STAT443 Assignment 2.

Question1.

$$\phi(B)(1-B)^d X_t = \theta(B) Z_t \quad \text{for } \{Z_t\} \sim WN(0,\sigma^2)$$

$W_t = X_t + A_0 + A_1 t + \cdots + A_{d-1} \cdot t^{d-1}$, where $A_0, A_1, \ldots, A_{d-1}$ are random variables.

We want to prove that: $\phi(B)\cdot(1-B)^d W_t = \theta(B) Z_t$

$$\iff \phi(B)(1-B)^d \cdot (X_t + A_0 + A_1 t + \cdots + A_{d-1} \cdot t^{d-1}) = \theta(B) \cdot Z_t$$

$$\iff \phi(B)\cdot(1-B)^d X_t + \phi(B)\cdot(1-B)^d \cdot (A_0 + \cdots + A_{d-1} \cdot t^{d-1}) = \theta(B) \cdot Z_t$$

$$\iff \cancel{\theta(B) \cdot Z_t} + \phi(B)\cdot(1-B)^d \cdot (A_0 + \cdots + A_{d-1} \cdot t^{d-1}) = \cancel{\theta(B) \cdot Z_t}$$

$$\iff \phi(B)(1-B)^d \cdot (A_0 + \cdots + A_{d-1} t^{d-1}) = 0$$

We want to prove that $(1-B)^d (A_0 + \cdots + A_{d-1} \cdot t^{d-1}) = 0$, which is $\nabla^d (A_0 + \cdots + A_{d-1} \cdot t^{d-1}) = 0$, and $A_0, A_1, \ldots, A_{d-1}$ are arbitary.

Induction Method:

Step1: When $d=1$, $\nabla^1 A_0 = A_0 - A_0 = 0$

Step2: Assume: for $d=k$, $\nabla^k (A_0 + \cdots + A_{k-1} t^{k-1}) = 0$ is always true for arbitary $A_0, A_1, \ldots, A_{k-1}$

Step3: let $d = k+1$

$$\nabla^{k+1}(A_0 + A_1 t + \cdots + A_k t^k) = \nabla^k \left(\nabla (A_0 + A_1 t + \cdots + A_k t^k) \right)$$

$$= \nabla^k \left[(A_0 + A_1 t + \cdots + A_k t^k) - (A_0 + A_1(t-1) + \cdots + A_k (t-1)^k) \right]$$

Notice that $(A_0 + A_1(t-1) + \cdots + A_k(t-1)^k)$ can be written as a k-order polynomial, which is $(B_0 + B_1 t + \cdots + B_{k-1} t^{k-1} + A_k t^k)$ where $B_0, B_1, \ldots, B_{k-1}$ are arbitary since $A_0, A_1, \ldots, A_k$ are arbitary.

So $\nabla^k \left[(A_0 + A_1 t + \cdots + A_{k-1} t^{k-1} + A_k t^k) - (A_0 + A_1(t-1) + \cdots + A_k(t-1)^k) \right]$

$$= \nabla^k \left[(A_0 + A_1 t + \cdots + A_{k-1} t^{k-1} + \cancel{A_k t^k}) - (B_0 + B_1 t + \cdots + B_{k-1} t^{k-1} + \cancel{A_k t^k}) \right]$$

$= \nabla^k \left[(A_0 - B_0) + (A_1 - B_1) t + \cdots + (A_{k-1} - B_{k-1}) t^{k-1} \right]$ where $(A_0 - B_0), (A_1 - B_1), \ldots, (A_{k-1} - B_{k-1})$ are arbitary since $A_0, A_1, \ldots, A_{k-1}$ and $B_0, B_1, \ldots, B_{k-1}$ are arbitary.

$= \nabla^k (C_0 + C_1 t + \cdots + C_{k-1} \cdot t^{k-1})$ We use $C_0 = A_0 - B_0$, $C_1 = A_1 - B_1$, $\ldots$, $C_{k-1} = A_{k-1} - B_{k-1}$ and $C_0, C_1, \ldots, C_{k-1}$ are arbitary.

$= 0$ based on the assumption that $\nabla^k (A_0 + \cdots + A_{k-1} t^{k-1}) = 0$ for arbitary $A_0, A_1, \ldots, A_{k-1}$

Step4: Therefore, we proved that $\nabla^d (A_0 + \cdots + A_{d-1} t^{d-1})$ for $\forall d \geq 0$

Since $\nabla^d (A_0 + \cdots + A_{d-1} t^{d-1}) = (1-B)^d (A_0 + \cdots + A_{d-1} t^{d-1}) = 0$

so $\phi(B) \cdot (1-B)^d (A_0 + \cdots + A_{d-1} t^{d-1}) = 0$

so $\phi(B) \cdot (1-B)^d W_t = \theta(B) \cdot Z_t$ as mentioned at beginning of the proof.

Therefore, $\phi(B) \cdot (1-B)^d W_t = \theta(B) Z_t$ proved.

Question 2.

$$C_{ij} = \sum_{k=-\infty}^{\infty} \left(\rho_{k+i}\rho_{k+j} + \rho_{k-i}\cdot\rho_{k+j} - 2\rho_i \rho_k\cdot \rho_{k+j} - 2\cdot\rho_j\cdot\rho_k \rho_{k+i} + 2\rho_i\cdot\rho_j\cdot\rho_k^2 \right)$$

For MA(1), we known that $\rho_k = \begin{cases} 1, & k=0 \\ \frac{-\theta}{1+\theta^2}, & k=1 \\ 0, & k>1 \end{cases}$

We found that if $\rho_{k+i}\cdot\rho_{k+j}$ not zero, then $|j-i| \le 2$; if $\rho_{k-i}\cdot\rho_{k+j}$ not zero, then $|j+i| \le 2$; if $\rho_i\cdot\rho_k\cdot\rho_{k+j}$ not zero, then $i \le 1$,

if $\rho_j\cdot\rho_k\cdot\rho_{k+i}$ not zero, then $j \le 1$; if $\rho_i\cdot\rho_j\cdot\rho_k^2$ not zero, then $i \le 1, j \le 1$.

Case 1: $i=j=1$

$$C_{11} = \sum_{k=-\infty}^{\infty} \left(\rho_{k+1}^2 + \rho_{k-1}\cdot\rho_{k+1} - 2\cdot\rho_1\cdot\rho_k\cdot\rho_{k+1} - 2\cdot\rho_1\cdot\rho_k\cdot\rho_{k+1} + 2\cdot\rho_1^2\cdot\rho_k^2 \right)$$

$$= (\rho_{-1}^2 + \rho_0^2 + \rho_1^2) + (\rho_{-1}\cdot\rho_1) - 4\cdot(\rho_1\cdot\rho_{-1}\cdot\rho_0 + \rho_1\cdot\rho_0\cdot\rho_1) + 2\cdot\rho_1^2\,(\rho_{-1}^2 + \rho_0^2 + \rho_1^2)$$

$$= (1+2\rho_1^2) + \rho_1^2 - 8\cdot\rho_1^2 + 2\rho_1^2\cdot(1+2\cdot\rho_1^2)$$

$$= 1 + 3\rho_1^2 - 8\rho_1^2 + 2\rho_1^2 + 4\rho_1^4 = 1 - 3\rho_1^2 + 4\rho_1^4$$

Case 2: $i=1, j=2$

$$C_{12} = (\rho_{-1}\cdot\rho_0 + \rho_0\cdot\rho_1) + 0 - 2\cdot\rho_1\rho_{-1}\cdot\rho_{-1+2}$$

$$= (\rho_1 + \rho_1) - 2\cdot\rho_1^3$$

$$= 2\cdot\rho_1(1-\rho_1^2)$$

Case 3: $i=1, j=3$

$$C_{13} = \rho_{-1}\rho_1 + 0 - 0 - 0 + 0$$

$$= \rho_1^2$$

Case 4: $i>1$

4.1: $i=j>1$

$$C_{ii} = \sum_{k=-\infty}^{\infty} \left(\rho_{k+i}^2 + \rho_{k-i}\cdot\rho_{k+i} - \cancel{2\rho_i\cdot\rho_k\cdot\rho_{k+i}} \xrightarrow{} 0 - \cancel{2\rho_i\cdot\rho_k\cdot\rho_{k+i}} \xrightarrow{} 0 + \cancel{2\rho_i^2\cdot\rho_k^2} \xrightarrow{} 0 \right)$$

$$= (\rho_{-1}^2 + \rho_0^2 + \rho_1^2) + 0 + 0 + 0 = 1 + 2\rho_1^2$$

4.2: $i>1, j=i+1$

$$C_{i,i+1} = \rho_{-1}\cdot\rho_0 + \rho_0\cdot\rho_1 + 0 = \rho_{-1}\cdot(1) + \rho_1\cdot(1) = 2\cdot\rho_1$$

4.3: $i>1,\ j=i+2$

$$C_{i,i+2} = \rho_{-1}\cdot\rho_1 + 0 + 0 = \rho_1^2$$

Case 5: Other cases.

$$C_{ii} = 0 + 0 - 0 - 0 + 0 = 0$$

In conclusion:

$$C_{ij} = \begin{cases} 1-3\rho_1^2+4\rho_1^4 & (i=j=1) \\ 2\rho_1 - 2\rho_1^3 & (i=1,\ j=2) \\ \rho_1^2 & (j=i+2,\ i\geq 1) \\ 1+2\rho_1^2 & (i=j>1) \\ 2\rho_1 & (j=i+1,\ i>1) \\ 0 & (\text{other cases}) \end{cases}$$