Exponential and Logarithmic Functions

4.1 Exponential Functions

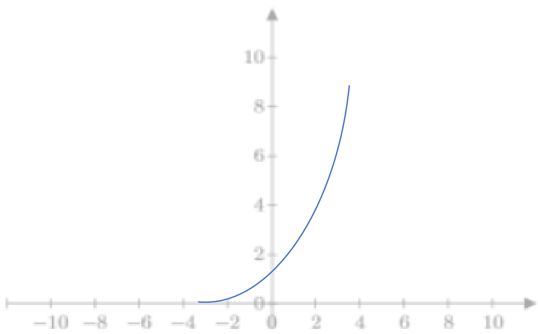
A function that has a variable in an exponent, such as f(x) $=2^{x}$, is called an exponential function. The number being raised to the power is called the base.

f(x) $=2^{x}$ Exponent

Base

The table below shows some values of the exponential function f(x) $=2^{x}$, and its graph (based on these points) is shown on the right.

x	$y=2^{x}$
-3	$2^{-3}=\frac{1}{8}$
-2	$2^{-2}=\frac{1}{4}$
-1	$2^{-1}=\frac{1}{2}$
0	$2^{0}=1$
1	$2^{1}=2$
2	$2^{2}=4$
3	$2^{3}=8$



Compound Interest

For P dollars invested at annual interest rate r compounded m times a year for t years,

$$\binom{\text{Value after}}{t \text{ years}} = \mathrm{P} \cdot \left(1+\frac{r}{m}\right)^{mt}$$

For example, for monthly compounding we would use m =12 and for daily compounding m =365 (the number of days in the year).

Example 1. FINDING A VALUE UNDER COMPOUND INTEREST

Find the value of $4000 invested for 2 years at 12% compounded quarterly.

< solution >

$$4000 \cdot \left(1+\frac{0.12}{4}\right)^{4 \cdot 2} \approx 5067.08$$

The value after 2 years will be $5067.08.

PRESENT VALUE

For a future payment of P dollars at annual interest rate r compounded m times a year to be paid in t years,

$$\begin{pmatrix} Present \\ Value \end{pmatrix} = \frac{P}{\left(1+\frac{r}{m}\right)^{mt}}$$

Example 2. FIDING PRESENT VALUE

Find the present value of $5000 to be paid 8 years from now at 10% interest compounded semiannually.

< solution >

For semiannual compounding (m = 2), the formula gives

$$\frac{P}{\left(1+\frac{r}{m}\right)^{mt}} = \frac{5000}{\left(1+\frac{0.10}{2}\right)^{2\cdot 8}} \approx 2290.56$$

Therefore, the present value of the $5000 is just $2290.56.

Continuous Compounding

For P dollars invested at annual interest rate r compounded continuously for t years,

$$\begin{pmatrix} Value\ after \\ t\ years \end{pmatrix} = \mathrm{P}e^{rt}$$

Example 3. FINDING VALUE WITH CONTINUOUS COMPOUNDING

Find the value of $1000 at 8% interest compounded continuously for 20 years.

< solution >

We use the formula $\mathrm{P}e^{rt}$ with P = 1000, r = 0.08, and t= 20.

$\mathrm{P}e^{rt} = 1000 \cdot e^{0.08 \cdot 20} \approx \$4953.03$

Present Value with Continuous Compounding

For a future payment of P dollars at annual interest rate r compounded continuously to be paid in t years,

$$\begin{pmatrix} Present \\ Value \end{pmatrix} = \frac{P}{e^{rt}} = \mathrm{P}e^{-rt}$$

Example 4. **FINDING PRESENT VALUE WITH CONTINUOUS COMPOUNDING**

The present value of $5000 to be paid in 10 years at 7% interest compounded continuously is ?

< solution >

$\frac{5000}{e^{0.07 \cdot 10}} = \frac{5000}{e^{0.7}}$ ≈$2482.93

4.2 Logarithmic Function

※$log_a x = y \leftrightarrow a^y = x$

The natural logarithm：

$y = log_e x = lnx \leftrightarrow e^y = x$ (for x>0)

Graph = p290

EXAMPLE

$log_{10} \frac{1}{10}$ = y$\leftrightarrow$ $10^y = \frac{1}{10}$

$\rightarrow$ $log_{10} \frac{1}{10}$ = -1

$log_9 3$ = y$\leftrightarrow$ $9^y = 3 \leftrightarrow log_9 3 = \frac{1}{2}$

Properties of natural loganthms

1. ln1 = 0
2. lne = 1
3. ln$e^x = x$
4. $e^{lnx} = x$
5. ln(M · N) = lnM+lnN
6. ln($\frac{M}{N}$) = lnM-lnN
7. ln($\frac{1}{N}$) = -lnN
8. ln(M^N) = NlnM

Example 9.

A sum is invested at 12% interest camponnded quarterly.
How soon will it double in value?

< solution >

r=$\frac{1}{4}\cdot 12\% = 0.03$

p$(1+0.03)^n$=2p
1.03^n = 2
ln$(1.03)^n$=ln2
nln=(1.03) =ln2

n=$\frac{\ln(1.03)}{ln2} \approx 23.4$

$\frac{23.4}{4} \approx$ 5.9years

Example 10.
A sum is invested at 15% compounded continuously. How soon will it triple?

< solution >
P$e^{0.15n} = 3P$
$e^{0.015n}$ =3
ln($e^{0.015n}) = ln3$

n = $\frac{\ln 3}{0.15}$ = 7.3 years

4.3 Differentiation of Logarithmic and Exponential Functions

1. $\frac{d}{dx} lnx = \frac{1}{x}$ (for x>0)

< solution >

pf : $\frac{d}{dx} lnx = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}$

$= \lim_{h\to 0}\frac{1}{h}\ln(\frac{x+h}{x})$

$= \lim_{h\to 0}\frac{1}{x}\frac{1}{h}\ln(1+\frac{h}{x})$

$= \lim_{h\to 0} \frac{1}{x} \ln(1+\frac{h}{x})^{\frac{x}{h}}$

Let $\frac{x}{h} = n$, as h=0^+ , we have n→ ∞

$(*) = \lim_{h\to 0} \frac{1}{x} \ln(1+\frac{1}{h})^{n}$

$= \frac{1}{x} \cdot lne = \frac{1}{x}$

2. $\frac{d}{dx}\ lnf(x) = \frac{f'(x)}{f(x)}$

Example 2.

$\frac{d}{dx}\ln(x^2+1)$= ?

< **solution** >

$\frac{d}{dx}\ln(x^2+1) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$

3. $\frac{d}{dx}e^x = e^x$

< **solution** >

pf：$\ln e^x = x$

$\frac{d}{dx}(lne^x) = \frac{d}{dx}x$

$\frac{\frac{d}{dx}e^x}{e^x} = 1$

$\rightarrow \frac{d}{dx}e^x = e^x$

4. $\frac{d}{dx}e^{f(x)}$=?

< **solution** >

$\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot f'(x)$

Example 4.

$\frac{\mathrm{d}}{dx}\left(\frac{e^x}{x}\right)$= ?

< solution >

$\frac{\mathrm{d}}{dx}\left(\frac{e^x}{x}\right)=\frac{x\cdot e^x-1\cdot e^x}{x^2}=\frac{x\cdot e^x-e^x}{x^2}$

5. $\frac{\mathrm{d}}{dx}a^x$=?

< solution >

$\frac{\mathrm{d}}{dx}a^x=\frac{\mathrm{d}}{dx}e^{lnax}=\frac{\mathrm{d}}{dx}e^{xlna}=e^{xlna}\cdot lna=lna\cdot(a^x)$

6. $\frac{\mathrm{d}}{dx}a^{f(x)}$ =?

< solution >

$\frac{\mathrm{d}}{dx}a^{f(x)}=(lna)\cdot a^{f(x)}\cdot f'(x)$

ex.

$\frac{\mathrm{d}}{dx}2^x$=(ln2) 2^x

$\frac{\mathrm{d}}{dx}5^{3x^x+1}$= (ln5) $5^{3x^2+1}\cdot 6x$

=(ln5)$6x\cdot 5^{3x^2+1}$

7. $\frac{\mathrm{d}}{dx}\log_a x=\frac{\mathrm{d}}{dx}\left(\frac{lnx}{lna}\right)=\frac{1}{lna}\frac{\mathrm{d}}{dx}lnx=\frac{1}{(lna)\cdot x}$

$\frac{\mathrm{d}}{dx}\log_a f(x)=\frac{f'(x)}{(lna)\cdot f(x)}$

ex.

$\frac{\mathrm{d}}{dx}\log_5 x=\frac{1}{(ln5)\cdot x}$

$\frac{\mathrm{d}}{dx}\log_2(x^3+1)=\frac{3x^2}{(ln2)(x^3+1)}$

4.4 Two Applications to Economics: Relative

Rates and Elasticity of Demand

- absolute rate of change：$f'(x)$
- relstire rate or change：$\frac{d}{dx}lnf(x) = \frac{f'(x)}{f(x)}$

Example 1.

G(t)=8.2$e^{\sqrt{t}}$

ln G(t)=ln8.2$e^{\sqrt{t}}$

= ln8.2+$t^{-\frac{1}{2}}$

$\frac{d}{dt}$(ln8.2+$t^{-\frac{1}{2}}$)=$\frac{1}{2}t^{-\frac{1}{2}}$

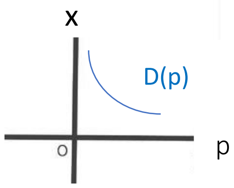
at t=25

$\frac{d}{dt}$lnG(t) = $\frac{1}{2} \cdot \frac{1}{\sqrt{25}}$ =0.1

→in 25 years the G(t) will be increasing at the relatire rate of 0.1 per year.

- **Demaid function**

The demand function x=D(p) gives the quantity x of an item that will be demanded by consumers if the price is p.



For a demand function D(p), let us calcutate the relatire rate of change of demand dirided by the relatire rate of change of price.

※$\frac{\text{the relatire rate of change of demand}}{\text{the relatire rate of change of price}}$

$=\frac{\frac{d}{dp}lnD(p)}{\frac{d}{dp}lnp}$

$=\frac{\frac{D'(p)}{D(p)}}{\frac{1}{p}}$

$=\frac{pD'(p)}{D(p)}$ (sornetimes, $D'(p) < 0$)

- **Elasticity of Demand**

For a demand function D(p), the elasticity of demand is

E(p)= $\frac{-pD'(p)}{D(p)}$

Demand is elastic if E(p) > 1 and inelastic if E(p) < 1.

Example 2

D(p)=81-p^2 $(0 \leq p \leq 9)$

E(p)= $\frac{-pD'(p)}{D(p)} = \frac{-p(-2p)}{81-p^2}$

= $\frac{2p^2}{81-p^2}$

(a) Evaluating at p = 3 gives

E(3)=$\frac{2\cdot 3^2}{81-3^2} = \frac{1}{4} < 1$

→Interpretation: The elasticity is less than 1, so demand for tickets is inelastic at a price of $3. This means that a small price change (up or down from this level) will cause only a slight change in demand. More precisely, elasticity of means that a 1% price change will cause only about a % change in demand.

(b) At the price of $6, the elasticity of demand is

E(6)=1.6> 1

→Interpretation: The elasticity is greater than 1, so demand is elastic at a price of $6. This means that a small change in price (up or down from this level) will cause a relatively large change in demand. In particular, elasticity of 1.6 means that a price change of 1% will cause about a 1.6% change in demand.

- **Using Elasticity to Increase Revenue**

R=$P \cdot x$

= $P \cdot D(p)$

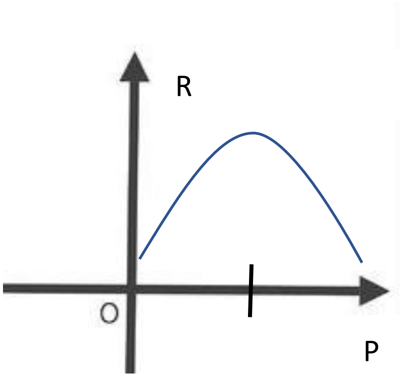
$R'(p)$=D(p)+ $PD'(p)$

=D(p)$[1 + \frac{PD'(p)}{D(p)}]$

= D(p)$[1 - \frac{-PD'(p)}{D(p)}]$

= D(p)$[1 - E(p)]$

E(p)	< 1	=1	>1
$R'(p)$	>0	=0	< 0
R(p)	↑	max	↓



- **To increase revenue:**

If demand is elastic ($E > 1$), you should lower prices.
If demand is inelastic ($E < 1$), you should raise prices.