· The Taylor series of f(x) of x=0 is

$$f(x) = f(0) + \frac{f\prime(0)}{1!}x + \frac{f\prime\prime(0)}{2!}x^2 + \frac{f\prime\prime\prime(0)}{3!}x^3 + \cdots + \frac{f''(0)}{n!}x^n + \cdots$$

· The Taylor series of f(x) of x=a is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f''(a)}{3!}x(x-a)^3 + \cdots +$$

$$\frac{f''(a)}{n!}x(x-a)^n + \cdots$$

Example 2

Find the Taylor series at x=0 for e^x, and find its interval of convergence.

<solution>

$f(x) = e^x$

$f'(x) = f''(x) = f'''(x) = \cdots = e^x$

$f(0) = f'(0) = f''(0) = \cdots = e^0 = 1$

$$\Rightarrow e^x = f(0) \quad + \frac{f\prime(0)}{1!}x + \frac{f\prime\prime(0)}{2!}x^2 + \frac{f\prime\prime\prime(0)}{3!}x^3 + \cdots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$r = \lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1$$

The radins of convergence is R=∞

The interval of convergence is $-\infty < x < \infty$

Note :

$$e = 1 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \cdots$$

Example 3

Find the Taylor series at x=0 for $\sin x$

<solution>

$f(x) = sin\, x$ $\qquad f(0) =$
0

$f'(x) = cos\, x$ $\qquad f'(0) = 1$

$$f''(x) = -\sin x \qquad \Rightarrow \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$$

$$\Rightarrow \sin x = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}$$

$$r = \lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1} \cdot x^{2n+3}}{(2n+3)!}}{\frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty} \left|\frac{x^2}{(2n+2)(2n+3)}\right| = 0 < 1$$

The radins of convergence is R=∞

The interval of convergence is $-\infty < x < \infty$

Example 4

$$\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad , -\infty < x < \infty$$

$$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad , -\infty < x < \infty$$

H.W Find the Taylor series at x=0 for $\cos x$

Practice problem

$$\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad , -\infty < x < \infty$$

$$\Rightarrow \cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots \qquad , -\infty < x < \infty$$

Example 5

$$\int_0^x \frac{\sin t}{t} dt = \ ?$$

<solution>

$$\because \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

$$\Rightarrow \frac{\sin t}{t} = \frac{1}{t}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$$

$$\Rightarrow \int_0^x \frac{\sin t}{t} dt = \int_0^x (1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots) dt$$

$$= \left. t - \frac{t^3}{3\cdot3!} + \frac{t^5}{5\cdot5!} - \frac{t^7}{7\cdot7!} + \cdot\!\cdot\right|_0^x$$

$$= x - \frac{x^3}{3\cdot3!} + \frac{x^5}{5\cdot5!} - \frac{x^7}{7\cdot7!} + \cdots$$

$$\Rightarrow \int_0^2 \frac{sint}{t}dt \approx 2 \quad - \frac{2^3}{3\cdot3!} + \frac{2^5}{5\cdot5!} - \frac{2^7}{7\cdot7!} + \cdots \approx 1.605$$

Example 6

Find the Taylor series at x=1 for lnx

<solution>

$f(x) = ln\,x$ $f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\Rightarrow$ $f'(1) = 1$

$f''(x) = -x^{-2}$ $f''^{(1)} = -1$

$f'''(x) = 2x^{-3}$ $f'''(x) = 2$

. .

. .

$$\Rightarrow lnx = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f''(1)}{3!}x(x-1)^3 + \cdots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

$$r = \lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^n}{(n+1)}(x-1)^{n+1}}{\frac{(-1)^n}{n}(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{n(x-1)}{n+1}\right| = |x-1| < 1$$

$\Rightarrow 0 < x < 2$

The radins of convergence is R=1

The interval of convergence is $0 < x < 2$

Example 10.3

3. $\frac{x}{1\cdot2} + \frac{x^2}{2\cdot3} + \frac{x^3}{3\cdot4} + \cdots = ?$

<solution>

$C_n = \frac{x^n}{n(n+1)}$, $C_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$

$$\frac{C_{n+1}}{C_n}=\frac{\frac{x^{n+1}}{(n+1)(n+2)}}{\frac{x^n}{n(n+1)}}=\frac{x^{n+1}}{(n+1)(n+2)}\cdot\frac{n(n+1)}{x^n}=\frac{n}{n+2}x$$

$$r=\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|=\lim_{n\to\infty}\left|\frac{n}{n+2}\cdot x\right|=|x|<1$$

The radins of convergence is R=1

The interval of convergence is $-1<x<1$

9. $\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$ =?

<solution>

$$C_n=\frac{x^{2n}}{n!}\quad ,C_{n+1}=\frac{x^{2(n+1)}}{(n+1)!}$$

$$\frac{C_{n+1}}{C_n}=\frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}}=\frac{x^2}{n+1}$$

$$r=\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|=\lim_{n\to\infty}\left|\frac{x^2}{n+1}\right|=0$$

The radins of convergence is R=∞

The interval of convergence is $-\infty<x<\infty$

17. $f(x)=e^{\frac{x}{5}}$

(a) $f(x)=e^{\frac{x}{5}}$

$$f'(x)=\frac{1}{5}e^{\frac{x}{5}}$$

$$f''(x)=\frac{1}{25}e^{\frac{x}{5}}$$

.

.

The Taylor series is :

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

$$=1+\frac{1}{5}x+\frac{1}{5^2\cdot 2!}x^2+\frac{1}{5^3\cdot 3!}x^3+\cdots$$

(b) The Taylor series at x=0 for e^x

$$1+\frac{x}{1!}+\frac{x^2}{1!}+\frac{x^3}{1!}+\cdots$$

Substituting $\frac{x}{5}$ for x

$$\Rightarrow 1+\frac{x}{5}+\frac{1}{2!}\left(\frac{x^2}{5}\right)+\frac{1}{3!}\left(\frac{x^3}{5}\right)+\cdots$$

$$=1+\frac{1}{5}x+\frac{1}{5^2 2!}x^2+\frac{1}{5^2 3!}x^3+\cdots$$

21. Taylor series at x=0 for $\frac{1}{1-x}$ is

$$1+x+x^2+x^3+\cdots \qquad |x|<1$$

multiply the Taylor series for $\frac{1}{1-x}$ by x^2

$\Rightarrow$ The Taylor series at x=0 for $\frac{x^2}{1-x}$ is

$$x^2+x^3+x^4+x^5+\cdots$$

27. $sinx = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$

$$-sinx=-x+\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\cdots$$

$$x-sinx=\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\cdots$$

$$\Rightarrow \frac{x-sinx}{x^3}=\frac{1}{3!}-\frac{x^2}{5!}+\frac{x^4}{7!}-\cdots$$

29. $\frac{1}{1+x}=1-x+x^2-x^3+\cdots$ for $|x|<1$

$$\int\frac{1}{1+x}dx=\int(1-x+x^2-x^3+\cdots)dx$$

$$\Rightarrow \ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$$

33. $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

$$e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots$$

$\Rightarrow \frac{e^x+e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

45.

(a) $S_n - S_{n-1} = (C_1 + C_2 + \cdots + C_n) - (C_1 + C_2 + \cdots + C_{n-1}) = C_n$

(b) $\lim_{n\to\infty} S_n = a$

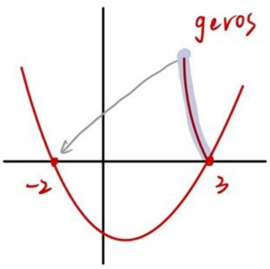
$\Rightarrow \lim_{n\to\infty} C_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = a - a = 0$

10.4 Newton's method

Example 1

Find the geros of $f(x) = x^2 - x - 6$

<solution>



$f(x) = (x-3)(x+2) = 0$

$\Rightarrow$ The geros are $\begin{cases} x = 3 \\ x = -2 \end{cases}$

Newton's method is a procedure for approximating the solutions in case the the exact solutions are difficient or impossible to find.

Given an initial approximation $x = x_0$, instand of solving $f(x) = 0$, we set the Taylor polynomial at $x = x_0$ equal to gero

$f(x_0) + f'(x_0)(x - x_0) = 0$

$\Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$

$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $\qquad f(x_0) \neq 0$

The sample formula can then be applied to the "better" approximate

$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $\qquad f'(x_1) \neq 0$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ $\qquad f'(x_2) \neq 0$

. .

. .

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $\qquad f'(x_n) \neq 0$

Newton's method

To approximate a solution to $f(x) = 0$, choose an intial approximate x_0, and caculate $x_1, x_2, x_3 \dots$ converge, the converge to a solution of $f(x) = 0$

Example 2

Approximate $\sqrt{2}$ by using three iterations of Newton's method.

<solution>

$f(x) = x^2 - 2$ $\qquad \Rightarrow f'(x) = 2x$

Take $x_0 = 1$

$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1^2-2}{2\cdot 1} = 1.5$

$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{f(1.5)}{f'(1.5)} \approx 1.4167$

$\Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(2)} = 1.4167 - \frac{f(1.4167)}{f'(1.4167)} \approx 1.4142$

$\therefore \sqrt{2} \approx 1.4142$

Example 3

Approximate the solution to $e^x = 2 - 2x$, continuing until two successive iterations agree to nine decimal place.

<solution>

$f(x) = e^x - 2 + 2x$ $\qquad \Rightarrow f'(x) = e^x + 2$

Take $x_0 = 0$

$\Rightarrow x_1 = 0 - \frac{e^0 - 2 + 2(0)}{e^0 + 2} = 0.333333333$

$\Rightarrow x_2 = 0.333333333 - \frac{e^{0.333333333} - 2 + 2(0.333333333)}{e^{0.333333333} + 2} \approx 0.3149922850$

$\Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.3149230588$

$$\Rightarrow x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.3149230578$$

$$\Rightarrow x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} \approx 0.3149230578$$

$$\therefore x \approx 0.3149230578$$