CH7
Calculus of several Variables

7.1. Functions of several variables

- Function of Two variables

A function f of two variables is a rule such that to each ordered pair (x,y) in the domain of f there corresponds one and only one number f(x,y)
ex.

f(x,y) = $\frac{\sqrt{x}}{y^2}$

g(u,v) = e^{ur} – ln u

- Relative maximum (minimum) value

A point (a,b,c) on a surface z=f(x,y) is a relative maximum point (relative minimum point) if f(a,b) >= f(x,y) (f(a,b) <= f(x,y)) for all (x,y) in some region surrounding (a,b)

- A <u>saddle point</u> is a point that is the highest point along one curve of the surface and the lowest point along another curve

Note : A saddle point is not a relative extreme point.

7.2 Partial derivatives

- Partial derivatives

(a) partial derivative of f(x,y) with respect to x is written with

$\frac{d}{dx}$f(x,y) = fx(x,y) = $\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$

= derivative of f with respect to x , with y held constant
=Instantaneous rate of charge of f with respect to x , when y is held constant.
(b) partial derivative of f(x,y) with respect to y is written with

$\frac{d}{dy}$f(x,y) = fy(x,y) = $\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$

= derivative of f with respect to y , with x held constant
= Instantaneous rate of charge of f with respect to y , when x is held constant.
<u>Example 1.2.</u>

$\frac{d}{dx}x^3y^4 = 3x^2y^4$

$\frac{d}{dy}x^3y^4 = x^3 4y^3 = 4x^3y^3$

Example 3

$\frac{d}{dx}y^4 = 0$ (y^4 : constant)

$\frac{d}{dy}x^2 = 0$

Example 4

$\frac{d}{dx}(2x^4 - 3x^3y^3 - y^2 + 4x + 1)$

$= 8x^3 - 9x^2y^3 - 0 + 4 + 0$

$= 8x^3 - 9x^2y^3 + 4$

Example 5

$f(x,y) = 5x^4 - 2x^2y^3 - 4y^2$

$\Rightarrow fx(x,y) = 20x^3 - 4xy^3$

$fy(x,y) = -3^2 - 8y$

example 6

$f = e^x \ln y$

$\Rightarrow fx = e^x \ln y$

$fy = e^x \frac{1}{y}$

example 7

$f = (xy^2 + 1)^4$

$fy = 4(xy^2 + 1)^3(x \times 2y)$

$= 8xy(xy^2 + 1)^3$

Example 8

$g = \frac{xy}{x^2+y^2}$

$\frac{dg}{dx} = \frac{(x^2+y^2)\times y - 2x \times xy}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}$

Example 9

$f(x,y) = \ln(x^2 + y^2)$

$fx(x,y) = \frac{2x}{x^2+y^2}$

$fy(x,y) = \frac{2y}{x^2+y^2}$

example 10

$f(x,y) = e^{x^2+y^2}$, find fy(1,3)

$fy(x,y) = e^{x^2+y^2} \times (2y)$

$fy(1,3) = e^{1^2+3^2}(2 \times 3) = 6e^{10}$

example 11

f(x,y,z) = $e^{x^2+y^2+z^2}$

fy(x,y,z) = $2ze^{x^2+y^2+z^2}$

fz(1,1,1) = $2 \times 1 \times e^{1+1+1} = 2e^3$

- Interpreting partials Geometrically.

(a) $\frac{df}{dx}$ gives the slope of the surface in the x-direction

(b) $\frac{df}{dy}$ gives the slope of the surface in the y-direction

Second-Order Partials

fxx $\frac{d^2}{dx^2}f$

fyy $\frac{d^2}{dy^2}f$

fxy $\frac{d^2}{dydx}f = \frac{d}{dy}(\frac{d}{dx}f)$

fyx $\frac{d^2}{dxdy}f = \frac{d}{dx}(\frac{d}{dy}f)$

ex15

f(x,y) = $x^4 + 2x^2y^2 + x^3y + y^4$

fx = $4x^3 + 4xy^2 + 3x^2y$

fxx = $12x^2 + 4y^2 + 6xy$

fxy = 8xy + $3x^2$

fy = $4x^2y + x^3 + 4y^3$

fyy = $4x^2 + 12y^2$

fyx = 8xy + $3x^2$

Note:

fxy = fyx is not true for all functions , but it is true for all the functions that we will encounter in this book.

7.3 Optimizing functions of several variables

- Critical point

(a,b) is a critical point of f(x,y) if fx(a,b)=0 and fy(a,b)=0

Note: Relative maximum and minimum values can occur only at critical points.

- D-Test

If (a,b) is a critical point of the function f(x,y) , and let

D(a,b) = fxx(a,b)fyy(a,b) – $[fxy(a,b)]^2$

1. If D(a,b) > 0 , and fxx(a,b) > 0
Then f(x,y) has a relative min at(a,b).
2. If D(a,b) > 0 , and fxx(a,b) < 0
Then f(x,y) has a relative max at(a,b).
3. If D(a,b) < 0 , then f(x,y) has a saddle point at(a,b).
4. If D(a,b) = 0 , then no conclusion can be drawn from this test.
Note: A saddle point is neither a relative max nor a relative min.
ex.1
f(x,y) = $3x^2 + y^2 + 3xy + 3x + y + 6$

$$\begin{cases} fx = 6x + 3y + 3 \\ fy = 2y + 3x + 1 \end{cases}$$

x = -1 , y = 1
C.P:(-1,1)
ex.2
f(x,y) = $3x^2 + y^2 - 6x - 12y$

Set$\begin{cases} fx = 6x + 3y + 3 = 0 \\ fy = 2y + 3x + 1 = 0 \end{cases}$

x = -1 , y = 1
C.P.(-1,1)
fxx = 6
fyy = 2
fxy = 3
D(-1,1) = (6)(2) - 3^2 = 3 > 0
and fxx(-1,1) = 6 > 0
f(x,y) has a relative min at (-1,1) and f(-1,1) = 5
ex.3
f(x,y) = $e^{x^2-y^2}$

Set$\begin{cases} fx = e^{x^2-y^2} \times (2x) = 0 \\ fy = e^{x^2-y^2} \times (-2y) = 0 \end{cases}$

x=0 , y=0
fxx = $e^{x^2-y^2}(2x)(2x) + e^{x^2-y^2}(2)$
fxy = $e^{x^2-y^2}(-2y)(-2y) + e^{x^2-y^2}(-2)$
fxy = $e^{x^2-y^2}(2x)(-2y)$
D(0,0) = fxx(0,0)fyy(0,0) – $[fxy(0,0)]^2$ = 2(-2)-0 = -4 < 0
f(x,y) has a saddle point at (0,0)

7.4

Let d1 , d2 and d3 stand for the vertical distances between the three points and the line

$y = ax + b$

The line that minimize the sum of the squares of these vertical derivations is called the least squares line or the regression line.

<u>Derivative of the Formula for the Least Squares Line</u>

For n points (x1 y1) (x2,y2) … (xn,yn)

$$S = ax1 + b - y1^2$$
$$+ ax2 + b - y2^2$$
$$+ \dots$$
$$+ axn + b - yn^2$$

$$\frac{ds}{da} = 2(ax1 + b - y1)x1$$
$$+ 2(ax2 + b - y2)x2$$
$$+ \dots$$
$$+ 2(axn + b - yn)^2$$
$$= 2a\sum xi^2 + 2b\sum xi + 2\sum xiyi$$

$$\frac{ds}{db} = 2(ax1 + b - y1)$$
$$+ 2(ax2 + b - y2)$$
$$+ \dots$$
$$+ 2(axn + b - yn)$$
$$= 2a\sum xi + 2bn + 2\sum yi$$

$$\text{Set } \begin{cases} \frac{ds}{da} = 0 \\ \frac{ds}{db} = 0 \end{cases}$$

$$\begin{cases} a\sum xi^2 + b\sum xi = \sum xiyi \\ a\sum xi + bn = \sum yi \end{cases}$$

$$a = \frac{\begin{vmatrix} \sum xiyi & \sum xi \\ \sum yi & n \end{vmatrix}}{\begin{vmatrix} \sum xi^2 & \sum xi \\ \sum xi & n \end{vmatrix}} = \frac{n\sum xiyi - (\sum xi)(\sum yi)}{n\sum xi^2 - (\sum xi)^2}$$

$$b = \frac{1}{n}(\sum yi - a\sum xi)$$

Least squares line $y = ax + b$

Ex1

x	y	xy	x^2
1	10	10	1
2	12	24	4
3	25	75	9
$\sum xi$	$\sum yi$	$\sum xiyi$	$\sum xi^2$

6 47 109 14

$a = \frac{3\times109-6\times47}{3\times14-6^2} = \frac{30}{4}$

$b = \frac{1}{3}(47 - \frac{30}{4} \times 6) = \frac{2}{3}$

7.5

- <u>The method of lagrage multipliers</u>

To optimize (find the relative extrema)

f(x,y) subject to g(x,y) = 0 .

Let F(x,y,λ) = f(x,y) + λg(x,y)

solve the equations

$$\begin{cases} Fx = 0 \\ Fy = 0 \\ F\lambda = 0 \end{cases}$$

f(x,y) : objection function

g(x,y) = 0 : constrained equation

F(x,y,λ) : Lagrange function

λ : Langrange multiplier

<u>ex1</u>

Maximize A = xy

Subject to zx + y = 400 (2x + 4y – 400 = 0)

Let F (x,y,λ) = xy + λ(2x + y - 400)

$$\text{Set}\begin{cases} Fx = y + 2\lambda = 0 \\ Fy = x + \lambda = 0 \\ F\lambda = 2x + y - 400 = 0 \end{cases}$$

$$\begin{cases} \lambda = \frac{-y}{2} \\ \lambda = -x \end{cases} \quad \frac{-y}{2} = -x$$

x = 100 , y = 200

<u>ex3</u>

Max P = $600L^{\frac{2}{3}}K^{\frac{1}{3}}$

s.t 40L + 100K – 3000 = 0

Let F(L,K,λ)

= $600L^{\frac{2}{3}}K^{\frac{1}{3}}$ + λ(40L + 100K – 3000)

Set $\begin{cases} FL = 400L^{\frac{-1}{3}}K^{\frac{1}{3}} + 40\lambda = 0 \\ FK = 200L^{\frac{2}{3}}K^{\frac{-2}{3}} + 100\lambda = 0 \\ F\lambda = 40L + 100K - 3000 = 0 \end{cases}$

K = 10 , L = 50

i.e Labor 50 units

Capital 10 units

<u>Interpretation of λ</u>

$|\lambda|$ = Number of additional objection units for each additional constraint unit.

In ex3

$\lambda = -10L^{\frac{-1}{3}}K^{\frac{1}{3}}$ 5.8

meaning the production increases by about 5.8 units for each additional dollar in the budget.

- To optimize (find the relative extrema) f(x,y,z) subject to g(x,y,z) = 0

Let F(x,y,z,λ)

F(x,y,z)+λg(x,y,z)

Solve equations

$$\begin{cases} Fx = 0 \\ Fy = 0 \\ Fz = 0 \\ F\lambda = 0 \end{cases}$$

- <u>Level curve</u>

The graph of z = f(x,y) is a surface in 3-dim space.

The level curve f(x,y) = c is the projection onto the xy plane of the curve format by the intersection of the surface z = f(x,y) with the horizontal plane z = c.

<u>Justification of Lagranges method</u>

Suppose the constraint curve g(x,y) = 0 and the level curves f(x,y) = c are drawn in the xy plane.

To maximize f(x,y) subject to g(x,y) = 0 , we must find the highest level curve of f that intersects the constraint curve (the point p) the two curves have the same slope.

- The slope of f = c is $\frac{-fx}{fy}$
- The slope of g = 0 is $\frac{-gx}{gy}$

$$\frac{-fx}{fy} = \frac{-gx}{gy}$$

$$\frac{-fx}{gx} = \frac{-fy}{gy} = \lambda$$

$$\begin{cases} fx + \lambda gx = 0 \\ fy + \lambda gy = 0 \end{cases}$$

$$\begin{cases} Fx = fx + \lambda gx = 0 \\ Fy = fy + \lambda gy = 0 \end{cases}$$

7.6

- <u>Differential of f(x)</u>

For a differentiable function f(x) , the differential df is

df = f'(x)dx

- <u>Total differential of f(x,y)</u>

For a function f(x,y) , the total differential df is

df = fxdx + fydy

<u>ex1</u>

f(x,y) = $5x^3 - 4xy^{-1} + 3y^4$

df = fxdx + fydy

= $(15x^2 - 4y^{-1})$dx + $(4xy^{-2} + 12y^3)$dy

<u>ex2</u>

z = $\ln(x^2 + y^3)$

dz = zxdx + zydy

= $\frac{2x}{x^2+y^3}$dx + $\frac{3y^2}{x^2+y^3}$dy

<u>Approximating changes by Total Differential</u>

- Change in f(x,y)

△f = f(x+△x , y+△y) – f(x,y)

- Differential Approximation Formula

f(x+$\vartriangle$ x , y $+\vartriangle$ y - f(x,y)

fx$\times\vartriangle$ x + fy $\times\vartriangle$ y

Change in f

Total differential of f

<u>Ex3</u>

For

f(x,y) = $x^2 + 4xy + y^3$

and values x=3 , y=2

$\vartriangle x = 0.2$, $\vartriangle y = -0.1$

(a)

$\vartriangle$ f

= f(x+$\vartriangle$ x , y $+\vartriangle$ y) - f(x,y)

=f(3+0.2 , 2-0.1) – f(3,2)

F(3.2 , 1.9) – f(3,2)

=0.419

(b)

df

$=f_x dx + f_y dy$

$=(2x + 4y)dx + (4x+3y^2)dy$

=0.4

Total differential of f(x,y,z)

For a function f(x,y,z) , the total differential df is

Df = fxdx + fydy +fzdz

$f_x \Delta x + f_y \Delta y + f_z \Delta z$

$\Delta f = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$

=df

Ex.

Let f(x,y,z) = $\sqrt{z}e^{x-y^2}$

Approximate f(4.1,1.9,9.1)

X=4 , y=2 , z=9

$\Delta x = 0.1$, $\Delta y = -0.1$, $\Delta z = 0.1$

fx = $\sqrt{z}e^{x-y^2}$

fx(4,2,9) = $\sqrt{9}e^{4-2^2}$= 3

fy = $\sqrt{z}e^{x-y^2}(-2y)$

fy(4,2,9) = -12

fz = $\frac{1}{2\sqrt{z}}e^{x-y^2}$

fz(4,2,9) = $\frac{1}{6}$

f(4.1,1.9,9.1) – f(4,2,9)

fx(4,2,9)(0.1) + fy(4,2,9)(-0.1) + fz(4,2,9)(0.1)

f(4.1,1.9,9.1) - $\sqrt{9}e^{4-2^2}$

= 3(0.1) + (-12)(-0.1) + $\left(\frac{1}{6}\right)$(0.1)

f(4.1,1.9,9.1) – 3 = 1.157

f(4.1,1.9,9.1) = 4.157

7.7

Double Integral

The double integral of a continuous function f(x,y) on a rectangular region R is

$\int\int f(x,y)\,dx\,dy$

$= \lim_{\Delta x,\Delta y\to 0} \sum f(xi, yj)\ \Delta x\ \Delta y$

Note: If f(x,y) is nonaegative on R , then the double integral gives the volume under f

over R.
ex 1

$$\int_0^1 \int_0^2 \left(3x^2 + 6xy^2\right)dxdy$$

$$=\int_0^1 (\int_0^2 \left(3x^2 + 6xy^2\right)dx)dy \quad (*)$$

$$(\int_0^2 \left(3x^2 + 6xy^2\right)dx)$$

$$= x^3 + \tfrac{6}{2}x^2y^2 \, |_0^2$$

$$= 8 + 12y^2$$

$$(*) = \int_0^1 (8 + 12y^2)dy$$

$$= 8y + 4y^3 \, |_0^1$$

$$= 12$$

$$\int_0^1 \int_0^2 \left(3x^2 + 6xy^2\right)dxdy = 12$$

Ex 2

$$\int_0^2 \int_0^1 \left(3x^2 + 6xy^2\right)dydx$$

$$= \int_0^2 (\int_0^1 \left(3x^2 + 6xy^2\right)dy)dx$$

$= 12$

- For a continuous function f(x,y)

$$\int_c^d \int_a^b f(x, y)dxdy$$

$$=\int_a^b \int_c^d f(x, y)dydx$$

- The double integral

$\iint f(x, y)dxdy$ over the
Region R = {(x,y)|a<=x<=b , c<=y<=d}
can be evaluated by

$$\int_c^d \int_a^b f(x, y)dxdy$$

or $\int_a^b \int_c^d f(x, y)dydx$

ex 3
1. Let R be the region in the xy-plane bounded by the graphs of y=g(x) , y=h(x) and

the vertical line x=a , x=b then

$$\iint f(x,y)dxdy$$

$$= \int_a^b [\int_{g(x)}^{h(x)} f(x,y)dy]dx$$

Note: If f(x,y)>=0 on R then the double integral gives the volume under the surface f(x,y) above R.

2.

$$\iint f(x,y)dxdy$$

$$= \int_a^b [\int_{g(x)}^{h(x)} f(x,y)dy]dx$$

Ex 6

Find the voiumn under the surface f(x,y) = 12xy , and above the region R

$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} 12xydydx$$

Where

$$\int_{x^2}^{\sqrt{x}} 12xydydx$$

$$= 12x \times \frac{1}{2}y^2 |_{x^2}^{\sqrt{x}}$$

$$= 6x^2 - 6x^5$$

$$V = \int_0^1 (6x^2 - 6x^5)dx$$

$$= 1$$

39.

$$\int_0^1 \int_x^{2x} e^y dydx$$

$$= \int_0^1 [e^y|_x^{2x}]dx$$

$$= \int_0^1 (e^{2x} - e^x)dx$$

$$= \frac{1}{2}e^{2x} - e^x|_0^1$$

$$= \frac{1}{2}e^2 - e + \frac{1}{2}$$

$$\int_0^1 \int_{\frac{y}{2}}^{y} e^y dxdy + \int_1^2 \int_{\frac{y}{2}}^{1} e^y dxdy$$

= (I) + (II)

$$(I) = \frac{1}{2}$$

$$(II) = \frac{1}{2}e^2 - e$$

$$(I) + (II) = \frac{1}{2}e^2 - e + \frac{1}{2}$$