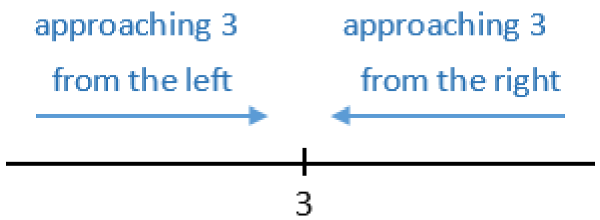
Derivatives And Their Uses

2-1 Limits and Continuity

The notation $x \to 3$ (read " x approaches ") means that x takes values closer and closer to 3 but never equals 3.



Limit of a function

$\lim_{x \to c} f(x) = \mathrm{L}$ means that $f(x)$ approaches the number L as x approaches the number c.

Formally, limits are defined as follows :

$$\lim_{x \to c} f(x) = \mathrm{L} \text{ if } \forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } |f(x) - L| < \varepsilon \text{ wherever } 0 < |x - c| < \delta$$

Example 1

Find $\lim_{x \to 3}(2x + 4)$

(I) Use tables

$x \to 3$ $(x < 3)$

x	$f(x) = 2x + 4$
2.9	9.8
2.99	9.98
2.999	9.998

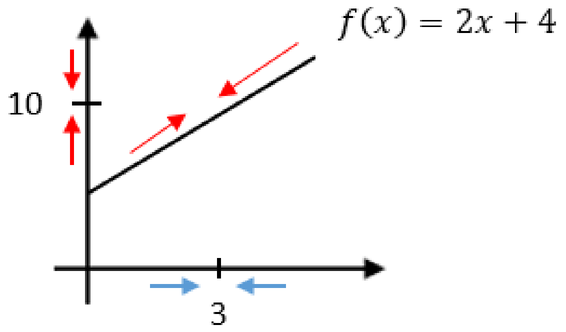
$x \to 3$ $(x > 3)$

x	$f(x) = 2x + 4$
3.1	10.2
3.01	10.02
3.001	10.002

Seem to approach 10

$\Rightarrow \lim_{x \to 3}(2x + 4) = 10$

(II) Use graph



$\Rightarrow \lim_{x \to 3}(2x + 4) \;= 10$

Rules of Limits

1. $\lim_{x \to c} a = a$

2. $\lim_{x \to c} x^n = c^n$

3. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$ $\quad (c > 0$ if n is even$)$

4. If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist, then

   a. $\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

   b. $\lim_{x \to c}[f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$

   c. $\lim_{x \to c}[f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

   d. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ (if $\lim_{x \to c} g(x) \neq 0$)

Summary of rules limits

For functions composed of additions, subtractions, multiplications, divisions, powers, and roots, limits may be evaluated by direct substitution, provided that the resulting expression is defined.

$$\lim_{x \to c} f(x) = f(c)$$

Example 3

a. $\lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$

b. $\lim_{x \to 6} \frac{x^2}{x+3} = \frac{6^2}{6+3} = \frac{36}{9} = 4$

One sided limits

$$x \to c \begin{cases} from\ \ the\ \ left & x \to c^- \\ from\ \ the\ \ right & x \to c^+ \end{cases}$$

$\lim_{x \to c^-} f(x)$ means the limit of $f(x)$ as $x \to c$ but with $x < c$

$\lim_{x \to c^+} f(x)$ means the limit of $f(x)$ as $x \to c$ but with $x > c$

$$\Rightarrow \lim_{x \to c} f(x) = L \iff \text{both} \begin{cases} \lim_{x \to c^-} f(x) = L \\ \lim_{x \to c^+} f(x) = L \end{cases}$$

(otherwise, $\lim_{x \to c} f(x)$ does not exist)

Example 5

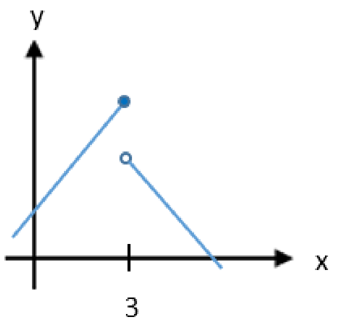
Find the piecewise linear function $f(x) = \begin{cases} x+1 & if\ \ x \le 3 \\ 8-2x & if\ \ x > 3 \end{cases}$ graphed on the left, find the following limits or state that they do not exist.

a. $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x+1) = 3+1 = 4$

b. $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (8-2x) = 8 - 2 \cdot 3 = 2$

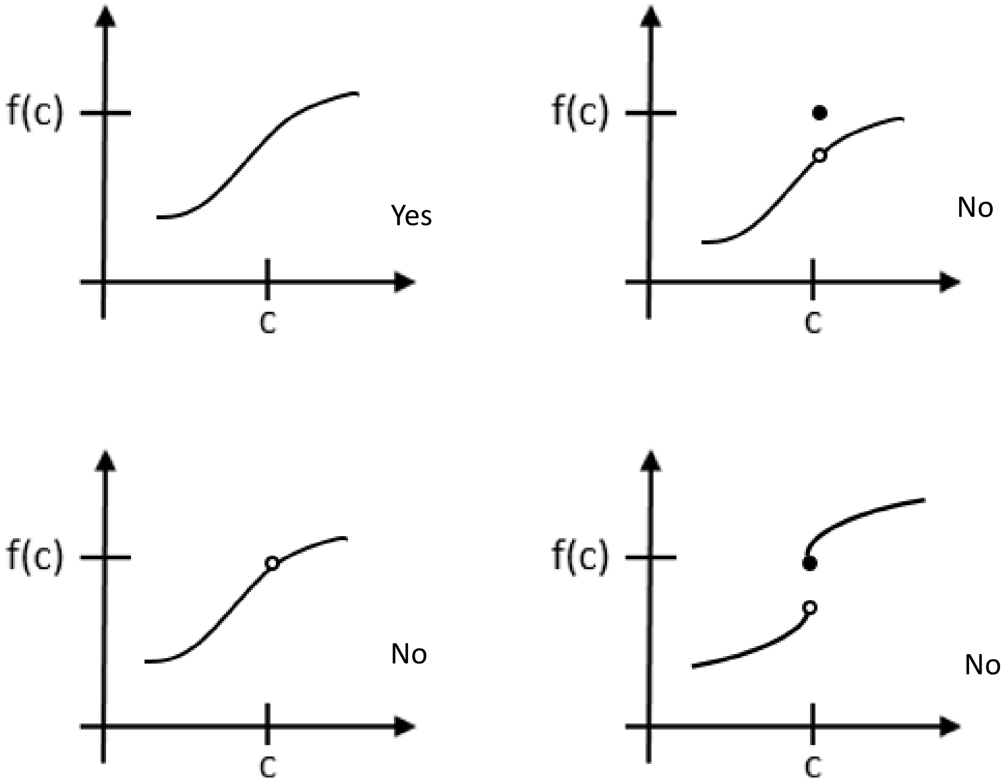
c. $\lim_{x \to 3} f(x)$ does not exist

(since $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$)



Continuity

Continuous at c ?



A function f is continuous at c if the following three conditions hold:

1. $f(c)$ is defined
2. $\lim_{n\to c} f(x)$ exists
3. $\lim_{n\to c} f(x) = f(c)$

f is discontinuous at c if one or more of these conditions fails to hold.

If f and g are continuous at c, than the following are also continuous at c :

1. $f \pm g$
2. $a \cdot f$
3. $f \cdot g$

4. $\frac{f}{g}$ $(g(c) \neq 0)$

5. $f(\mathrm{g}(x))$

1. Every polynomial function is continuous.
2. Every rational function is continuous except where the denominator is zero.

Example 9

Determine whether each function is continuous.

a. $f(x) = x^3 - 3x^2 - x + 3$

Is continuous since it is a polynomial.

b. $f(x) = \frac{1}{(x+1)^2}$

Is continuous at all x-value except x=-1

c. $f(x) = e^{x^2-1}$

Is continuous since it is a composition of e^x and $x^2 - 1$

2-2 Rates of Change, and Derivatives

The average rate of change of a function f between x and $x + h$ is

$$\frac{f(x+h) - f(x)}{h}$$

The instantaneous rate of change of a function f at the number x is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

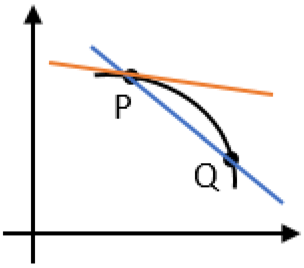
Example 1

Find the instantaneous rate of change of the temperature function $f(x) = x^2$ at $x = 1$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

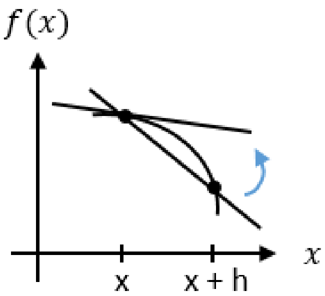
$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{f(1+h)^2 - 1^2}{h} = \lim_{h \to 0} \frac{h(2+h)}{h} = \lim_{h \to 0} 2 + h = 2$$

Secant and Tangent lines



Secant line to the curve at p

Tangent line to the curve at p



Secant lines approach the tangent lines as $h \to 0$

Derivative

For a function f, the derivative of f at x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that the limit exists.

Note:

Secant line : Average rate of change of f between x and $x + h$

= slope of the secant line through at x and $x + h$

$= \frac{f(x+h)-f(x)}{h}$

Tangent line : Instantaneous rate of f at x

$= \text{slope of the tangent line at } x$

$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \text{the derivative of } f \text{ at } x$

Example 4

Find the derivative of $f(x) = x^2 - 7x + 150$

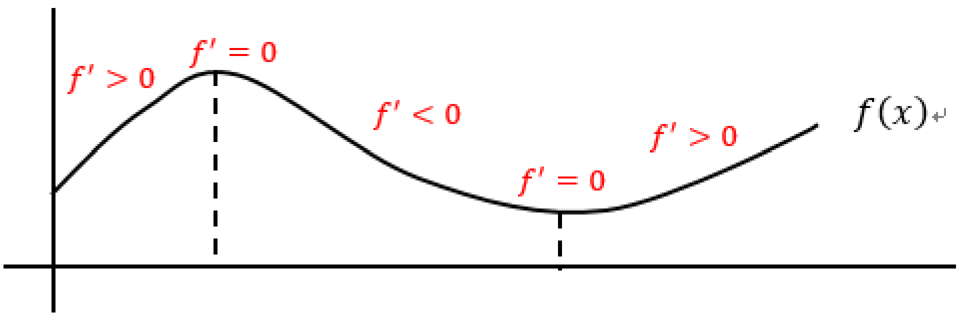
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^2-7(x+h)+150]-[x^2-7x+150]}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h-7)}{h} = \lim_{h\to 0}(2x+h-7) = 2x-7$$

Notation for the derivative

Prime notation		Leibniz's notation
$f'(x)$	=	$\frac{d}{dx}f(x) = \frac{df}{dx}$
$f'(z)$	=	$\left.\frac{df}{dz}\right\rvert_{x=z}$
y'	=	$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$



2-3 Some Differentiation Formulas

1. For any constant c, $\frac{d}{dx}c = 0$

$$f'(x) = \lim_{\mathrm{h}\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{\mathrm{h}\to 0}\frac{c-c}{h} = 0$$

2. For any constant exponent n,

$$\frac{d}{dx}x^n = nx^{n-1}$$

(1) $\frac{d}{dx}x^5 = 5x^4$

(2) $\frac{d}{dx}x^{100} = 100x^{99}$

(3) $\frac{d}{dx}\frac{1}{x^2} = \frac{d}{dx}x^{-2} = -2x^{-3}$

(4) $\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{\frac{-1}{2}}$

(5) $\frac{d}{dx}x = 1$

3. For any constant c,

$$\frac{d}{dx}\big(cf(x)\big) = c\cdot f'(x)$$

(1) $\frac{d}{dx}7x^9 = 7\frac{d}{dx}x^9 = 7\cdot 9x^8 = 63x^8$

(2) $\frac{d}{dx}5x^{-2} = -10x^{-3}$

4. $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

(1) $\frac{d}{dx}(x^5 + x^7) = 5\mathrm{x}^4 + 7x^6$

(2) $\frac{d}{dx}\left(x^{\frac{1}{2}} + x^{-2}\right) = \frac{1}{2}x^{\frac{-1}{2}} - 2x^{-3}$

(3) $f(x) = 2x^3 - 5x^2 + 7$, find $f'(2) = \left.\frac{df}{dx}\right|_{x=2}$

$$\Rightarrow f'(x) = 6x^2 - 10x$$

$$f'(2) = 6 \cdot 2^2 - 10 \cdot 2 = 4 = \left.\frac{df}{dx}\right|_{x=2}$$

5. $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$$\neq f'(x) \cdot g'(x)$$

(1) $\frac{d}{dx}(x^3 \cdot x^5)$

$= 3x^2 \cdot x^5 + x^3 \cdot 5x^4$
$= 3x^7 + 5x^7 = 8x^7$

(2) $\frac{d}{dx}[(x^2 - x + 2)(x^3 + 3)]$

$= (2x - 1)(x^3 + 3) + (x^2 - x + 2)(3x^2)$
$= 5x^4 - 4x^3 + 6x^2 + 6x - 3$

6. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - g'(x) \cdot f(x)}{[g(x)]^2}$

(1) $\frac{d}{dx}\left[\frac{x^9}{x^3}\right] = \frac{x^3 \cdot 9x^8 - 3x^2 \cdot x^9}{(x^3)^2} = \frac{9x^{11} - 3x^{11}}{x^6} = \frac{6x^{11}}{x^6} = 6x^5$

(2) $\frac{d}{dx}\left[\frac{x^2}{x+1}\right] = \frac{(x+1) \cdot 2x - 1 \cdot x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}$

2-4 Higher-Order Derivatives

For a function f, the derivative of f at x is defined as $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$, the second derivative is defined as $f''(x) = f'(f'(x))$

$$\vdots$$

the fourth derivative of f is defined as

$$f''''(x) = f'(f'''(x)) = f'(f'(f''(x))) = f'(f'(f'(f'(x))))$$

Notation

Prime notation		Leibniz's notation	
$f''(x)$	=	$\frac{d^2}{dx^2}f(x)$	Second derivative
y''	=	$\frac{d^2y}{dx^2}$	
$f'''(x)$	=	$\frac{d^3}{dx^3}f(x)$	Third derivative
y'''	=	$\frac{d^3y}{dx^3}$	
$f^{(n)}(x)$	=	$\frac{d^n}{dx^n}f(x)$	n^{th} derivative
$y^{(n)}$	=	$\frac{d^ny}{dx^n}$	

Application

If $s(x) =$ distance at time t, then

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h)-s(t)}{h} = \text{ velocity at time t}$$

$$a(x) = v'(t) = s''(t) = \text{ acceleration at time t}$$

Example 5

A delivery truck is driving along a straight road, and after t hours its distance (in miles) east of its starting point is

$$s(t) = 24t^2 - 4t^3 \quad \text{for} \quad 0 \leq t \leq 6$$

a. Find the velocity of the truck after 2 hours.

$v(t) = 48t - 12t^2$

$v(2) = 96 - 48 = 48$

b. Find the velocity of the truck after 5 hours.

$v(5) = 240 - 300 = -60$

c. Find the acceleration of the truck after 1 hour.

$a(t) = 48 - 24\mathrm{t}$

$a(1) = 48 - 24 = 24$

2-5 The Chain Rule and the Generalized Power Rule

$$\frac{d}{dx}x^5 = \ ?$$

$$\frac{d}{dx}(6x+1)^5 = \ ?$$

$$\frac{d}{d(6x+1)}(6x+1)^5 = \ ?$$

Composite function

Example 1

For $f(x) = x^2$ and $g(x) = 4 - x$, find $f\big(g(x)\big)$

$f\big(g(x)\big) = f(4-x) = (4-x)^2$

$g\big(f(x)\big) = g(x^2) = 4 - x^2$

The Chain Rule

$$\frac{d}{dx}f\big(g(x)\big) = f'(g(x)) \cdot g'(x)$$

Generalized Power Rule

$$\frac{d}{dx}[g(x)]^n = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

Example 3

Use the Chain Rule to find $\frac{d}{dx}(x^2 - 5x + 1)^{10}$

$$\frac{d}{dx}(x^2 - 5x + 1)^{10} = 10(\mathrm{x}^2 - 5\mathrm{x} + 1)^9(2x - 5)$$

2-6 Nondifferentiable Functions

Example 1

Show that $f(x) = |x|$ is not differentiable at $x = 0$

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$f'(x) = \lim_{\mathrm{h}\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(0) = \lim_{\mathrm{h}\to 0} \frac{f(0+h) - f(x0)}{h} = \lim_{\mathrm{h}\to 0} \frac{f(h) - f(0)}{h} = \lim_{\mathrm{h}\to 0} \frac{|h| - |0|}{h} = \lim_{\mathrm{h}\to 0} \frac{|h|}{h}$$

For positive h,

$$\lim_{\mathrm{h}\to 0^+} \frac{|h|}{h} = \lim_{\mathrm{h}\to 0^+} \frac{\mathrm{h}}{h} = \lim_{\mathrm{h}\to 0^+} 1 = 1$$

For negative h,

$$\lim_{h\to 0^-}\frac{|h|}{h} = \lim_{h\to 0^-}\frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$$

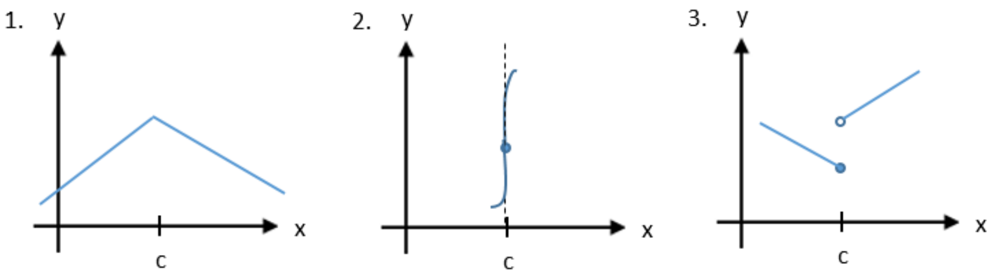
Since $\lim_{h\to 0^+}\frac{|h|}{h} \neq \lim_{h\to 0^-}\frac{|h|}{h}$, the limit $\lim_{h\to 0}\frac{|h|}{h}$ does not exist.

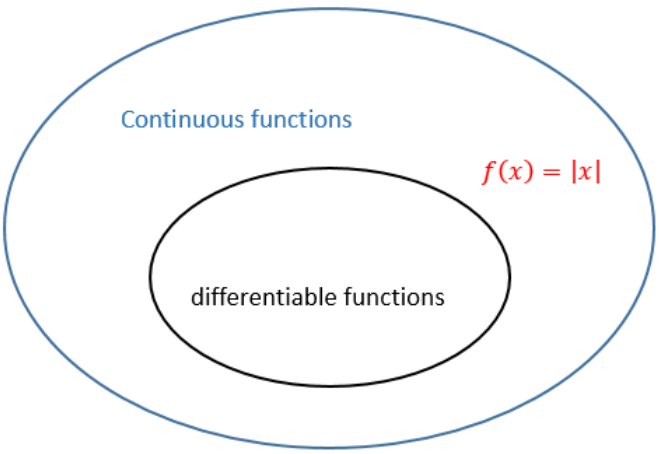
So the derivative does not exist.

If a function f satisfies any of the following conditions :

1. f has a corner point at $x = c$
2. f has a vertical tangent at $x = c$
3. f is discontinuous at $x = c$

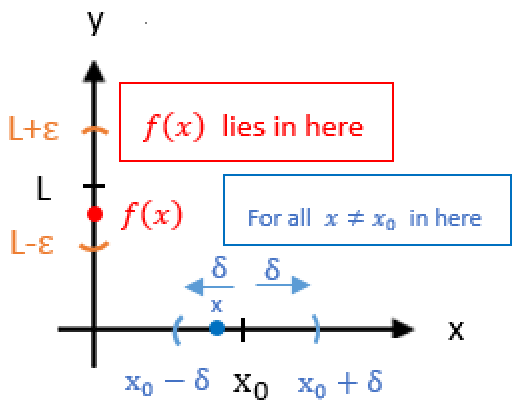
Then f will not be differentiable at c





Definition

$$\lim_{x\to x_0} f(x) = \mathrm{L} \text{ if } \forall\varepsilon > 0,\ \exists\delta > 0 \text{ s.t. } |f(x) - L| < \varepsilon \text{ wherever } 0 < |x - x_0| < \delta$$


y
$f(x)$ lies in here
L+ε
L
$f(x)$
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For all $x \neq x_0$ in here
δ
δ
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x
$x_0 - \delta$
x_0
$x_0 + \delta$