CH10
Sequences and Series

10.1 Geometric Series

- Sum of a finite geometric series

$\sum_{k=1}^{n-1} ar^k$ = a + ar +…+ ar^{n-1}

= $a\frac{1-r^n}{1-r}$ (r$\neq$ 1)

- Infinite Series

$\sum_{k=0}^{\infty} ak = a1 + a2 + \cdots$

- Sn = a1 + a2 + … + an is called the n^{th} partial sum (sum of the first n terms) of the series $\sum_{k=1}^{\infty} ak$

1. If there exist a finite number S , such that $\lim_{n\to\infty} Sn = S$, then $\sum_{k=1}^{\infty} ak = S$ (converges to S)

2. If $\lim_{n\to\infty} Sn$ diverge (or does not exist) then we say the series $\sum_{k=1}^{\infty} ak$ is divergent.

- Sum of an infinite geometric series $\sum_{k=0}^{\infty} ar^k$ = a + ar + ar^2 + … = $\frac{a}{1-r}$, for |r|<1

$\sum_{k=0}^{\infty} ar^k$ is divergent for |r|>=1

10.2 Taylor polynomials

- A polynomial of degree n is a function of the form p(x) = a0 + a1x + a2x^2 + … + anx^n

Where a0 … an are given numbers and an$\neq$ 0

- The n^{th} Taylor polynomial of f(x) at x=0 is the polynomial pn(x) defined by

pn(x) = f(0) + $\frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$

(pn(0) = f(0) , p’n(0) = f’(0) … $pn^{(n)}(0) = f^{(n)}(0)$)

<u>Example 2</u>

Let f(x) = e^x f(0) = f’(0) = f’’(0) = f’’’(0) = 1

the 1^{st} Taylor polynomial for f(x) at x=0 is

p1(x) = f(0) + $\frac{f'(0)}{1!}$x = 1+x

Note: 1. Higher-order Taylor polynomials generally approximate functions more closely.

2. Each approximation is more accurate closer to x=0 (exact at x=0)

(P757)

The remainder formula

- If pn(x) is the n^{th} Taylor polynomial of f(x) at x=0 then Rn(x) = f(x) – pn(x)

$= \frac{f^{n+1}(+)}{(n+1)!}x^{n+1}$, for some t between 0 and x

- If pn(x) is the n^{th} Taylor polynomial of f(x) at x=a

then

Rn(x) = f(x) – pn(x)

$= \frac{f^{n+1}(+)}{(n+1)!}x - a^{n+1}$, for some t between a and x

- Error in Taylor Approximation at x=0

$|Rn(x)| = |f(x)\text{-}pn(x)| <= \frac{M}{(n+1)!}|x|^{n+1}$

Where M is any number such that $|f^{n+1}(t)| <= M$

For all t between 0 and x

<u>Example 3</u>

Approximate $e^{0.5}$ using the $p3(x)$ for e^x, and estimate the error.

In example 2 , $p3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$

$e^{0.5} = p3(0.5) = 1 + 0.5 + \frac{1}{2!}(0.5)^2 + \frac{1}{3!}(0.5)^3 = 1.6458$

$f^{(4)}(t) = e^t$, and $0 \leq t \leq 0.5$

$|f^{(4)}(t) = |e^t \leq e^{0.5} < e^1 < 3 = M$

$|R3(0.5)| <= \frac{M}{(3+1)!}|0.5|^4$

$= \frac{3}{24}(0.0625)$

= 0.008

$e^{0.5} = 1.6458$ with an error less than 0.008

<u>Example 4</u>

Find the Taylor polynomial at x=0 that approximates

e^x with an error less than 0.005 on the interval $-1 \leq x \leq 1$

For $-1 \leq x \leq 1$, t between 0 and x

$Rn(x) = |\frac{f^{n+1}(t)}{(n+1)!}x^{n+1}|$

$= |\frac{e^t}{(n+1)!}x^{n+1}| <= \frac{e}{(n+1)!} <= \frac{3}{(n+1)!} <= 0.005$

n >= 5

$e^x = p5(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5$

<u>Example 5</u>

Approximate sin1.1 by using the fifth Taylor polynomial at x=0 for sinx and estimate the error

f(x)=sinx	f(0)=0
f'(x)=cosx	f'(0)=1
f"(x)=-sinx	f"(0)=0
f'''(x)=-cosx	f'''(0)=-1
$f^{(4)} = \sin x$	$f^{(4)}(0) = 0$
$f^{(5)} = \cos x$	$f^{(5)}(0) = 1$

$p5(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$

$= 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5$

$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

Sin1.1 = p5(1,1) = 1.1 - $\frac{1}{3!}(1,1)^3 + \frac{1}{5!}(1,1)^5 = 0.892$

$f^{(6)}(t) = -\sin t\,,\, 0 \le t \le 1.1$

=|R5(1,1)| = $\left|\frac{f^{(6)}(t)}{6!}(1,1)^6\right| <= \frac{1}{6!}(1,1)^6 = 0.0025$

Sin1.1 = 0.892 with an error less than 0.0025

· The Taylor series of f(x) of x=0 is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f''(0)}{n!}x^n + \cdots$$

· The Taylor series of f(x) of x=a is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f''(a)}{3!}x(x-a)^3 + \cdots +$$

$$\frac{f''(a)}{n!}x(x-a)^n + \cdots$$

Example 2

Find the Taylor series at x=0 for e^x, and find its interval of convergence.

<solution>

$f(x) = e^x$

$f'(x) = f''(x) = f'''(x) = \cdots = e^x$

$f(0) = f'(0) = f''(0) = \cdots = e^0 = 1$

$$\Rightarrow e^x = f(0) \ + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$r = \lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1$$

The radins of convergence is R=∞

The interval of convergence is $-\infty < x < \infty$

Note :

$$e = 1 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \cdots$$

Example 3

Find the Taylor series at x=0 for $\sin x$

<solution>

$f(x) = sin\, x$ $f(0) = 0$

$f'(x) = cos\, x$ $f'(0) = 1$

$f''(x) = -\sin x \qquad \Rightarrow \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$\Rightarrow \sin x = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7 + \cdots$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}$

$$r = \lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}\cdot x^{2n+3}}{(2n+3)!}}{\frac{(-1)^n\cdot x^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+3)}\right| = 0 < 1$$

The radins of convergence is R=∞

The interval of convergence is $-\infty < x < \infty$

Example 4

$\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad , -\infty < x < \infty$

$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad , -\infty < x < \infty$

H.W Find the Taylor series at x=0 for $\cos x$

Practice problem

$\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad , -\infty < x < \infty$

$\Rightarrow \cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots \qquad , -\infty < x < \infty$

Example 5

$\int_0^x \frac{\sin t}{t} dt = \ ?$

<solution>

$\because \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$

$\Rightarrow \frac{\sin t}{t} = \frac{1}{t}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$

$\Rightarrow \int_0^x \frac{\sin t}{t} dt = \int_0^x (1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots) dt$

$$= \left. t - \frac{t^3}{3\cdot 3!} + \frac{t^5}{5\cdot 5!} - \frac{t^7}{7\cdot 7!} + \cdots \right|_0^x$$

$$= x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots$$

$$\Rightarrow \int_0^2 \frac{sint}{t} dt \approx 2 \ - \frac{2^3}{3\cdot 3!} + \frac{2^5}{5\cdot 5!} - \frac{2^7}{7\cdot 7!} + \cdots \approx 1.605$$

Example 6

Find the Taylor series at x=1 for lnx

<solution>

$f(x) = ln\, x$ $\qquad f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1} \qquad \Rightarrow \qquad f'(1) = 1$

$f''(x) = -x^{-2} \qquad f''^{(1)} = -1$

$f'''(x) = 2x^{-3} \qquad f'''(x) = 2$

. .

. .

$$\Rightarrow lnx = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f''(1)}{3!}x(x-1)^3 + \cdots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

$$r = \lim_{n\to\infty} \left| \frac{C_{n+1}}{C_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{(n+1)}(x-1)^{n+1}}{\frac{(-1)^n}{n}(x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{n(x-1)}{n+1} \right| = |x-1| < 1$$

$\Rightarrow 0 < x < 2$

The radins of convergence is R=1

The interval of convergence is $0 < x < 2$

Example 10.3

3. $\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots = ?$

<solution>

$C_n = \frac{x^n}{n(n+1)}$, $C_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$

$$\frac{C_{n+1}}{C_n} = \frac{\frac{x^{n+1}}{(n+1)(n+2)}}{\frac{x^n}{n(n+1)}} = \frac{x^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^n} = \frac{n}{n+2}x$$

$$r = \lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty} \left|\frac{n}{n+2} \cdot x\right| = |x| < 1$$

The radins of convergence is R=1

The interval of convergence is $-1 < x < 1$

9. $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ =?

<solution>

$$C_n = \frac{x^{2n}}{n!} \quad , C_{n+1} = \frac{x^{2(n+1)}}{(n+1)!}$$

$$\frac{C_{n+1}}{C_n} = \frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}} = \frac{x^2}{n+1}$$

$$r = \lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty} \left|\frac{x^2}{n+1}\right| = 0$$

The radins of convergence is R=∞

The interval of convergence is $-\infty < x < \infty$

17. $f(x) = e^{\frac{x}{5}}$

(a) $f(x) = e^{\frac{x}{5}}$

$$f'(x) = \frac{1}{5}e^{\frac{x}{5}}$$

$$f''(x) = \frac{1}{25}e^{\frac{x}{5}}$$

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The Taylor series is :

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 1 + \frac{1}{5}x + \frac{1}{5^2 \cdot 2!}x^2 + \frac{1}{5^3 \cdot 3!}x^3 + \cdots$$

(b) The Taylor series at x=0 for e^x

$$1+\frac{x}{1!}+\frac{x^2}{1!}+\frac{x^3}{1!}+\cdots$$

Substituting $\frac{x}{5}$ for x

$$\Rightarrow 1+\frac{x}{5}+\frac{1}{2!}\left(\frac{x^2}{5}\right)+\frac{1}{3!}\left(\frac{x^3}{5}\right)+\cdots$$

$$=1+\frac{1}{5}x+\frac{1}{5^2 2!}x^2+\frac{1}{5^2 3!}x^3+\cdots$$

21. Taylor series at x=0 for $\frac{1}{1-x}$ is

$$1+x+x^2+x^3+\cdots \qquad |x|<1$$

multiply the Taylor series for $\frac{1}{1-x}$ by x^2

$\Rightarrow$ The Taylor series at x=0 for $\frac{x^2}{1-x}$ is

$$x^2+x^3+x^4+x^5+\cdots$$

27. $sinx=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$

$$-sinx=-x+\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\cdots$$

$$x-sinx=\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\cdots$$

$$\Rightarrow \frac{x-sinx}{x^3}=\frac{1}{3!}-\frac{x^2}{5!}+\frac{x^4}{7!}-\cdots$$

29. $\frac{1}{1+x}=1-x+x^2-x^3+\cdots$ for $|x|<1$

$$\int\frac{1}{1+x}dx=\int(1-x+x^2-x^3+\cdots)dx$$

$$\Rightarrow \ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$$

33. $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

$$e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots$$

$\Rightarrow \frac{e^x+e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

45.

(a) $S_n - S_{n-1} = (C_1 + C_2 + \cdots + C_n) - (C_1 + C_2 + \cdots + C_{n-1}) = C_n$

(b) $\lim_{n\to\infty} S_n = a$

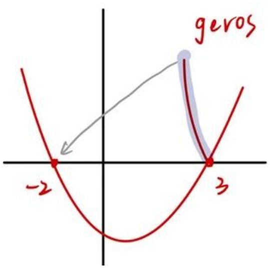
$$\Rightarrow \lim_{n\to\infty} C_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = a - a = 0$$

10.4 Newton's method

Example 1

Find the geros of $f(x) = x^2 - x - 6$

<solution>



$$f(x) = (x-3)(x+2) = 0$$

$$\Rightarrow \text{ The geros are } \begin{cases} x = 3 \\ x = -2 \end{cases}$$

Newton's method is a procedure for approximating the solutions in case the the exact solutions are difficient or impossible to find.

Given an initial approximation $x = x_0$, instand of solving $f(x) = 0$, we set the Taylor polynomial at $x = x_0$ equal to gero

$$f(x_0) + f'(x_0)(x - x_0) = 0$$

$$\Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad f(x_0) \neq 0$$

The sample formula can then be applied to the "better" approximate

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad f'(x_1) \neq 0$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \qquad\qquad f'(x_2) \neq 0$$

$\vdots \qquad\qquad \vdots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad\qquad f'(x_n) \neq 0$$

Newton's method

To approximate a solution to $f(x) = 0$, choose an intial approximate x_0, and caculate $x_1, x_2, x_3 \ldots$ converge, the converge to a solution of $f(x) = 0$

Example 2

Approximate $\sqrt{2}$ by using three iterations of Newton's method.

<solution>

$f(x) = x^2 - 2 \qquad\qquad \Rightarrow f'(x) = 2x$

Take $x_0 = 1$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1^2 - 2}{2 \cdot 1} = 1.5$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{f(1.5)}{f'(1.5)} \approx 1.4167$$

$$\Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(2)} = 1.4167 - \frac{f(1.4167)}{f'(1.4167)} \approx 1.4142$$

$\therefore \sqrt{2} \approx 1.4142$

Example 3

Approximate the solution to $e^x = 2 - 2x$, continuing until two successive iterations agree to nine decimal place.

<solution>

$f(x) = e^x - 2 + 2x \qquad\qquad \Rightarrow f'(x) = e^x + 2$

Take $x_0 = 0$

$$\Rightarrow x_1 = 0 - \frac{e^0 - 2 + 2(0)}{e^0 + 2} = 0.333333333$$

$$\Rightarrow x_2 = 0.333333333 - \frac{e^{0.333333333} - 2 + 2(0.333333333)}{e^{0.333333333} + 2} \approx 0.3149922850$$

$$\Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.3149230588$$

$\Rightarrow x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.3149230578$

$\Rightarrow x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} \approx 0.3149230578$

$\therefore x \approx 0.3149230578$