 - \cos\theta_{14}} = 2\pi \frac{c^2(1 - b)^2 + [a^2 f + 2ac(1 - b)](f - \sqrt{f^2 - c^2})}{c^2 \sqrt{f^2 - c^2}}
$$

where $\epsilon = \frac{m_\psi^2}{2p_1p_4}$, $f = 1 - d + \epsilon$,

$$
\begin{aligned}
a &= \sin\theta_1 \sin\theta_3 = \frac{E^2 - p^2}{p^2 p_1 p_3} \sqrt{(p_1 - \frac{E-p}{2})(\frac{E+p}{2} - p_1)(p_3 - \frac{E-p}{2})(\frac{E+p}{2} - p_3)} \\
b &= \cos\theta_1 \cos\theta_3 = \frac{E^2}{p^2 p_1 p_3}(p_1 - \frac{E^2 - p^2}{2E})(p_3 - \frac{E^2 - p^2}{2E}) \\
c &= \sin\theta_1 \sin\theta_4 = \frac{E^2 - p^2}{p^2 p_1 (E - p_3)} \sqrt{(p_1 - \frac{E-p}{2})(\frac{E+p}{2} - p_1)(p_3 - \frac{E-p}{2})(\frac{E+p}{2} - p_3)} \\
d &= \cos\theta_1 \cos\theta_4 = \frac{E^2}{p^2 p_1 (E - p_3)}(p_1 - \frac{E^2 - p^2}{2E})(\frac{E^2 + p^2}{2E} - p_3)
\end{aligned}
\tag{15}
$$