Antiderivatives and its applications

5.1 Antiderivatives and Indefinite Integrals

·Antiderivative

If g'(x) = f(x), then we say that g(x) is an antidervative of f(x).

ex. f(x) = 2x $\Longrightarrow$ g(x) = $x^2 + c$ is an antiderivative of f(x) =2x for any constant c.

·Indefinite integral

$\int f(x)dx = g(x) + c$ · The integral of f(x) is g(x)+c

$\longleftrightarrow$ g'(x) = f(x) · The derivative of g(x) is f(x)

ex. $\int 2xdx = x^2 + c$

·Integration Rules

1. $\int x^n dx = \frac{1}{n+1}x^{n+1}$ $\left(\frac{d}{dx}x^n = nx^{n-1}\right)$

2.$\int 1dx = x + c$

3.$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

4.$\int kf(x)dx = k\int f(x)dx$

Example 1

$\int x^3 dx = \frac{1}{3+1}x^{3+1} + c = \frac{1}{4}x^4 + c$

$\downarrow$ $\downarrow$

n=3 $\frac{1}{n+1}x^{n+1}$

Example 2

·$\int \sqrt{x}\, dx =?$

<solution>

$$\int \sqrt{x}\, dx = \int x^{\frac{1}{2}} dx = \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + c = \frac{2}{3}x^{\frac{3}{2}} + c$$

Example 3

$\cdot \int \frac{1}{x^2} dx = ?$

<solution>

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{1}{-2+1} x^{-2+1} = -x^{-1} + c$$

Example 4

$\cdot \int 1 dx = ?$

<solution>

$$\int 1 dx = \int x^0 dx = \frac{1}{0+1} x^{0+1} + c = x + c$$

Example 5

$\cdot \int t^3 dt = ?$

<solution>

$$\int t^3 dt = \frac{1}{3+1} t^{3+1} + c = \frac{1}{4} t^4 + c$$

$\cdot \int u^{-4} du = ?$

<solution>

$$\int u^{-4} du = \frac{1}{-4+1} u^{-4+1} + c = -\frac{1}{3} u^{-3} + c$$

Example 6

$\cdot \int (x^2 + x^3) dx \ = \ ?$

<solution>

$$\int (x^2 + x^3) dx \overset{\text{by 3}}{=} \int x^2 dx + \int x^3 dx = \frac{1}{3} x^3 + \frac{1}{4} x^4 + c$$

Example 7

$\cdot \int 6x^2 dx \ = \ ?$

<solution>

$$\int 6x^2 dx \underset{\text{by 4}}{=} 6 \int x^2 dx = 6 \cdot \frac{1}{3} x^3 + c = 2x^3 + \mathrm{c}$$

Example 8

$\cdot\int 7dx = ?$

<solution>

$\int 7dx = 7\int 1dx = 7x + c$

<u>Example 9</u>

$\cdot\int(6x^2 - 3x^{-2} + 5)dx = ?$

<solution>

$= 6\int x^2dx - 3\int x^{-2}dx + 5\int 1dx$

$= 6\cdot\frac{1}{3}x^3 - 3\cdot\frac{1}{-1}x^{-1} + 5x + \mathrm{c} = 2x^3 + 3x^{-1} + 5x + \mathrm{c}$

$= 2x^3 + 3x^{-1} + 5x + \mathrm{c}$

Example10~11

<u>Example 12</u>

$\cdot\int x^2(x+6)^2dx = \int x^2(x^2+12x+36)dx = \int(x^4+12x^3+36x^2)dx$

$= \frac{1}{5}\mathrm{x}^5 + 3\mathrm{x}^4 + 12\mathrm{x}^3 + \mathrm{c}$

Q : $\int \blacksquare^{10}\ \triangle^{9}\ dx = \ ?$

<u>Example 13</u>

$\mathrm{MC(x)} = 6\sqrt{x}$, and the fixed cost is \$1,000. Find the cost function.

<solution>

$\mathrm{c(x)} = \int MC(x)dx = \int 6\sqrt{x}dx = 6\int x^{\frac{1}{2}}dx = 4x^{\frac{3}{2}} + c$

$\because c(0) = 1000, \therefore c = 1000 \Longrightarrow c(x) = 4x^{\frac{3}{2}} + 1000$

<u>Example 14</u>

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5.2 Integration Using Logarithmic and Exponential Functions

1.$\int e^{ax}dx = \frac{1}{a}e^{ax} + c$ $(\frac{d}{dx}e^{ax} =$

$ae^{ax})$

Example 1.2

$\cdot \int e^{2x}dx = \frac{1}{2}e^{2x} + c$

$\cdot \int e^{\frac{1}{2}x}dx = \frac{1}{1/2}e^{\frac{1}{2}x} + c = 2e^{\frac{1}{2}x} + c$

$\cdot \int 6e^{-3x}dx = 6\int e^{-3x}dx = 6 \cdot \frac{1}{-3}e^{-3x} + c = -2e^{-3x} + c$

$\cdot \int e^{x}dx = 1 \cdot e^{x} + c = e^{x} + c$

Example 3

- Rate of $12e^{a2t}$ new cares per day $f(t) = \int 12e^{0.2t}dt = 12 \cdot \frac{1}{0.2}e^{0.2t} + c =$

  $60e^{0.2t} + c$

  Began with 4 cares $f(0) = 60e^{0.2\cdot 0} + c = 60 + c \Rightarrow c = -56 \Rightarrow f(t) =$ $60e^{0.2t} - 56$
- At t=30, $f(30) = 60e^{0.2(30)} - 56 \approx 24150$

2. $\int \frac{1}{x}dx = \int x^{-1}dx = ln|x| + c$

If $x > 0,$ $\frac{d}{dx}ln|x| = \frac{1}{x}$

If $x < 0,$ lnx is undefined

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$$

$\cdot \Rightarrow \int \frac{1}{x}dx = ln|x| + c$

Example 4.5

$\cdot \int \frac{5}{2x}dx = ?$

<solution>

$\int \frac{5}{2x}dx = \frac{5}{2}\int \frac{1}{x}dx = \frac{5}{2}ln|x| + c$

$\cdot \int (x^{-1} + x^{-2}) dx = ?$

<solution>

$\int (x^{-1} + x^{-2}) dx = \int (x^{-1}) dx + \int (x^{-2}) dx = ln|x| - x^{-1} + c$

Example 6

S(t)= $\int \frac{25}{t} dt$

<solution>

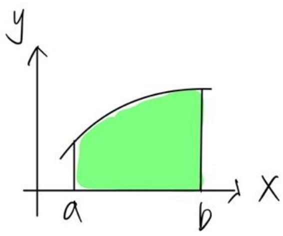
S(t)= $\int \frac{25}{t} dt = 25 \int \frac{1}{t} dt = 25lnx + c$

S(1)= $25ln1 + c = c \Rightarrow c = 0$

S(t)= $25lnt$

at $\mathrm{t} = 12, \mathrm{S}(12) = 25\ln(12) = 62$

5.3 Definite Integrals and Areas



Approximate the area by rectanyles.

Example 1

Approximate the area under $f(x) = x^2$ from 1 to 2 by five rectangles.

$\cdot \triangle\ x = \frac{2-1}{5} = \frac{1}{5} = 0.2$

A= $1^2 \cdot 0.2 + 1.2^2 \cdot 0.2 + 1.4^2 \cdot 0.2 + 1.6^2 \cdot 0.2 + 1.8 \cdot 0.2 = 2.04$

- Area under f from a to b approximated by a left rectangles.

1. Calculate $\triangle x = \frac{b-a}{n}$

2. Find $x_1, x_2 \ldots\ldots x_n$ by successive additions of $\triangle x$ beginning with $x_1 = a$

3. Calculate the sum $f(x_1) \cdot \triangle x + f(x_2) \cdot \triangle x + \cdots + f(x_n) \cdot \triangle x$

Note :

- The sum in step 3 is called a Riemann Sum.
- The limit of the Riemann Sum as $n \to \infty$ gives the area under f(f : nonnegative),

  is called the definite integral of f from a to b written $\int_a^b f(x)dx$

Definite Integral

- Let f be a continuous function on [a,b].

  The definite integral of f from a to b is defined as $\int_a^b f(x)dx = \lim_{n\to\infty}[f(x_1) \triangle x + f(x_2) \triangle x + \cdots + f(x_n) \triangle x]$
- If f is nonnegative on [a,b], then the definite integral gives the area under f from a to b.

  $\lim_{n\to\infty}[f(x_1) \triangle x + f(x_2) \triangle x + \cdots + f(x_n) \triangle x]$

  $= \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j) \triangle x$

  $= F(x)\Big|_a^b = F(b) - F(a)$

  Where F(x) is an antiderivative of f(x)

Example 3

$\int_1^2 x^2 dx = ?$

<solution>

$\int_1^2 x^2 dx = \frac{1}{3}x^3 \Big|_1^2 = \frac{1}{3} \cdot 2^3 - \frac{1}{3} \cdot 1^3 = \frac{7}{3}$

Example 4

Find the area under $y = e^{2x}$ from x=0 to x=1, since e^{2x} is nonnegative and continuous on [0,1]

<solution>

$$\Rightarrow A = \int_0^1 e^{2x}dx = \frac{1}{2}e^{2x}\Big|_0^1 = \frac{1}{2}e^2 - \frac{1}{2}e^0 = \frac{1}{2}e^2 - \frac{1}{2}$$

Example 5

Find the area under $f(x) = \frac{1}{x}$ from x =1 to x = e, since $\frac{1}{x}$ is nonnegative and contionous on [1,e]

<solution>

$$\Rightarrow A = \int_1^e \frac{1}{x}dx = lnx\Big|_1^e = lne - ln1 = 1 - 0 = 1$$

Example 6

Find the area under $y = 24 - 6x^2$ from -1 to 1, since $y = 24 - 6x^2$ is nonnegative on [-1,1] (check)

<solution>

$$\Rightarrow A = \int_{-1}^1 (24 - 6x^2)dx = 24x - 2x^3\Big|_{-1}^1 = (24 - 2) - (-24 + 2) = 44$$

Example 7

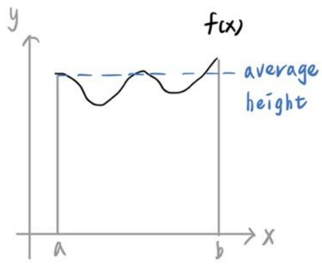
$MC(x) = \frac{75}{\sqrt{x}}$,Total cost of units 100 to 400 ?

<solution>

$$= \int_{100}^{400} \frac{75}{\sqrt{x}}dx = \int_{100}^{400} 75x^{\frac{-1}{2}}dx = 75 \cdot 2 \cdot x^{\frac{1}{2}}\Big|_{100}^{400} = 1500$$

$\Rightarrow$The cost of producing units 100 to 400 is $1500

5.4 Further Applications of Definite Integrals: Average Value and Area Between Curves



- Average value of a function

  Average value of f on [a,b] = $\frac{1}{b-a}\int_a^b f(x)dx$

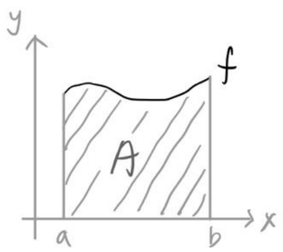
Example 1

Find the average value of $f(x) = \sqrt{x}$ from x=0 to x=9

<solution>

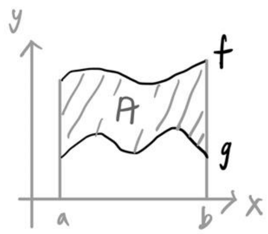
average value =$\frac{1}{9-0}\int_0^9 \sqrt{x}dx = \frac{1}{9}\int_0^9 x^{\frac{1}{2}}\,dx = \frac{1}{9}\cdot\frac{2}{3}x^{\frac{3}{2}}\Big|_0^9 = 2$

In 5.3

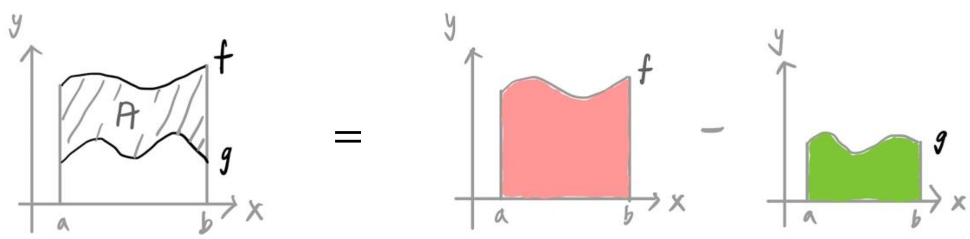


$$A = \int_a^b f(x)dx$$

but the area between curres ?



A= ?



= $\int_a^b f(x)dx$ −

$\int_a^b g(x)dx$

· The area between two contionous curre $f(x) \geq g(x)$ on [a,b] is $\int_a^b [f(x) - g(x)]dx$

Example 3

Find the area between $y = 3x^2 + 4$ and $y = 2x - 1$ from x = -1 to x = 2

$(3x^2 + 4) > (2x - 1)$ on (-1,2] <u>check !</u>

<solution>

$\Rightarrow A = \int_{-1}^{2} [(3x^2 + 4) - (2x - 1)]dx$

$= \int_{-1}^{2} (3x^2 - 2x + 5)dx = x^3 - x^2 + 5x\Big|_{-1}^{2} = 21$

Example 5

Find the area bounded by the curres

$y = 3x^2 - 12$ and $y = 12 - 3x^2$

<solution>

solve :

$3x^2 - 12 = 12 - 3x^2$

$6x^2 - 24 = 0$

$6(x - 2)(x + 2) = 0$

$x = 2, -2$

since :

$12 - 3x^2 > 3x^2 - 12 \qquad on(-2, 2] \qquad (why\ ?)$

$A = \int_{-2}^{2} [(12 - 3x^2) - (3x^2 - 12\ )]dx$

$= \int_{-2}^{2} (24 - 6x^2)dx$

$= 24x - 2x^3\Big|_{-2}^{2} = 64$

5.6 Integration by Substitution

$\frac{df}{dx} = f'$

$\Rightarrow af = f'(x)dx$

· For a differentiable function f(x), the differential af is af = f'(x)dx

Example 1

f(x)	differential df
$f(x) = x^2$	$df = 2xdx$
$f(x) = 10x$	$df = \frac{1}{x}dx$
$f(x) = e^{x^2}$	$df = e^{x^2} \cdot (2x)dx$
$f(x) = x^4 - 5x + 2$	$df = (4x^3 - 5)dx$

· $\frac{d}{dx}\left[F\left(g(x)\right)\right] = F'\left(g(x)\right) \cdot g'(x) = f\left(g(x)\right) \cdot g'(x)$, where $F'(x) = f(x)$

$\Rightarrow \int f\left(g(x)\right) \cdot g'(x)dx = F\left(g(x)\right) + c$

Example 3

$\int (x^2 + 1)^3 \cdot 2xdx = ?$

<solution>

Let $u = x^2 + 1 \Rightarrow du = 2xdx$

$\therefore \int (x^2 + 1)^3 \cdot 2xdx = \int u^3 du = \frac{1}{4}u^4 + c = \frac{1}{4}(x^2 + 1)^4 + c$

Example 4

$\int (x^2 + 1)^3 \cdot xdx = ?$

<solution>

Let $u = x^2 + 1 \Rightarrow du = 2xdx$ $\qquad (xdx = \frac{1}{2}du)$

$\therefore \int (x^2 + 1)^3 \cdot xdx = \int u^3 \cdot \frac{1}{2}du = \frac{1}{2}\int u^3 du = \frac{1}{8}u^4 + c = \frac{1}{8}(x^2 + 1)^4 + c$

Example 5

$\int e^{x^3} \cdot x^2 dx = ?$

<solution>

Let $u = x^3 \Rightarrow du = 3x^2 dx$ $\qquad (x^2 dx = \frac{1}{3}du)$

$\therefore \int e^{x^3} \cdot x^2 dx = \int e^u \cdot \frac{1}{3} du = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + c = \frac{1}{3} e^{x^3} + c$

Example 6

$MC(x) = \frac{x^3}{x^4+1}$, and fixed costs are are $1,000, find the cost function.

<solution>

$C(x) = \int MC(x)dx = \int \frac{x^3}{x^4+1} dx$

Let $u = x^4 + 1 \Rightarrow du = 4x^3 dx$ $\qquad (x^3 dx = \frac{1}{4} du)$

$\therefore \int \frac{x^3}{x^4+1} dx = \int \frac{1}{u} \cdot \frac{1}{4} du = \frac{1}{4} \int \frac{1}{u} du = \frac{1}{4} ln|u| + c = \frac{1}{4} \ln(x^4 + 1) + c$

Since $C(0) = C = 1000 \Rightarrow C(x) = \frac{1}{4} \ln(x^4 + 1) + 1000$

Example 7

$\int \sqrt{x^3 - 3x}(x^2 - 1)dx = ?$

<solution>

Let $u = x^3 - 3x \Rightarrow du = (3x^2 - 3)dx = 3(x^2 - 1)$ $\qquad ((x^2 - 1)dx = \frac{1}{3} du)$

$\Rightarrow \int \sqrt{x^3 - 3x}(x^2 - 1)dx = \int u^{\frac{1}{2}} \cdot \frac{1}{3} du = \frac{2}{9} u^{\frac{3}{2}} + c = \frac{2}{9} (x^3 - 3x)^{\frac{3}{2}} + c$

Example 8

$\int e^{\sqrt{x}} \cdot x^{\frac{-1}{2}} dx = ?$

<solution>

Let $u = x^{\frac{1}{2}} \Rightarrow du = \frac{1}{2} x^{\frac{-1}{2}} dx$ $\qquad (x^{\frac{-1}{2}} dx = 2du)$

$\Rightarrow \int e^{\sqrt{x}} \cdot x^{\frac{-1}{2}} dx = 2 \int e^u du = 2e^u + c = 2e^{\sqrt{x}} + c$

Example 9

$\int_4^5 \frac{1}{3-x} dx =?$

<solution>

Let $u = 3 - x \Rightarrow du = -dx$

$$\Rightarrow \int_4^5 \frac{1}{3-x} dx = -\int_{-1}^{-2} \frac{1}{u} du = -ln|u| \Big|_{-1}^{-2} = -ln2$$