CH6

6-1 Integration by parts

For differentiable function u and v

$$\int udv = uv - \int vdu$$

pf:

$(uv)' = u'v + uv'$

$\Rightarrow uv = \int u'v + \int uv' = \int vdu + \int udv$

$\Rightarrow \int udv = uv - \int vdu$

How to choose the u and the dv

1. Choose dv to be the most complicated part of the integral that can be integrated easily.
2. Choose u so that u' is simpler than u.

Summary

	Choose :
$\int x^n e^{ax} dx$	$u = x^n$, $dv = e^{ax}dx$
$\int x^n lnx dx$	$u = lnx$, $dv = x^n dx$
$\int (x+a)(x+b)^n dx$	$u = (x+a)$, $dv = (x+b)^n dx$

Example 1

Find $\int xe^x dx$

Let $u = x \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = \int e^x dx$

$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$

(Check : $\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x$)

Example 2

Find $\int x^2 lnx dx$

Let $u = lnx \Rightarrow du = \frac{1}{x}dx$

$dv = x^2 dx \Rightarrow v = \int x^2 dx = \frac{1}{3}x^3$

$\int x^2 lnx dx = (lnx) \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x}dx = \frac{1}{3}x^3 lnx - \frac{1}{9}x^3 + C$

Example 3

Find $\int (x-2)(x+4)^8 dx$

Let $u = x - 2 \Rightarrow du = dx$

$dv = (x+4)^8 dx \Rightarrow v = \int (x+4)^8 dx = \frac{1}{9}(x+4)^9$

$\Rightarrow \int (x-2)(x+4)^8 dx = (x-2) \cdot \frac{1}{9}(x+4)^9 - \int \frac{1}{9}(x+4)^9 dx$

$= \frac{1}{9}(x-2)(x+4)^9 - \frac{1}{90}(x+4)^{10}$

Example 4

Find $\int 2te^{-0.1t} dt$

Let $u = 2t \Rightarrow du = 2dt$

$dv = e^{-0.1t} dt \Rightarrow v = \int e^{-0.1t} dt = -10e^{-0.1t}$

$\int 2te^{-0.1t} dt = (2t) \cdot (-10e^{-0.1t}) - \int (-10e^{-0.1t}) 2dt$

$= -20te^{-0.1t} - 200e^{-0.1t} + C$

We have seem that $\int f(x)g(x)dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeated without difficulty.
In situations like this, this is a way organize the calculation that saves a great deal with work.

Example

Find $\int x^2 e^x dx$

x^2	e^x	
$2x$	e^x	$+$
2	e^x	$-$
0	e^x	$+$

$$\Rightarrow \int x^2 e^x = x^2 e^x - 2xe^x + 2e^x + C$$

Example

Find $\int xe^{3x} = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

x	e^{3x}	
1	$\frac{1}{3}e^{3x}$	$+$
0	$\frac{1}{9}e^{3x}$	$-$

$$\Rightarrow \int xe^{3x} = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

6-2 Integration Using Tables

Example 1

Find $\int \frac{1}{x^2-4} dx$

$(\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} ln \left|\frac{x-a}{x+a}\right| + C)$

Take $a = 2$

$\Rightarrow \int \frac{1}{x^2-4} dx = \frac{1}{4} ln \left|\frac{x-2}{x+2}\right| + C$

Example 2

Find $s(x) = \int \frac{x}{\sqrt{x+9}} dx$

$(\int \frac{x}{\sqrt{ax+b}} dx = \frac{2ax-4b}{3a^2} \sqrt{ax+b} + C)$

Take $a = 1,\ b = 9$

$\int \frac{x}{\sqrt{x+9}} dx = \frac{2x-36}{3} \sqrt{x+9} + C$

$\Rightarrow$ Since $s(0) = 0$

$\Rightarrow -36 + C = 0,\ C = 36$

$\therefore s(x) = \left(\frac{2}{3}x - 12\right) \sqrt{x+9} + 36$

Example 3

Find $n = 2.5 \int_{0.2}^{0.5} \frac{1}{q^2(1-q)} dq$

$(\int \frac{1}{x^2(ax+b)} dx = -\frac{1}{b}(\frac{1}{x} + \frac{a}{b} ln \left|\frac{x}{ax+b}\right|) + C)$

Take $a = -1,\ b = 1$

$\Rightarrow \int \frac{1}{q^2(1-q)} dq = -(\frac{1}{q} - ln \left|\frac{q}{1-q}\right|) + C$

$n = -2.5(\frac{1}{q} - ln \left|\frac{q}{1-q}\right|)\Big|_{0.2}^{0.5} = -2.5[\left(\frac{1}{0.5} - ln \left|\frac{0.5}{1-0.5}\right|\right) - \left(\frac{1}{0.2} - ln \left|\frac{0.2}{1-0.2}\right|\right)] \approx 11$

Example 4

Find $\int \frac{x}{\sqrt{x^4+1}} dx$

$(\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = ln\left|x + \sqrt{x^2 \pm a^2}\right| + C)$

Let $u = x^2 \Rightarrow du = 2xdx$

$\int \frac{x}{\sqrt{x^4+1}} dx = \int \frac{1}{\sqrt{u^2+1}} \frac{1}{2} du$

$= \frac{1}{2} \int \frac{1}{\sqrt{u^2+1}} du = \frac{1}{2} ln\left|u + \sqrt{u^2 + 1}\right| + C = \frac{1}{2} ln\left|x^2 + \sqrt{x^4 + 1}\right| + C$

Example 5

Find $\int \frac{e^{-2t}}{e^{-t}+1} dt$

$(\int \frac{x}{x+1} dx = x - ln|x + 1| + C)$

Let $x = e^{-t} \Rightarrow dx = -e^{-t}dt$

$\int \frac{e^{-2t}}{e^{-t}+1} dt = \int \frac{-x}{x+1} dx = -x + ln|x + 1| + C = -e^{-t} + ln|e^{-t} + 1| + C$

6-3 Improper Integrals

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

If $\int_a^{\infty} f(x)dx$ is a finite number, then we say that $\int_a^{\infty} f(x)dx$ is convergent ;

If $\int_a^{\infty} f(x)dx$ has no limit, then we say that $\int_a^{\infty} f(x)dx$ is divergent.

$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$

$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx$

$= \lim_{a\to-\infty} \int_a^0 f(x)dx + \lim_{b\to\infty} \int_0^b f(x)dx$

Example 3

$\int_1^\infty \frac{1}{x^2} dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2} dx = \lim_{b\to\infty} (-\frac{1}{x})\Big|_1^b = \lim_{b\to\infty}(-\frac{1}{b} + 1) = 1$

Example 4

$\int_1^\infty \frac{1}{\sqrt{x}} dx = \lim_{b\to\infty} \int_1^b x^{-\frac{1}{2}} dx = \lim_{b\to\infty} (2x^{\frac{1}{2}})\Big|_1^b = \lim_{b\to\infty}\left(2\sqrt{b} - 2\right)$ does not exist.

$\Rightarrow \int_1^\infty \frac{1}{\sqrt{x}} dx$ is divergent.

Example 6

$\int_2^\infty \frac{x}{(x^2+1)^2} dx$

Let $u = x^2 + 1 \Rightarrow du = 2xdx$

$\int_2^\infty \frac{x}{(x^2+1)^2} dx = \int_5^\infty \frac{1}{2u^2} du = \lim_{b\to\infty} \int_5^b \frac{1}{2u^2} du$

$= \lim_{b\to\infty} (-\frac{1}{2}u^{-1})\Big|_5^b = \lim_{b\to\infty}\left(-\frac{1}{2b} + \frac{1}{10}\right) = \frac{1}{10}$

Example 7

$\int_{-\infty}^3 4e^{2x} dx = \lim_{a\to-\infty} \int_a^3 4e^{2x} dx = \lim_{a\to-\infty} (2e^{2x})|_a^3 = \lim_{a\to-\infty} (2e^6 - 2e^{2a}) = 2e^6$

L'Hôpital's Rule

Let $f(x)$ and $g(x)$ are differentiable on an open interval containing c, $\lim_{x\to c} f(x) =$

$\lim_{x\to c} g(x) = 0$ (or $\pm\infty$)

Then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

Example

$\lim_{x \to 1} \frac{lnx}{x-1} \ (\frac{0}{0})$

$= \lim_{x \to 1} \frac{1/x}{1} = 1$

Example

$\lim_{x \to \infty} xe^{-2x} = \lim_{x \to \infty} \frac{x}{e^{2x}} \ (\frac{\infty}{\infty})$

6-4 Numerical Integration

Suppose $f(x)$ is continuous on (a, b)

$\Delta x = \frac{b-a}{n}$

End point : $x_1 = a, \ x_2 = a + \Delta x, \ x_3 = x_2 + \Delta x, \ \cdots, \ x_{n+1} = b$

1. Trapezoidal Approximation

$\int_a^b f(x)dx \approx \left[\frac{1}{2} f(x_1) + f(x_2) + \cdots + f(x_n)\right) + \frac{1}{2} f(x_{n+1})]\Delta x$

Error $\leq \frac{(b-a)^3}{12n^2} \max_{a \leq x \leq b} |f''(x)|$

Example 1

Approximate $\int_1^2 x^2 dx$ using four trapezoids.

$a = 1,\ b = 2,\ \Delta x = \frac{2-1}{4} = 0.25$

Endpoints : 1, 1.25, 1.5, 1.75, 2

$\int_1^2 x^2 dx \approx \left[\frac{1}{2}(1)^2 + (1.25)^2 + (1.5)^2 + (1.75)^2\right) + \frac{1}{2}(2)^2] \cdot 0.25 = 2.34375$

Error $\leq \frac{(2-1)^3}{12 \cdot 4^2} \max_{1 \leq x \leq 2} |2x| = \frac{1}{12 \cdot 4^2} \cdot 2 = 0.01047$

$(\int_1^2 x^2 dx = \frac{1}{3}\mathrm{x}^3|_1^2 = \frac{7}{3} \approx 2.33)$

2. Simpson's Rule

$\int_a^b f(x)dx \approx [f(x_1) + 4f(x_2) + 2f(x_3) + \cdots + 4f(x_n) + f(x_{n+1})]\frac{\Delta x}{3}$

n must be even.

Error $\leq \frac{(b-a)^5}{180n^4} \max_{a \leq x \leq b} \left|f^{(4)}(x)\right|$

Example 2

$\Delta x = \frac{5-3}{4} = 0.5$

Endpoints : 3, 3.5, 4, 4.5, 5

$\int_3^5 \frac{1}{x} dx \approx \left[\frac{1}{3} + 4\left(\frac{1}{3.5}\right) + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{4.5}\right) + \frac{1}{5}\right] \cdot \frac{0.5}{3} = 0.51084$

Error $\leq \frac{(5-3)^5}{180 \cdot 4^4} \max_{3 \leq x \leq 5} |24x^{-5}| = \frac{2^5}{180 \cdot 4^4} \cdot 24 \cdot 3^{-5} \approx 0.00007$