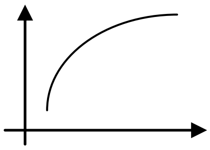
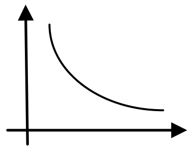
CH3

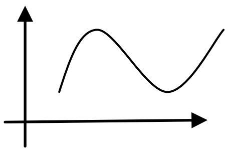
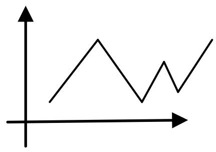
Further Applications of Derivatives

3.1 Graphing using the first derivative

- A function is said to be increasing if its graph is rising as x increase ; and decreasing if its graph falling as x increases.
- If f'(x)>0 for a<x<b , then f(x) is increasing for a<x<b.
  If f'(x)<0 for a<x<b , then f(x) is decreasing for a<x<b.

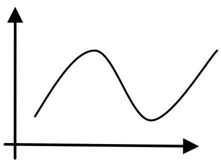
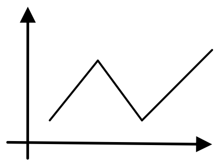



- On a graph , a relative maximum point is a point is a point that is at least as high us the neighboring points of the curve on either side ; and a relative minimum point is a point that is at least as low as the neighboring points on either side.

<u>Critical number</u>

A critical number of a function f is an x-value in the domain of f at which either f'(x)=0 or f'(x) is undefined.
(Derivative is zero or undefined)

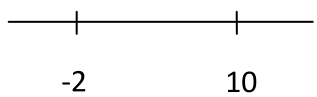



<u>Example 1</u>

Determine where the function $f(x)=x^3-12x^2-60x+36$ is increasing and where is decreasing , find its relative extrema , and draw the graph.

$$f'(x)=3x^2-24x-60 = 3(x+2)(x-10)=0$$

$\longrightarrow$ critical numbers C.N. $\begin{cases} x = -2 \\ x = 10 \end{cases}$



intervals	sign of f’	increasing or decreasing
x<-2	+	$\uparrow$
-2<x<10	-	$\downarrow$
10<x	-	$\downarrow$

1. f(x) is $\uparrow$ for x<-2 and x>10

   f(x) is $\downarrow$ for -2<x<10

2. f(x) has a relative maximum at x=-2 and f(-2)=100

   f(x) has a relative minimum at x=10 and f(10)=-764

<u>First-derivative Test</u>

If a function f has a critical number c then at x=c , the function has a

$$\begin{cases} \text{relative maximum if } f' > 0 \; just\ before\ c\ and\ f' < 0 \; just\ after\ c \\ \text{relative minimum if } f' < 0 \; just\ before\ c\ and\ f' > 0 \; just\ after\ c \end{cases}$$

<u>Example 2</u>

Determine where the function $f(x)=-x^4+4x^3-20$ is increasing and where is decreasing , find its relative extrema , and draw the graph.

$$f'(x)=-4x^3+12x^2=-4x^2(x-3)=0$$

$\rightarrow$ C.N. $\begin{cases} x = 0 \\ x = 3 \end{cases}$

intervals	sign of f’	increasing or decreasing
x<0	+	$\uparrow$
0<x<3	+	$\uparrow$
x>3	-	$\downarrow$

1. f(x) is increasing for x<3 and decreasing for x>3
2. f(x) has a relative maximum at x=3 , and f(3)=7

   No relative minimum
3. graph : p.193

<u>Example 3</u>

$f(x)= \frac{1}{x^2-4x}$ is undefined at x=0 and x=4

$f'(x)= \frac{-2(x-2)}{(x^2-4x)^2}$ is zero at x=2 and undefined at x=0 and x=4

intervals	sign of f’	increasing or decreasing
x<0	+	$\uparrow$
0<x<2	+	$\uparrow$
2<x<4	-	$\downarrow$
x>4	-	$\downarrow$

1. f(x) is increasing for x<2 and 0<x<2
   f(x) is decreasing for 2<x<4 and x>4

2. f(x) has a relative maximum at x=2 and f(2)= $\frac{-1}{4}$

No relative minimum.

3.2 Graphing using the first and second derivatives

- If f''(x)>0 on I , then f'(x) increases on I
  and the graph of f(x) is concave up on I
  If f''(x)<0 on I , then f'(x) decreases on I
  and the graph of f(x) is concave down on I

<u>Concavity and inflection points</u>

On an interval,
f''(x)>0 means that f is concave up
f''(x)<0 means that f is concave down
An inflection point is where the concaving changes
(f'' must be zero or undefined)

<u>Example 1</u>

$f(x)= x^3 - 9x^2 + 24x$

$f'(x)= 3x^2 - 18x + 24 = 3(x-2)(x-4)=0$ => 1^{st} order C.N. $\begin{cases} x = 2 \\ x = 4 \end{cases}$

$f''(x)= 6x - 18 = 0$ => 2^{nd} order C.N. x = 3

interval	sign of f'	sign of f''	$\uparrow$ or $\downarrow$	concavity	shape
x<2	+	-	$\uparrow$	down	
2<x<3	-	-	$\downarrow$	down	
3<x<4	-	+	$\downarrow$	up	
x>4	+	+	$\uparrow$	up	

1. f(x) has a relative maximum at x=2 , and f(2)=20
   f(x) has a relative minimum at x=4 , and f(4)=16
2. Inflection point : (3,18)
3. graph , p.185

<u>Example 2</u>

$f(x) = 18x^{\frac{1}{3}}$

$f'(x) = 6x^{\frac{-2}{3}}$ is undefined at x=0

=> 1^{st} order C.N. x=0

f''(x) = -4x$^{\frac{-5}{3}}$ is undefined at x=0

=> 2^{nd} order C.N. x=0

interval	sign of f'	sign of f''	$\uparrow$ or $\downarrow$	concavity	shape
x<0	+	+	$\uparrow$	up	
x>0	+	-	$\uparrow$	down	

1. No relative extrema
2. Inflection point : (0,0)
3. graph : p.206

Second-derivative test

If x=c is a critical number of f at which f'' is defined , then

1. f''(c)>0 means that f has a relative minimum at x=c
2. f''(c)<0 means that f has a relative maximum at x=c
3. f''(c)=0 means that the test is inconclusive

Example 4

f(x) = x^3 - $9x^2$ + 24x

f'(x) = $3x^2$ - 18x + 24 = 3(x-2)(x-4) = 0

=> C.N.$\begin{cases} x = 2 \\ x = 4 \end{cases}$

f''(x) = 6x-18

f''(2) = -6<0 => f(x) has a relative maximum at x=2

f''(4) = 6>0 => f(x) has a relative minimum at x=4

Note:

- If (c , f(c)) is an inflection point , then either f''(c)=0 or f''(c) is undefined
- If f''(c)>0 or f''(c) is undefined

3.3 Optimization

- The absolute maximum value of a function is the largest value of the function on its domain , the absolute minimum value of a function is the smallest value of the function on its domain.
- How to find the absolute extrema of a continuous function f(x) on a close interval [a,b]

  1. Find all critical numbers of f(x) on [a,b]
  2. Evaluate f(x) at the critical numbers and at the endpoints a and b

The largest and smallest values found in step 2 will be the absolute maximum and minimum values of f(x) on [a,b]

Example 1

Find the absolute extrema values of f(x) = $x^3 - 9x^2$ + 15x on [0,3]

f'(x)=$3x^2$ - 18x + 15 = 3(x-1)(x-5)=0 => C.N. $\begin{cases} x = 1 \\ \cancel{x = 5} \end{cases}$

$$\begin{cases} f(0) = 0 \\ f(1) = 7 \\ f(3) = 9 \end{cases}$$

=> f(x) has a absolute max at x=1 , and f(1)= 7

f(x) has a absolute min at x=3 , and f(1)= -9

<u>example 2</u>

V(t) = $96t^{\frac{1}{2}}$ - 64 t>0

V'(t) = $48t^{\frac{-1}{2}}$ – 6 = 0 => t=64

V''(t) = $-24t^{\frac{-3}{2}}$, V''(64) = $-24(64)^{\frac{-3}{2}}$ < 0

=> V(t) has absolute maximum at t=64 , and V(64)=384

<u>Application : maximizing profit</u>

p(x) : price function

C(x) : cost function

R(x) : revenue function

P(x) : profit function

<u>Example 3</u>

p(x) = 22000 - 70x

C(x) = (22000 - 70x)x = 22000x - $70x^2$

R(x) = 8000x + 20000

P(x) = $-70x^2$ + 14000x – 20000

(a) p'(x) = -140x +14000 = 0 => x=100

p''(x)=-140 , p''(100)= -140 < 0

=> p(x) has maximum at x = 100

(b) Selling price

p(100) = 22000 – 70(100) = 15000

(c) Max profit

P(100) = $-70(100)^2$+14000(100) – 20000 = 680000

- To maximize the profit function P(x)=R(x)-C(x)

  If P(x) has a maximum at x=a , then P'(a)=0

  => P'(a) = R'(a) – C'(a) = 0

=> R’(a) = C’(a)

Thus , profit is maximized at a production level foe which marginal revenue equals marginal cost.

3.6 Implict differentiation and related rates

- A function is said to be defined explicitly , weaning that y is defined by a rule or formula f(x) in x alone is to be defined implicitly , meaning that y is defined by an equation in x and y.

- Finding $\frac{dy}{dx}$ by implicit differentiation

1. Differentiation both sides of the equation with respect to x when differentiating a y , include $\frac{dy}{dx}$.

2. Collect all terms involving $\frac{dy}{dx}$ on one side , and all others on the other side.

3. Factor out the $\frac{dy}{dx}$ and solve for it by dividing.

Example 1.2

$x^2 + y^2 = 25$

$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25$

$2x + 2y\frac{dy}{dx} = 0 \quad => \frac{dy}{dx} = \frac{-x}{y}$

The slope of the circle $x^2 + y^2 = 25$

at(3,4) is $\frac{dy}{dx} = \frac{-x}{y} = \frac{-3}{4}$

at(3,-4) is $\frac{dy}{dx} = \frac{-x}{y} = \frac{3}{4}$

example 4

$y^4 + x^4 - 2x^2y^2 = 9$

$4y^3 \frac{dy}{dx} + 4x^3 - 4xy^2 - 4x^2y \frac{dy}{dx} = 0$

$(4y^3 - 4x^2y)\frac{dy}{dx} = -4x^3 + 4xy^2$

$=> \frac{dy}{dx} = \frac{-x^3+xy^2}{y^3-4x^2y}$

At(2,1) , $\frac{dy}{dx} = \frac{-6}{-3} = 2$

example 5

demand function $x=\sqrt{1900 - p^3}$

$x^2 = 1900 - p^3$

$\frac{d}{dx}(x)^2 = \frac{d}{dx}(1900 - p^3)$

$2x = -3p^2 \frac{dp}{dx}$ => $\frac{dp}{dx} = \frac{-2x}{3p^2}$

At p = 10 , x = $\sqrt{1900 - 10^3}$ = 30

$\frac{dp}{dx} = \frac{-60}{300}$ = -0.2

It says that the rate of change of price with respect to quantity is -0.2.