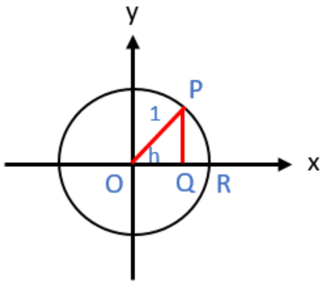
CH8

- If h is measured in radians, then

(a) $\lim\limits_{h \to 0} \frac{\sin h}{h} = 1$

(b) $\lim\limits_{h \to 0} \frac{\cos h}{h} = 0$



pf:

(a) $h =$ length of arc PR

sinh $=$ length of PQ

As $h \to 0$, then (length of PQ) $\approx$ (length of arc PR)

$\Rightarrow \lim\limits_{h \to 0} \frac{\sin h}{h} = 1$

(b) $\lim\limits_{h \to 0} \frac{\cos h - 1}{h} = \lim\limits_{h \to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)} = \lim\limits_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim\limits_{h \to 0} \frac{-\sin^2 h}{h(\cos h + 1)}$

$$= -\lim_{h \to 0} \frac{sinh}{h} \cdot \frac{sinh}{(\cos h + 1)} = -1 \cdot \frac{0}{1+1} = 0$$

- If t is measured in radians, then

(a) $\frac{d}{dt} \sin t = \cos t$

(b) $\frac{d}{dt} \cos t = -\sin t$

pf:

(a) $\frac{d}{dt} \sin t = \lim\limits_{h \to 0} \frac{\sin(t+h) - \sin t}{h} = \lim\limits_{h \to 0} \frac{\sin t \cos h + \cos t \sin h - \sin t}{h}$

$= \lim\limits_{h \to 0} \sin t \cdot \frac{\cos h - 1}{h} + \cos t \cdot \frac{\sin h}{h} = \sin t \cdot 0 + \cos t \cdot 1 = \cos t$

(b) $\frac{d}{dt} \cos t = \lim\limits_{h \to 0} \frac{\cos(t+h) - \cos t}{h} = \lim\limits_{h \to 0} \frac{\cos t \cos h - \sin t \sin h - \cos t}{h}$

$= \lim\limits_{h \to 0} \cos t \cdot \frac{\cos h - 1}{h} - \sin t \cdot \frac{\sin h}{h} = \cos t \cdot 0 - \sin t \cdot 1 = -\sin t$

- If t is measured in radians, then

(a) $\frac{d}{dt}\sin f(t) = [\cos t\, f(t)] \cdot f'(t)$

(b) $\frac{d}{dt}\cos f(t) = [-\sin t\, f(t)] \cdot f'(t)$

Example 1

Differentiate $f(t) = \frac{\sin t}{t}$

$\frac{d}{dt}\left(\frac{sint}{t}\right) = \frac{t\cdot\cos t - 1\cdot\sin t}{t^2} = \frac{t\cos t - \sin t}{t^2}$

Example 2

Differentiate $g(t) = t^3 \cdot \cos t$

$\frac{d}{dt}(t^3 \cdot \cos t) = 3t^2 \cdot \cos t + t^3 \cdot (-\sin t) = 3t^2 \cos t - t^3 \sin t$

Example 3

Differentiate $g(t) = \cos(t^2 + 1)$

$\frac{d}{dt}\cos(t^2 + 1) = -\sin(t^2 + 1) \cdot 2t$

Example 4

Differentiate $g(t) = \sin 4t$

$\Rightarrow g'(t) = \cos(4t) \cdot 4 = 4\cos 4t$

$\Rightarrow g''(t) = \frac{d}{dt}(4\cos 4t) = 4 \cdot (-\sin(4t) \cdot 4) = -16\sin 4t$

Example 6

$f \circ \mathrm{z} = e^{\sin z}$, find $f'(\pi)$

$f'(z) = e^{\sin z} \cdot \cos z$

$f'(\pi) = e^{\sin \pi} \cdot \cos \pi = e^{0} \cdot (-1) = -1$

Example 7

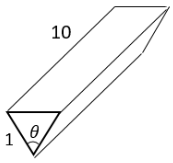
$f(t) = \sin^2 t^3 = (\sin t^3)^2$

$\Rightarrow f'(t) = 2\,(\sin t^3)^1 \cdot (\cos t)^3 \cdot 3t^2 = 6t^2 \sin t^3 \cos t^3$

Example 9

Find the angle θ that maximize the volume of the trough

max volume = max $\bigtriangledown$



$\mathrm{A} = \frac{1}{2} \cdot 1 \cdot \sin\theta = \frac{1}{2}\sin\theta$

$\Rightarrow A' = \frac{1}{2}\sin\theta = 0 \quad \Rightarrow \theta = \frac{\pi}{2} \quad (0 < \theta < \pi)$

$$\left(\begin{array}{c} \mathrm{A}'' = -\frac{1}{2}\sin\theta \\ A''\left(\frac{\pi}{2}\right) = -\frac{1}{2}\sin\frac{\pi}{2} = -\frac{1}{2} < 0 \\ \therefore\ A\ is\ maximized\ at\ \theta = \frac{\pi}{2} \end{array}\right)$$

$\therefore$ An angle of $\frac{\pi}{2}$ (90°) will maximize the volume of the trough.

- (a) $\frac{d}{dt}\tan t = \sec^2 t$

  (b) $\frac{d}{dt}\cot t = -\csc^2 t$

  (c) $\frac{d}{dt}\sec t = \sec t \cdot \tan t$

  (d) $\frac{d}{dt}\csc t = -\csc t \cot t$

pf:

(a) $\frac{d}{dt}\tan t = \frac{d}{dt}(\frac{\sin t}{\cos t}) = \frac{\cos t \cdot \cos t - \sin t \cdot (-\sin t)}{\cos^2 t}$

$= \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t$

(b) $\frac{d}{dt}\sec t = \frac{d}{dt}(\frac{1}{\cos t}) = \frac{0 - 1 \cdot (-\sin t)}{\cos^2 t}$

$= \frac{\sin t}{\cos^2 t} = \frac{1}{\cos t} \cdot \frac{\sin t}{\cos t} = \sec t \cdot \tan t$

Example

Find $\frac{d}{dt}\tan(t^3 + 1)$

$\frac{d}{dt}\tan(t^3 + 1) = \sec^2(t^3 + 1) \cdot (3t^2) = 3\mathrm{t}^2 \sec^2(t^3 + 1)$

Example

Find $\frac{d}{dx}\sec(\pi x + 1)$

$\frac{d}{dx}\sec(\pi x + 1) = \sec(\pi x + 1)\tan(\pi x + 1) \cdot \pi$

Example

Find $\frac{d}{dz}\csc^4 \sqrt{z}$

$\frac{d}{dz}\csc^4 \sqrt{z} = \frac{d}{dz}\left(\csc \sqrt{z}\right)^4$

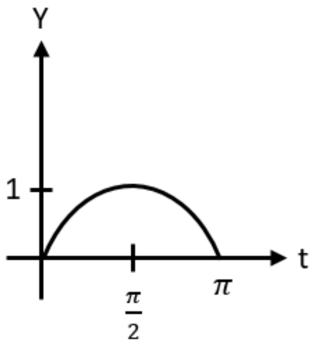
$= 4\csc^3 \sqrt{z}\left[-\csc \sqrt{z} \cdot 10 + \sqrt{z} \quad \left(\frac{1}{2} z^{-\frac{1}{2}}\right)\right] = -2z^{-\frac{1}{2}} \csc^4 \sqrt{z} \cot \sqrt{z}$

Integrals of trigonometric functions

1. $\int \sin t \, dt = -\cos t + C$
2. $\int \cos t \, dt = \sin t + C$

Example 1

Find the area under one arch of $y = \sin t$



$$A = \int_0^{\pi} \sin t \, dt$$
$$= -\cos t|_0^{\pi}$$
$$= -\cos \pi - (-\cos 0)$$
$$= 1 + 1 = 2$$

Example 2

Find $\int x^2 \sin x^3 \, dx$

Let $u = x^3 \Rightarrow du = 3x^2 dx$, $x^2 dx = \frac{1}{3} du$

$\Rightarrow \int x^2 \sin x^3 \, dx = \int \sin u \cdot \frac{1}{3} du$

$= \frac{1}{3} \int \sin u \, du = -\frac{1}{3} \cos u + C = -\frac{1}{3} \cos x^3 + C$

Example 3

Find $\int \sin x^5 \, 2t \cdot \cos 2t \, dt$

Let $u = \sin 2t \Rightarrow du = \cos 2t \cdot 2dt$, $\cos 2t \, dt = \frac{1}{2} du$

$\Rightarrow \int \sin x^5 \, 2t \cdot \cos 2t \, dt = \int u^5 \cdot \frac{1}{2} du$

$= \frac{1}{2} \int u^5 \, du = \frac{1}{12} u^6 + C = \frac{1}{12} \sin^6 2t + C$

Example 4

Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin z} \cos z \, dz$

Let $u = \sin z \Rightarrow du = \cos z \, dz$

$z = 0 \Rightarrow u = \sin 0 = 0$

$z = \frac{\pi}{2} \Rightarrow u = \sin\frac{\pi}{2} = 1$

$\Rightarrow \int_0^{\frac{\pi}{2}} e^{\sin z} \cos z \, dz = \int_0^1 e^u \, du = e^u |_0^1 = e^1 - e^0 = e - 1$

Example 5

Find $\int \frac{1+\cos t}{t+\sin t} dt$

Let $u = t + \sin t \Rightarrow du = (1 + \cos t)dt$

$\Rightarrow \int \frac{1+\cos t}{t+\sin t} dt = \int \frac{1}{u} du = \ln|u| + C = \ln|t + \sin t| + C$

3. $\int \sec^2 t \, dt = \tan t + C$ $\quad (\frac{d}{dt} \tan t = \sec^2 t)$

4. $\int \csc^2 t \, dt = -\cot t + C$ $\quad (\frac{d}{dt}(-\cot t) = \csc^2 t)$

5. $\int \sec t \tan t \, dt = \sec t + C$ $\quad (\frac{d}{dt} \sec t = \sec t \tan t)$

6. $\int \csc t \cot t \, dt = -\csc t + C$ $\quad (\frac{d}{dt}(-\csc t) = \csc t \cot t)$

Example 9

Find $\int \sec^2 5t \, dt$

Let $u = 5t \Rightarrow du = 5dt$, $dt = \frac{1}{5} du$

$\Rightarrow \int \sec^2 5t\, dt = \int \sec^2 u \cdot \frac{1}{5} du = \frac{1}{5} \int \sec^2 u\, du$

$= \frac{1}{5} \tan u + C = \frac{1}{5} \tan 5t\, C$

Example 10

Find $\int \tan^3 t \sec^2 t\, dt$

Let $u = \tan t \Rightarrow du = \sec^2 t\, dt$

$\Rightarrow \int \tan^3 t \sec^2 t\, dt = \int \mathrm{u}^3\, du = \frac{1}{4} \mathrm{u}^4 + C = \frac{1}{4} \tan^4 t + C$

Integrals of $\tan t$, $\cot t$, $\sec t$, $\csc t$

7. $\int \tan t\, dt = -\ln|\cos t| + C$
8. $\int \cot t\, dt = \ln|\sin t| + C$
9. $\int \sec t\, dt = \ln|\sec t + \tan t| + C$
10. $\int \csc t\, dt = \ln|\csc t - \cot t| + C$

pf:

7.

$\int \tan t\, dt = \int \frac{\sin t}{\cos t} dt$

Let $u = \cos t \Rightarrow du = -\sin t\, dt$

$\Rightarrow \int \tan t\, dt = \int \frac{\sin t}{\cos t} dt = -\int \frac{1}{u} du = -\ln|u| + C = -\ln|\cos t| + C$

9.

$\int \sec t\, dt = \int \sec t \frac{\sec t + \tan t}{\sec t + \tan t} dt = \int \frac{\sec^2 t + \sec t \tan t}{\sec t + \tan t} dt$

Let $u = \tan t + \sec t \Rightarrow du = (\sec^2 t + \sec t \tan t) dt$

$\Rightarrow \int \frac{1}{u} du = \ln|u| + C = \ln|\sec t + \tan t| + C$