Probability of Scoring in Billiards

Research Question: How does the mathematics in physics help in understanding the probability of scoring in pocket billiards?

Mathematics

Extended Essay

3998 Words

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Introduction

What does billiards have to do with mathematics?

Billiards is a “cue” sport that requires “players” to utilize and “score” a number of “coloured” balls for the goal of reaching an appropriate winning condition (Government). Though it may not seem as much while playing, one must have significant amounts of knowledge on physical mathematics in order to answer the research question: How does the mathematics in physics help in understanding the probability of scoring in pocket billiards? Collisions itself requires the understanding of vectors and trigonometry to not only calculate the trajectory of a billiard ball after its collision with a cue ball, but also for determining certain conditions at which the cue ball and the billiard ball collides. These conditions are looked upon as certain probabilities that can be calculated mathematically by integrating probability density functions.

Why is this even important?

In the end, any sport in general is a game of probability; one wins, while the other does not, or it may even result in a tie. A greater level of skill in a sport only increases the chances of winning. This is prevalent in sports such as baseball, where “sabermetrics” is used to represent each player’s “performance” in recent games as “statistical” numbers, utilizing this information to increase the chances of winning (Birnbaum). Pocket billiard also acts similarly in a sense that players assess the probability of scoring different billiard balls in a specific situation in order to gain the advantage against their opponents. However, intuition is a subjective way of understanding the difficulty of scoring a specific billiard ball. For inexperienced players or watchers, it is not easy to fathom how difficult scoring is. By representing these choices of scoring with probability, it can not only allow players to objectively compare the options of scoring, but it can also represent the difficulty of scoring in a certain way, increasing the excitement of this sport when a player does manage to beat the odds and score.

The Approach

The setting

In this approach, there will be several calculations regarding the collisions between the cue ball and the billiard ball. It is clear that there are several physics concepts that can be considered just involving the collisions between two spheres. Unfortunately, this will overcomplicate calculations, leading to conclusions unrelated to answering the research question. Therefore, there are a few assumptions needed to take into consideration while taking the approach:

1. The collisions between the balls are "elastic", in which the "energy" is "transferred" only in terms of the "energy" of "its motion" (Homer and Bowen-Jones 66, 76-77).
2. Forces such as friction will not be considered during calculations.
3. All balls possess identical radii.
4. The billiard ball is considered to be scored after half of it travels over the pocket.

The diagram below shows the model that the approach will be based off of in order to calculate the probability of scoring (see Figure 1).

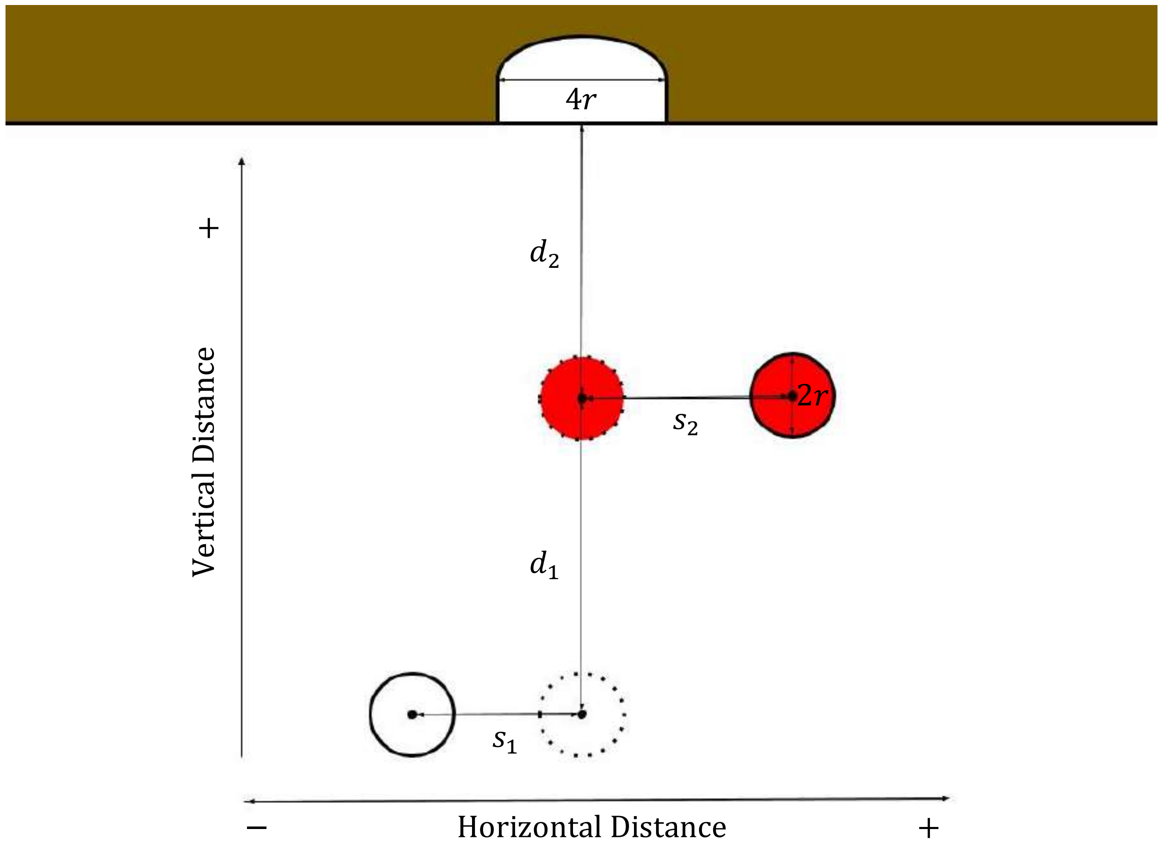

Figure 1: Model for finding the probability with the factor of position

$$d_1 = vertical\ distance\ from\ the\ centers\ of\ the\ cue\ ball\ to\ the\ billiard\ ball$$

$$d_2 = vertical\ distance\ from\ the\ centers\ of\ the\ billiard\ ball\ to\ the\ pocket$$

$$r = radius\ of\ the\ balls$$

$$s_1 = horizontal\ displacement\ from\ the\ center\ of\ the\ cue\ ball$$

$$s_2 = horizontal\ displacement\ from\ the\ center\ of\ the\ billiard\ ball$$

During this approach, we will be dealing with a lot of angles, since it determines the trajectory at which the cue ball or the billiard ball is headed. Therefore, it is important to instruct how angles will be defined in this process. When defining angles of the trajectory, they will be treated similar to how bearings are defined (see Figure 2).

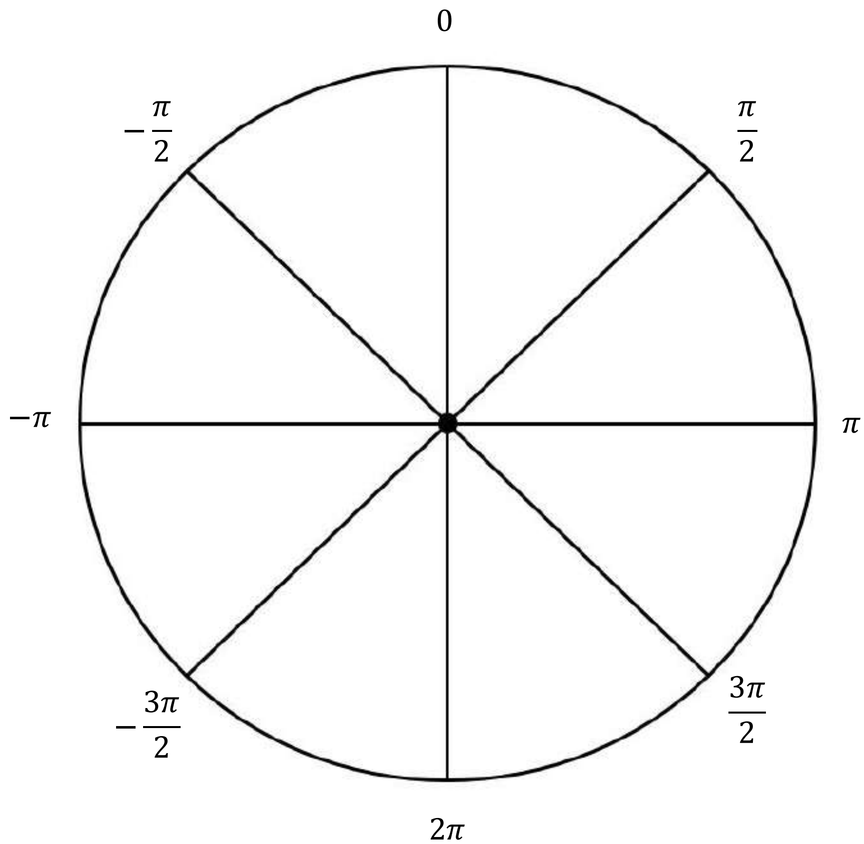

Figure 2: Angles of the trajectory of the cue ball represented similarly to bearings

The methodology

In this methodology, the formula for calculating the probability of scoring will not be formulated using a single probability. This is because there are two conditions that must be satisfied in order to score:

1. An appropriate collision between the cue ball and the billiard ball.
2. The billiard ball entering the pocket after the collision.

This can be understood fully through the use of a tree diagram (see Figure 3).

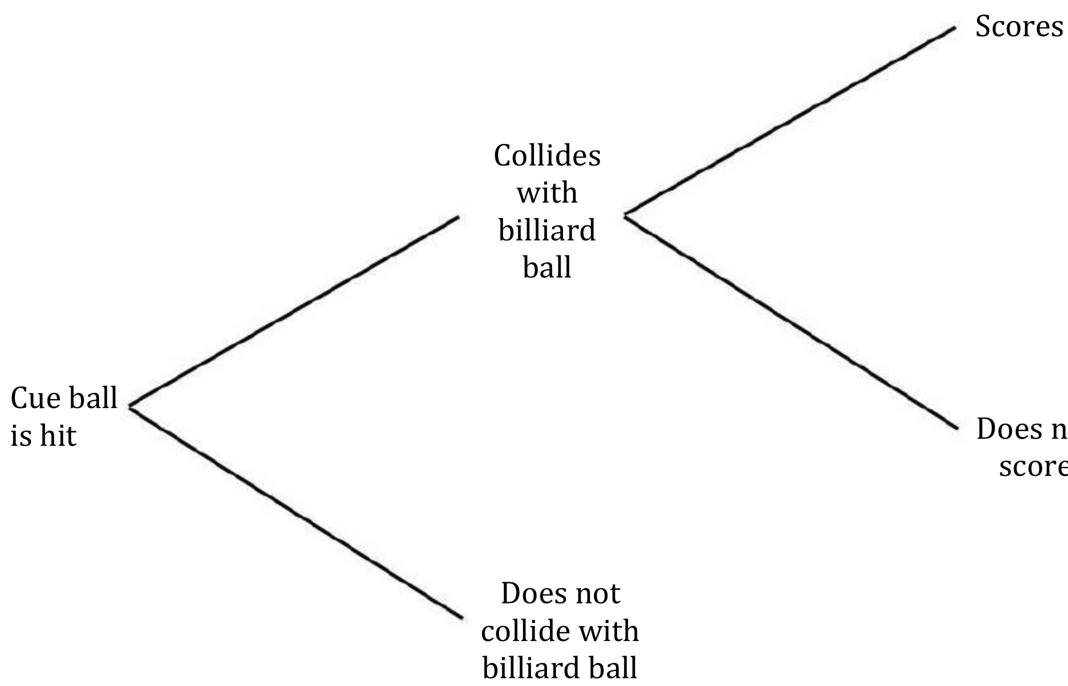

Figure 3: Tree diagram for the probability of scoring the billiard ball

Each condition will be evaluated by using the understanding of two-body collisions to find a range of angles at which a condition is satisfied. Then, the range of angles that satisfies the condition can be expressed as a probability density function relative to the player's skill level. The product of the two expressions that express the probability of each condition will be found to end with a formula that calculates the probability of scoring.

Why this methodology?

A probability density function is a statistical method used to calculate the probability of outcomes from negative to positive infinity (Harcet et al. 522). However, why is it appropriate to use probability density functions when answering the research question? Think of it this way: in a game of pocket billiards, no matter how accurate a player tries to be when shooting the cue ball, each consecutive shot will not collide with the same exact angle in every single attempt. This would mean that the cue ball can be shot by the player at any angle or an infinite number of angles no matter how precise one tries to be. Therefore, the probability must have a "continuous random variable"—in which the "values" are "uncountable"—making the methodology appropriate for answering the research question (Harcet et al. 520-521).

Trajectories that Allow the Collision to Occur

Let's first ask address the question: what are the factors that determine whether the cue ball collides with the billiard? So far, two factors are thought of: force applied on the cue ball and the angle at which this force is applied. Due to the assumption of neglecting forces, the only factor that affects the collision in this problem is the angle. However, how can the angle of the trajectory of the cue ball determine whether it collides with the billiard ball or not? In simple terms, collisions are the "interaction" between bodies at which "momentum transfers" from the center of mass from one to another as their radii overlap (Homer and Bowen-Jones 75). When two bodies collide, the vector can be divided into two directions relative to the "collision plane": the "normal component" and the "tangential component" (Townsend).

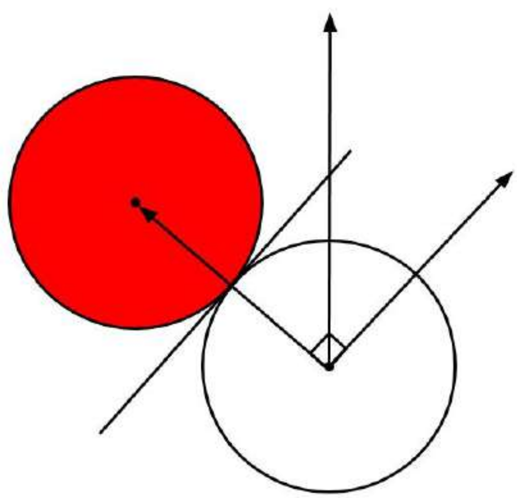

Figure 4: Interaction of the vector between two bodies

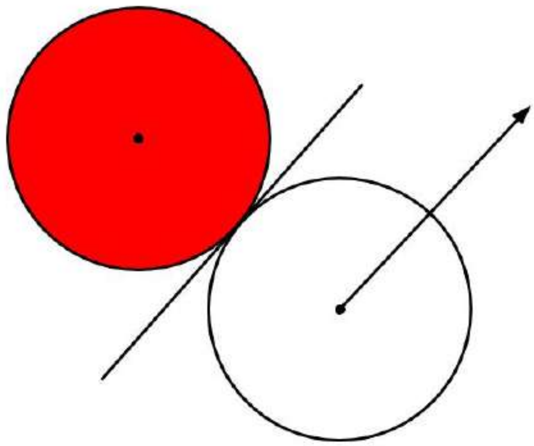

Figure 5: Initial vector component shown to be parallel to the collision plane

In short, the collision would not occur if the initial direction of the vector were parallel to the collision plane since it would not be divided into two different components (see Figure 5). Using this, the diagram for when the cue ball just misses the billiard ball can be constructed (see Figure 6).

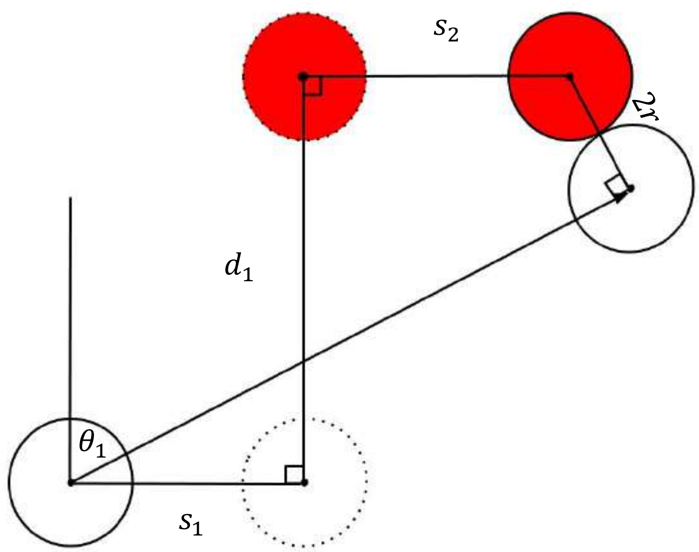

Figure 6: Collision does not occur between the cue ball and the billiard ball

We can now understand that any trajectory less than angle (θ_1) would allow for the collision between the cue ball and billiard ball to occur (see Figure 6). Using the understanding of trigonometry, the diagram can be reconstructed to provide a visual aid in evaluating angle (θ_1) (see Figure 7).

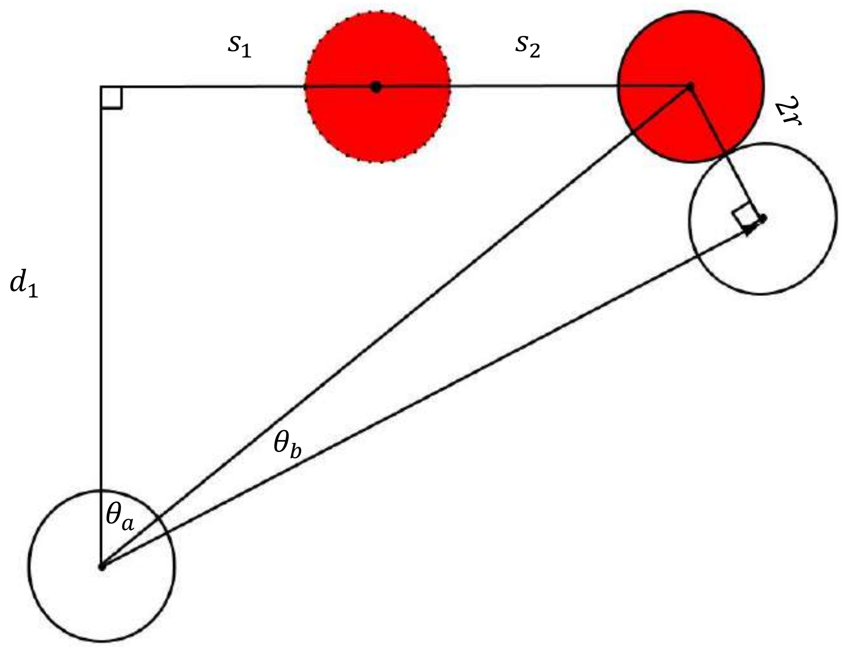

Figure 7: Reconstructed diagram for calculating angle (θ_1)

As shown in the diagram above, angle (θ_1) is divided into two different angles: angle (θ_a) and angle (θ_b). Through this we can say that:

$$\theta_1 = \theta_a + \theta_b$$

Angle (θ_a) and angle (θ_b) will be calculated separately, in which the sum of the two will be calculated to find angle (θ_1). This is because both angle (θ_a) and angle (θ_b) are acute angles of right-angled triangles, which simplifies the calculation of angle (θ_1) through different trigonometric formulae and ratios. The calculation for angle (θ_a) is shown below:

$$\tan\theta = \frac{opp}{adj}$$

Since in this situation, the horizontal translation of the cue ball—s_2—is negative relative to the original position of the billiard ball, and the horizontal translation of the billiard ball—s_1—is positive relative to its original position, a trigonometric ratio can be used to calculate angle (θ_a) (see Figure 7).

$$\tan\theta_a = \frac{s_2 - s_1}{d_1}$$

$$\theta_a = \arctan\frac{s_2 - s_1}{d_1}$$

Similarly, angle (θ_b) can be calculated using another trigonometric ratio since angle (θ_b) is also part of a right triangle:

$$\sin\theta = \frac{opp}{hyp}$$

$$\sin\theta_b = \frac{2r}{hyp}$$

We can understand that the hypotenuse of the triangle with angle (θ_b) and the hypotenuse of the triangle with angle (θ_a) are equal (see Figure 7). By using the base and height of the triangle with angle (θ_a), the hypotenuse of the triangle with angle (θ_b) can be calculated using the Pythagorean theorem:

$$hyp = \sqrt{(s_2 - s_1)^2 + (d_1)^2}$$

This hypotenuse is substituted back into the ratio for solving angle (θ_b):

$$\sin\theta_b = \frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

$$\theta_b = \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

Since it was established that:

$$\theta_1 = \theta_a + \theta_b$$

By substituting the expressions for angle (θ_a) and angle (θ_b), we can say that:

$$\theta_1 = \arctan\frac{s_2 - s_1}{d_1} + \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

When finding the probability of an outcome, there must be a certain range of an outcome for it to be probable. In this specific case, angle (θ_1) is one of the ranges that allow scoring the billiard ball to be probable. Therefore, the range of this angle to allow the collision to occur is defined. Since the calculation to evaluate angle (θ_1) is at its maximum, it can be said that:

$$\theta_{1_{max}} = \arctan\frac{s_2 - s_1}{d_1} + \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

To show a range of trajectories, we also need to consider the scenario where the cue ball is shot to the opposite side of the billiard ball (see Figure 8).

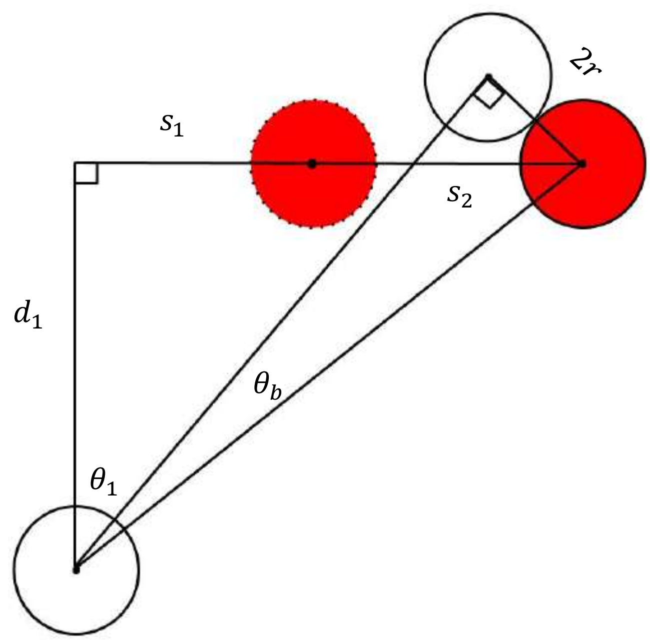

Figure 8: Another angle of (θ_1) where the collision does not occur between the cue ball and the billiard ball

The diagram above shows another angle (θ_1) at which the cue ball barely misses the billiard ball. Since the only changes made in this scenario is to move the cue ball to the opposite side of the billiard ball, angle (θ_a) can be defined (see Figure 8).

$$\theta_a = \theta_1 + \theta_b$$

Otherwise, angle (θ_1) can be defined as:

$$\theta_1 = \theta_a - \theta_b$$

Since it was calculated in the previous scenario that:

$$\theta_a = \arctan\frac{s_2 - s_1}{d_1}$$

$$\theta_b = \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

We can substitute each angle into the equation for angle (θ_1):

$$\theta_1 = \arctan\frac{s_2 - s_1}{d_1} - \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

Since angle (θ_1) in this scenario is the minimum angle at which the collision does not occur, it can be said that for the collision to occur:

$$\theta_{1_{min}} = \arctan\frac{s_2 - s_1}{d_1} - \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

Therefore, the range of angle (θ_1) can be evaluated as:

$$\arctan\frac{s_2 - s_1}{d_1} - \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}} < \theta_1$$

$$< \arctan\frac{s_2 - s_1}{d_1} + \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

Trajectories that Allow the Billiard Ball to Enter the Pocket

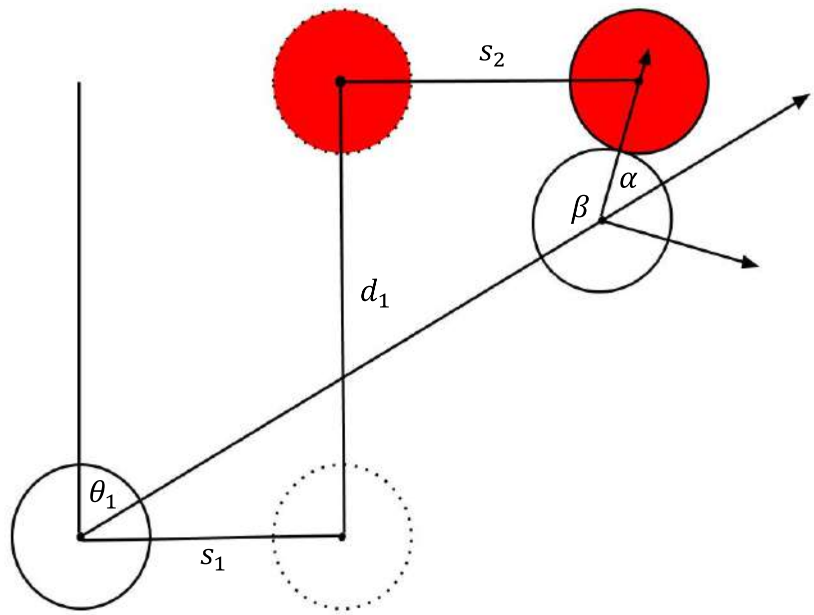

Figure 9: Collision vectors constructed between the cue ball and the billiard ball as it collides

Now that the condition for the collision between the cue ball and the billiard ball is evalauted, the angle of the billiard ball after the collision—angle (θ_2)—can be calculated in terms of the angle (θ_1). Let's first assume that the cue ball collides with the billiard ball. Two conditions are already established to help in calculating the formula for angle (θ_2) (see Figure 9):

$$\theta_2 = \theta_1 - \alpha$$

$$\alpha + \beta = \pi$$

Similar to when finding the range of angle (θ_1), the angles will be calculated using trigonometry:

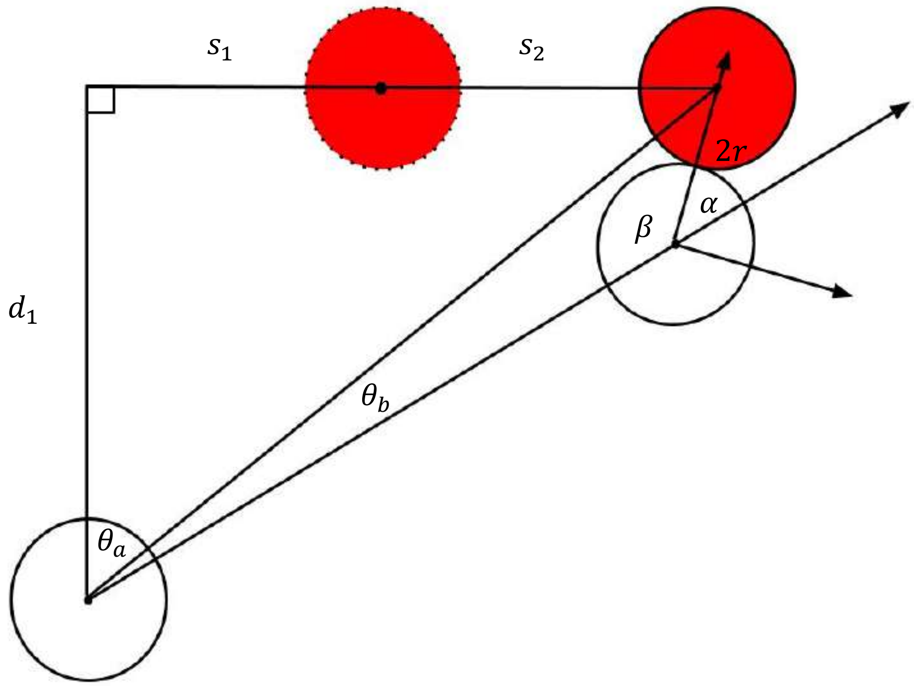

Figure 10: Reconstructed diagram for calculating the range of angle (θ_2)

Using ths sine rule, angle (β) can be evaluated inside the triangle (see Figure 10):

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \beta}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}} = \frac{\sin \theta_b}{2r}$$

Angle (β) can be evaluated through simplification and inverse trigonometry:

$$\beta = \arcsin\left[\frac{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}{2r} \sin \theta_b\right]$$

Since we know that (see Figure 10):

$$\theta_1 = \theta_a + \theta_b$$

$$\theta_b = \theta_1 - \theta_a$$

The formula for angle (β) can be rewritten through the substitution of angle (θ_b):

$$\beta = \arcsin\left[\frac{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}{2r}\sin(\theta_1 - \theta_a)\right]$$

Using the appropriate trigonometric ratio, a formula for angle (θ_a) is formulated:

$$\tan\theta_a = \frac{s_2 - s_1}{d_1}$$

$$\theta_a = \arctan\frac{s_2 - s_1}{d_1}$$

By substituting the equation for angle (θ_a), we get:

$$\beta = \arcsin\left[\frac{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}{2r}\sin\left(\theta_1 - \arctan\frac{s_2 - s_1}{d_1}\right)\right]$$

By rearranging the equation of each condition required for the collisions to occur, the formula for angle (θ_2) is formulated using the following relationship (see Figure 9):

$$\alpha = \pi - \beta$$

The value of angle (α) is substituted into the other condition for the collision (see Figure 9):

$$\theta_2 = \theta_1 - \alpha$$

$$\theta_2 = \theta_1 - \pi + \beta$$

Finally, the formula for angle (β) is substituted to lead to a formula for solving the minimum possible angle of angle (θ_2):

$$\theta_{2_{min}} = \theta_1 - \pi + \arcsin\left[\frac{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}{2r}\sin\left(\theta_1 - \arctan\frac{s_2 - s_1}{d_1}\right)\right]$$

We can then use the same information to evaluate the maximum possible angle of angle (θ_2). This can be done in a similar fashion as to when finding the range of angle (θ_1). For that reason, for the scenario where angle (θ_2) is at its maximum, we know that:

$$\theta_1 = \theta_a - \theta_b$$

Or otherwise:

$$\theta_b = \theta_a - \theta_1$$

Therefore, in this case, angle (β) can be written as:

$$\beta = \arcsin\left[\frac{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}{2r}\sin(\theta_a - \theta_1)\right]$$

By substituting the equation of angle (θ_a), we get

$$\beta = \arcsin\left[\frac{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}{2r}\sin\left(\arctan\frac{s_2 - s_1}{d_1} - \theta_1\right)\right]$$

Thus, the maximum possible angle of angle (θ_2) can be shown as:

$$\theta_{2_{max}} = \theta_1 - \pi + \arcsin\left[\frac{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}{2r}\sin\left(\arctan\frac{s_2 - s_1}{d_1} - \theta_1\right)\right]$$

Therefore, the range of angle (θ_2) is evaluated:

$$\theta_1 - \pi + \arcsin\left[\frac{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}{2r}\sin\left(\theta_1 - \arctan\frac{s_2 - s_1}{d_1}\right)\right] \leq \theta_2$$

$$\leq \theta_1 - \pi + \arcsin\left[\frac{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}{2r}\sin\left(\arctan\frac{s_2 - s_1}{d_1} - \theta_1\right)\right]$$

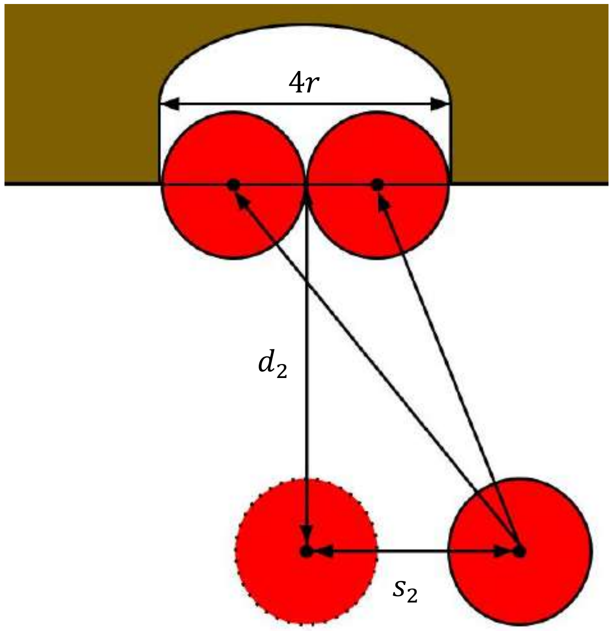

Figure 11: Two different outcomes where the billiard ball barely enters the pocket

The next step is to find the range of angle (θ_{2s}) at which the billiard ball is scored into the pocket. Since the assumption that the billiard ball is scored if half of the billiard ball enters the pocket is made, the range of angle (θ_{2s}) can be defined by finding the maximum and minimum possible angle at which the billiard ball enters the pocket (see Figure 11). Let's assign the maximum and minimum as angle ($\theta_{2s_{max}}$) and angle ($\theta_{2s_{min}}$). Using trigonometric ratios, the appropriate angles can be calculated (see Figure 12):

$$\tan\theta = \frac{opp}{adj}$$

$$\tan\theta_{2s_{max}} = \frac{s_2 + r}{d_2}$$

$$\theta_{2s_{max}} = \arctan\frac{s_2 + r}{d_2}$$

Similarly, angle ($\theta_{2s_{min}}$) is calculated:

$$\tan\theta_{2s_{min}} = \frac{s_2 - r}{d_2}$$

$$\theta_{2s_{min}} = \arctan\frac{s_2 - r}{d_2}$$

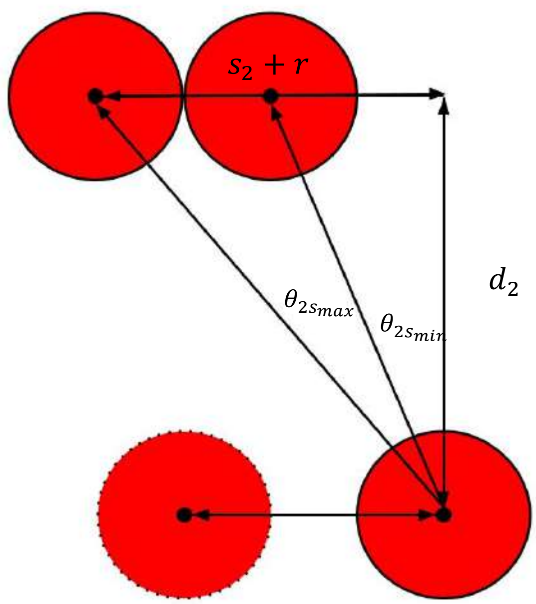

Figure 12: Reconstructed diagram for calculating the range of angle (θ_{2s}) that allow the billiard ball to enter the pocket

Therefore, since:

$$\theta_{2s_{min}} \leq \theta_{2s} \leq \theta_{2s_{max}}$$

Thus:

$$\arctan\frac{s_2 - r}{d_2} \leq \theta_{2s} \leq \arctan\frac{s_2 + r}{d_2}$$

Finding the Formula for the Probability of Scoring

Using the range of angles that allow the billiard ball to be scored, the probability for satisfying each condition that leads to scoring the billiard ball can be formulated using probability density functions:

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$where\ \sigma = standard\ deviation$$

$$where\ \mu = mean$$

(Weisstein)

This will help in understanding how the mathematics in physics can help in calculating the probability of scoring in different situations. Since the area under a probability density graph is one unit squared, calculating the area between a certain range under the graph will give the probability (Harcet et al. 522). To start, the probability of the first condition will be formulated:

1. The probability of the cue ball colliding with the billiard ball.

Apart from the position of the balls, another factor that affects this probability is the player's skill level. This can be determined by determining how much their shot deviates from a selected target. Let's assume that the player is proficient enough to hit the cue ball with a deviation of:

$$\theta_p \; radians$$

Since the player's hit can also deviate in the same angle towards the opposite side of the target, the angle at which the player can shoot will stay with this range:

$$-\theta_p \leq p \leq \theta_p$$

Using this, the standard deviation can be found. Standard deviation is a number that defines the average spread of numbers "from the mean" within a collection (Harcet et al. 296). In this case, we can consider that angle (θ_p) is the maximum and average spread of when the player hits the cue ball. Thus:

$$\sigma = \theta_p \; radians$$

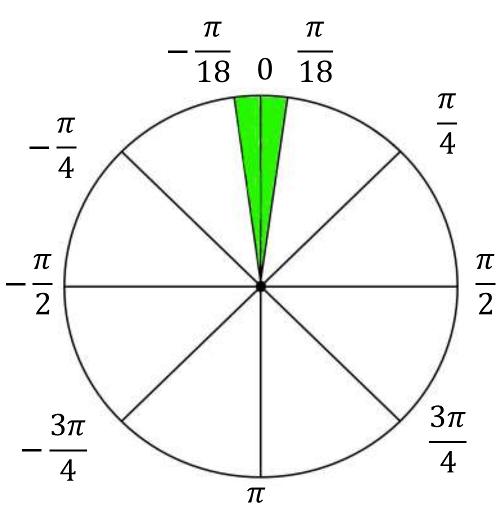

Figure 13: Visual representation of the range of angle (θ_p)

The next step is to find the mean of all the angles between the range of angle (θ_p). Normally, a formula would be used to calculate the mean, but in this case, it can be found easily (see Figure 13). Since the end ranges of angle (θ_p) are equal and opposite to each other, finding the sum of all the angles in the range would simply cancel each other out. Therefore:

$$\mu = 0$$

The standard deviation and mean are then substituted into the probability density function formula:

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$P(x) = \frac{1}{(\theta_p)\sqrt{2\pi}} e^{-\frac{(x-0)^2}{2(\theta_p)^2}}$$

$$P(x) = \frac{1}{(\theta_p)\sqrt{2\pi}} e^{-\frac{x^2}{2(\theta_p)^2}}$$

Since the first probability is to find whether the player is able to allow the cue ball to collide with the billiard ball, the angle (θ_p) must fall into a range of angles that allow the collision to occur or rather the range of angle (θ_1). Using this range, the upper and lower bound of this certain area under this graph can be set:

$$\arctan\frac{s_2 - s_1}{d_1} - \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}} < \theta_1$$
$$< \arctan\frac{s_2 - s_1}{d_1} + \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

The area under the function can be evaluated through integration:

$Probability\ of\ Cue\ Ball\ Colliding$

$$= \int_{\arctan\frac{s_2-s_1}{d_1}-\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}}^{\arctan\frac{s_2-s_1}{d_1}+\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}} \frac{1}{(\theta_p)\sqrt{2\pi}} e^{-\frac{x^2}{2(\theta_p)^2}}\, dx$$

$$= \frac{1}{(\theta_p)\sqrt{2\pi}} \int_{\arctan\frac{s_2-s_1}{d_1}-\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}}^{\arctan\frac{s_2-s_1}{d_1}+\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}} e^{-\frac{x^2}{2(\theta_p)^2}}\, dx$$

Use u-substitution to derive the equation:

$$u = \frac{x}{2\theta_p}$$

$$du = \frac{1}{2\theta_p} dx$$

$$= \left[\frac{1}{(\theta_p)\sqrt{2\pi}}\right](2\theta_p)\int_{\arctan\frac{s_2-s_1}{d_1}-\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}}^{\arctan\frac{s_2-s_1}{d_1}+\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}} e^{-u^2}\,du$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{\pi}}\int_{\arctan\frac{s_2-s_1}{d_1}-\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}}^{\arctan\frac{s_2-s_1}{d_1}+\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}} e^{-u^2}\,du\right)$$

When integrating a probability density function, an unconventional method has to be used to integrate the function due to the degree of x. This method is through the use of error functions:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

(Andrews 110)

Substitute the variables from the integrand appropriately into the formula:

$$= \left[\frac{1}{\sqrt{2}}\operatorname{erf}(u)\right]_{x=\arctan\frac{s_2-s_1}{d_1}-\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}}^{x=\arctan\frac{s_2-s_1}{d_1}+\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}}$$

Substitute u back into the equation and evaluate the definite integral:

$$= \left[\frac{1}{\sqrt{2}}\operatorname{erf}\left(\frac{x}{2\theta_p}\right)\right]_{x=\arctan\frac{s_2-s_1}{d_1}-\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}}^{x=\arctan\frac{s_2-s_1}{d_1}+\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}}$$

$$= \frac{1}{\sqrt{2}}\operatorname{erf}\left[\frac{\arctan\frac{s_2-s_1}{d_1}+\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}}{2\theta_p}\right]$$

$$-\frac{1}{\sqrt{2}}\operatorname{erf}\left[\frac{\arctan\frac{s_2-s_1}{d_1}-\arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}}{2\theta_p}\right]$$

To keep the formula from becoming too lengthy, angle ($\theta_{1_{max}}$) and angle ($\theta_{1_{min}}$) will be substituted into the formula. Since:

$$\theta_{1_{max}} = \arctan\frac{s_2 - s_1}{d_1} + \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

$$\theta_{1_{min}} = \arctan\frac{s_2 - s_1}{d_1} - \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

Therefore:

$$= \frac{1}{\sqrt{2}}\operatorname{erf}\left(\frac{\theta_{1_{max}}}{2\theta_p}\right) - \frac{1}{\sqrt{2}}\operatorname{erf}\left(\frac{\theta_{1_{min}}}{2\theta_p}\right)$$

The next step is to find the second probability necessary in order to calculate the probability of the billiard ball scoring:

2. The probability that the billiard ball, knowing that the collision occurs.

The method is similar to when finding the previous probability. The range of angle (θ_2)—the angles at which the cue ball collides with the billiard ball—are considered to help find the probability:

$$\theta_1 - \pi + \arcsin\left[\frac{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}{2r}\sin\left(\theta_1 - \arctan\frac{s_2 - s_1}{d_1}\right)\right] \le \theta_2$$

$$\le \theta_1 - \pi + \arcsin\left[\frac{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}{2r}\sin\left(\arctan\frac{s_2 - s_1}{d_1} - \theta_1\right)\right]$$

The maximum of angle (θ_1) will be substituted into the upper range of angle (θ_2) and the miniumum of angle (θ_1) will be substituted into the lower range of angle (θ_2). This will result in a range of angle (θ_2) after the cue ball collides with the billiard ball:

$$\mathit{Upper\ Range\ of}\ \theta_2 = \arctan\frac{s_2 - s_1}{d_1} + \arcsin\frac{2r}{\sqrt{(s_1 + s_2 + 4r)^2 + (d_1 + 2r)^2}} - \pi$$

$$Lower\ Range\ of\ \theta_2 = \arctan\frac{s_2 - s_1}{d_1} - \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}} - \pi$$

Using the upper and lower range, the range of angle (θ_2) is reevaluated:

$$\arctan\frac{s_2 - s_1}{d_1} - \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}} - \pi \le \theta_2$$

$$\le \arctan\frac{s_2 - s_1}{d_1} + \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}} - \pi$$

To avoid the range from becoming to complicated, the maximum and minimum of angle (θ_1) is substituted back into the upper and lower ranges appropriately, allowing the range of angle (θ_2) to be written as:

$$\theta_{1_{min}} - \pi \le \theta_2 \le \theta_{1_{max}} - \pi$$

Similar to the previous method, the upper or lower range of angle (θ_2) is considered as the maximum and average spread of when the billiard ball travels towards the pocket. Therefore:

$$\sigma = \frac{\theta_{1_{max}} - \theta_{1_{min}}}{2}$$

Due to the angles being equal and opposite to each other, the mean will also be equal to zero:

$$\mu = 0$$

By substituting all the necessary values into the probability density function formula, we get:

$$P(x) = \frac{1}{\left(\frac{\theta_{1_{max}} - \theta_{1_{min}}}{2}\right)\sqrt{2\pi}} e^{-\frac{x^2}{2\left(\frac{\theta_{1_{max}} - \theta_{1_{min}}}{2}\right)^2}}$$

Since this time, the probability of whether the billiard ball enters the pocket is being evaluated, the range of the angle at which the billiard ball enters the pocket will be used to set the upper and lower bounds:

$$\arctan\frac{s_2 - r}{d_2} \leq \theta_{2s} \leq \arctan\frac{s_2 + r}{d_2}$$

Therefore, the following is integrated using the provided upper and lower bounds with u-substitution:

$$= \int_{\arctan\frac{s_2-r}{d_2}}^{\arctan\frac{s_2+r}{d_2}} \frac{1}{\left(\frac{\theta_{1_{max}} - \theta_{1_{min}}}{2}\right)\sqrt{2\pi}} e^{-\frac{x^2}{2\left(\frac{\theta_{1_{max}}-\theta_{1_{min}}}{2}\right)^2}} dx$$

$$u = \frac{x}{\sqrt{2}\left(\frac{\theta_{1_{max}} - \theta_{1_{min}}}{2}\right)}$$

$$du = \frac{1}{\sqrt{2}\left(\frac{\theta_{1_{max}} - \theta_{1_{min}}}{2}\right)} dx$$

$$= \left(\frac{1}{\sqrt{\pi}}\right) \int_{\arctan\frac{s_2-r}{d_2}}^{\arctan\frac{s_2+r}{d_2}} e^{-u^2}\, du$$

Substitute the error function:

$$= \frac{1}{2}\left[\operatorname{erf}(u)\right]_{x=\arctan\frac{s_2-r}{d_2}}^{x=\arctan\frac{s_2+r}{d_2}}$$

This will lead to the error function:

$$= \left(\frac{1}{2}\right)\left\{\left[\operatorname{erf}\left(\frac{\sqrt{2}\arctan\frac{s_2 + r}{d_2}}{\theta_{1_{max}} - \theta_{1_{min}}}\right)\right] - \left[\operatorname{erf}\left(\frac{\sqrt{2}\arctan\frac{s_2 - r}{d_2}}{\theta_{1_{max}} - \theta_{1_{min}}}\right)\right]\right\}$$

Since the formulas for the two probabilities:

1. The cue ball hits the billiard ball, relative to the player's skill level.

2. The billiard ball is scored, with a condition that the collision occurs.

The probability of scoring the billiard ball can be calculated by finding the product of the two probabilities:

$$Probability\ of\ Scoring$$

$$= \left(\frac{1}{4}\right)\left[\operatorname{erf}\left(\frac{\theta_{1_{max}}}{2\theta_p}\right) - \operatorname{erf}\left(\frac{\theta_{1_{min}}}{2\theta_p}\right)\right]\left\{\left[\operatorname{erf}\left(\frac{\sqrt{2}\arctan\frac{s_2+r}{d_2}}{\theta_{1_{max}}-\theta_{1_{min}}}\right)\right] - \left[\operatorname{erf}\left(\frac{\sqrt{2}\arctan\frac{s_2-r}{d_2}}{\theta_{1_{max}}-\theta_{1_{min}}}\right)\right]\right\}$$

$$where\ \theta_p = skill\ level\ of\ the\ player\ measured\ in\ radians$$

$$where\ d_1 = vertical\ distance\ between\ the\ center\ of\ the\ balls$$

$$where\ d_2 = vertical\ distance\ between\ billiard\ ball\ and\ the\ pocket$$

$$where\ r = radius\ of\ the\ balls$$

$$where\ s_1 = horizontal\ displacement\ from\ the\ center\ of\ the\ cue\ ball$$

$$where\ s_2 = horizontal\ displacement\ from\ the\ center\ of\ the\ billiard\ ball$$

$$where\ \theta_{1_{max}} = \arctan\frac{s_2-s_1}{d_1} + \arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}$$

$$where\ \theta_{1_{min}} = \arctan\frac{s_2-s_1}{d_1} - \arcsin\frac{2r}{\sqrt{(s_2-s_1)^2+(d_1)^2}}$$

Usage of the Formula

Using the formula, the relationship between each variable of the formula—d_1, d_2, s_1, s_2—and the probability of scoring the billiard ball can be shown. By doing so, we can understand how physical mathematics can help in understanding how the probability is affected in different ways. All values in the following comparisons will be set to make the calculations simpler. To make a fair comparison, the default scenario will be set (see Figure 14).

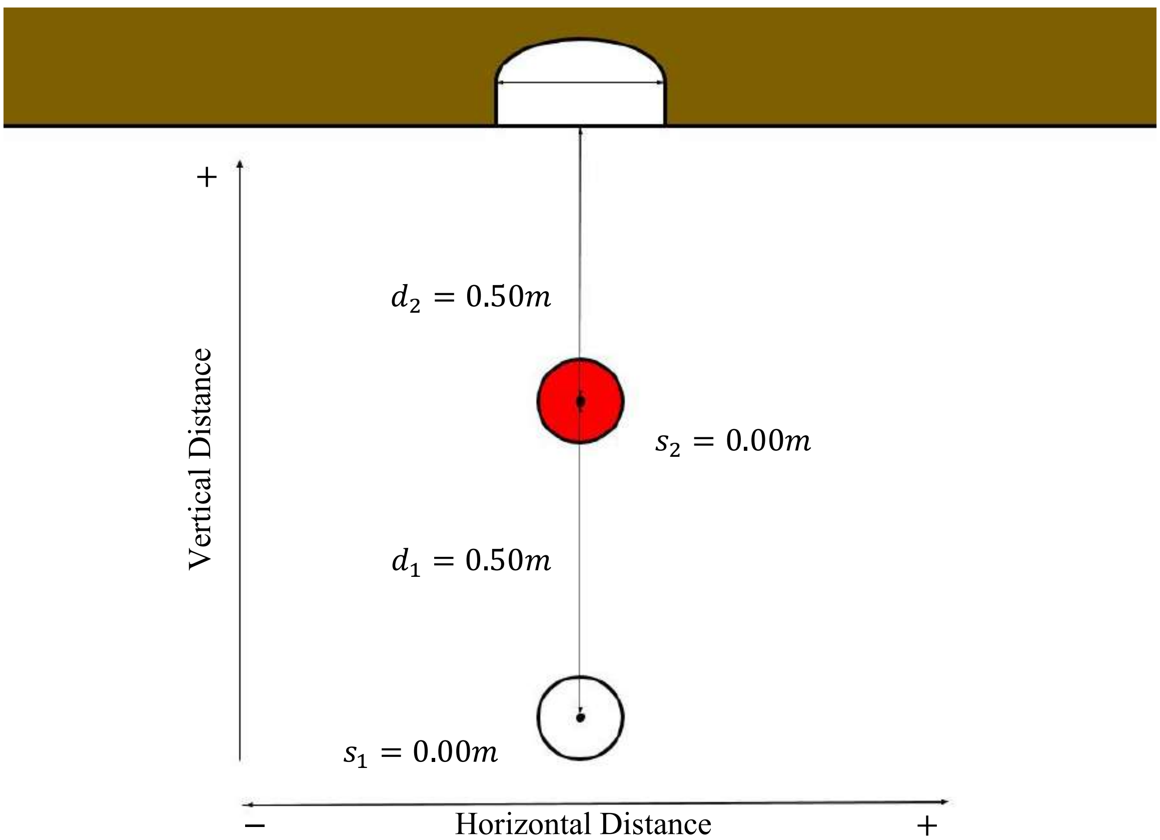

Figure 14: Default scenario without change in the value of variables

When finding the relationship between each variable, only one variable will be changed at a time, while other variables would be inputted as a the values that are shown in the figure above. For example: when the relationship between d_1 and the probability of scoring is to be considered, the values of each variable will be substituted into the formula as such (see Table 1):

d_1 (m)	d_2 (m)	s_1 (m)	s_2 (m)
0.10	0.50	0.00	0.00
0.20	0.50	0.00	0.00
0.30	0.50	0.00	0.00
0.40	0.50	0.00	0.00
0.50	0.50	0.00	0.00
0.60	0.50	0.00	0.00
0.70	0.50	0.00	0.00
0.80	0.5	0.0	0.0
0.90	0.5	0.0	0.0
1.00	0.5	0.0	0.0

Table 1: The values of each variable for when finding the relationship between d_1 and the probability of scoring

To maintain the fairness of each relationship, the skill level of the player and the type of billiard ball used will be constant. Therefore, we can first set the player's skill level or angle (θ_p) as:

$$\theta_p = 0.1$$

Even though different cue sports have billiard balls ranging in different sizes, we will be using the size of competitive pocket billiard balls; with the diameter of $52.5mm$ or $0.0525m$ (Government). Therefore the radius of the balls can be written as:

$$r = 0.02625m$$

Again, most values of each variable were set this way to make both the calculation and the comparisons convenient. By substituting the appropriate values for each relationship, the probability of scoring for each change can be calculated using the formula:

$Probability\ of\ Scoring$

$$= \left(\frac{1}{4}\right)\left[\operatorname{erf}\left(\frac{\theta_{1_{max}}}{2\theta_p}\right) - \operatorname{erf}\left(\frac{\theta_{1_{min}}}{2\theta_p}\right)\right]\left\{\left[\operatorname{erf}\left(\frac{\sqrt{2}\arctan\frac{s_2 + r}{d_2}}{\theta_{1_{max}} - \theta_{1_{min}}}\right)\right]\right.$$

$$\left. - \left[\operatorname{erf}\left(\frac{\sqrt{2}\arctan\frac{s_2 - r}{d_2}}{\theta_{1_{max}} - \theta_{1_{min}}}\right)\right]\right\}$$

As an example, the probability for when $d_1 = 0.10m$ will be calculated. For this reason, the other variables is known as constants (see Table 1):

$$d_2 = 0.50m$$

$$s_1 = 0.00m$$

$$s_2 = 0.00m$$

$$\theta_p = 0.1$$

Since angle ($\theta_{1_{max}}$) and angle ($\theta_{1_{min}}$) is defined as:

$$\theta_{1_{max}} = \arctan\frac{s_2 - s_1}{d_1} + \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

$$\theta_{1_{min}} = \arctan\frac{s_2 - s_1}{d_1} - \arcsin\frac{2r}{\sqrt{(s_2 - s_1)^2 + (d_1)^2}}$$

All the variables can be substituted appropriately into the formula to lead to the expression:

$Probability\ of\ Scoring$

$$= \left(\frac{1}{4}\right)[\operatorname{erf}(2.76) - \operatorname{erf}(-2.76)][\operatorname{erf}(0.0671) - \operatorname{erf}(-0.0671)]$$

Since error functions are odd functions:

$$\mathrm{erf}(-x) = -\mathrm{erf}\,(x)$$

(Andrews 110)

The expression can be simplified as:

$$Probability\ of\ Scoring \;\; = [\mathrm{erf}(2.76)][\mathrm{erf}(0.0671)]$$

Next, the error functions can be solved through the sigma notation below:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!\,(2n+1)}$$

$$where\ |x| < \infty$$

(Andrews 110)

The calculation of only one of the error functions will be shown due to the redundancies of showing both:

$$\mathrm{erf}(2.76) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n (2.76)^{2n+1}}{n!\,(2n+1)}$$

$$= \frac{2}{\sqrt{\pi}}\left(-\frac{(2.76)^3}{1!\,(3)} + \frac{(2.76)^5}{2!\,(5)} - \frac{(2.76)^7}{3!\,(7)} + \frac{(2.76)^9}{4!\,(9)} \cdots\right)$$

$$= 1.00$$

By calculating the remaining error function, the probability of scoring when $d_1 = 0.10m$ is found:

$$Probability\ of\ Scoring \;\; = (1.00)(0.0756)$$

$$= 0.0756$$

The probability of scoring is calculated for each value of each variable and displayed in a table (see Table 2):

Change in d_1		Change in d_2		Change in s_1		Change in s_2	
$d_1\ (m)$	Probability	$d_2\ (m)$	Probability	$s_1\ (m)$	Probability	$s_2\ (m)$	Probability
0.10	0.0756	0.10	0.535	0.10	0.0917	0.10	0.0379
0.20	0.147	0.20	0.426	0.20	9.15×10^{-3}	0.20	1.63×10^{-4}
0.30	0.184	0.30	0.322	0.30	3.17×10^{-4}	0.30	7.46×10^{-9}
0.40	0.201	0.40	0.253	0.40	6.61×10^{-6}	0.40	1.45×10^{-15}
0.50	0.207	0.50	0.205	0.50	1.28×10^{-7}	0.50	0
0.60	0.209	0.60	0.175	0.60	3.02×10^{-9}	0.60	0
0.70	0.208	0.70	0.151	0.70	9.68×10^{-11}	0.70	0
0.80	0.206	0.80	0.133	0.80	4.39×10^{-12}	0.80	0
0.90	0.202	0.90	0.119	0.90	2.78×10^{-13}	0.90	0
1.00	0.198	1.00	0.109	1.00	2.40×10^{-14}	1.00	0

Table 2: The relationship between each variable and the probability of scoring

These values can then be graphed to show a visual representation of how probability changes for each variable:

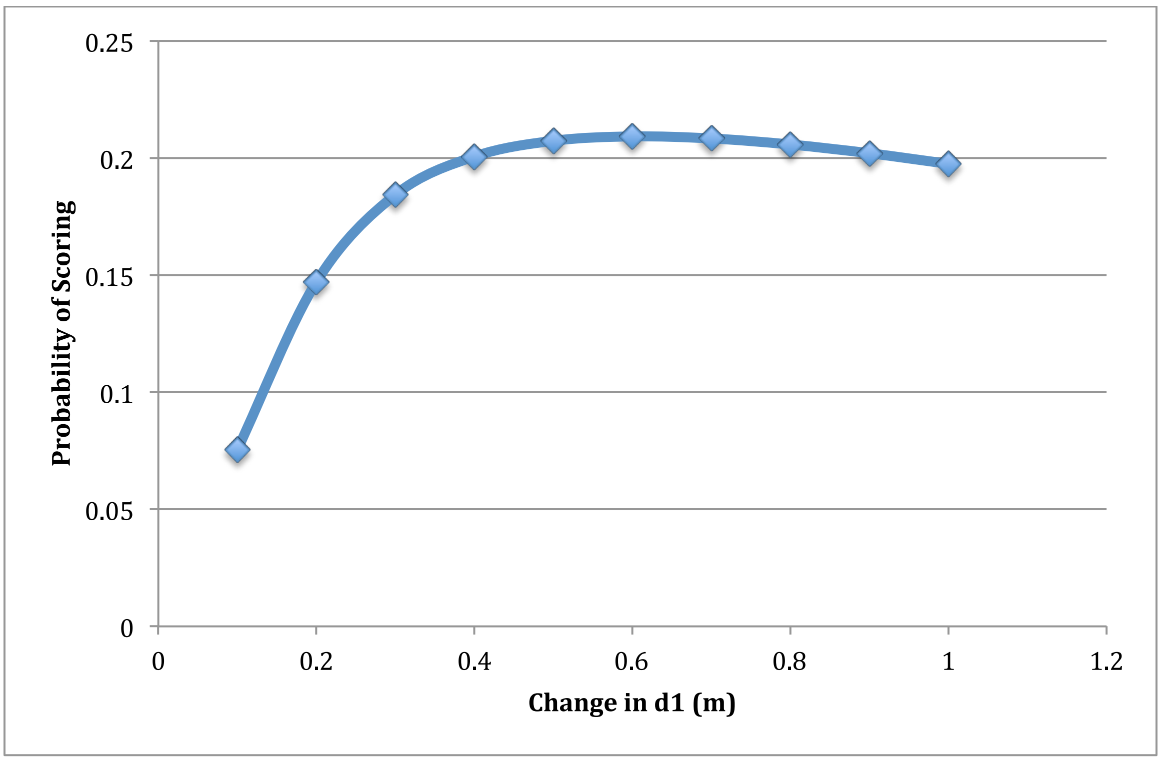

Figure 15: The relationship between d_1 and the probability of scoring

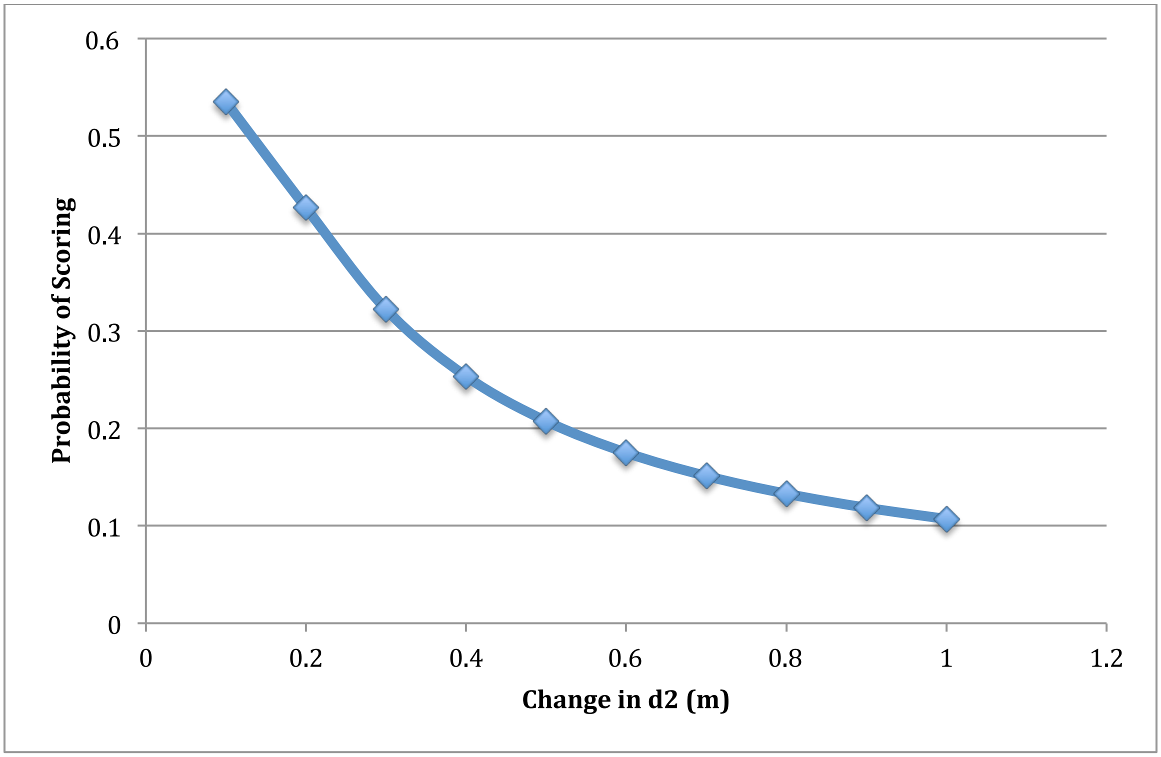

Figure 16: The relationship between d_2 and the probability of scoring

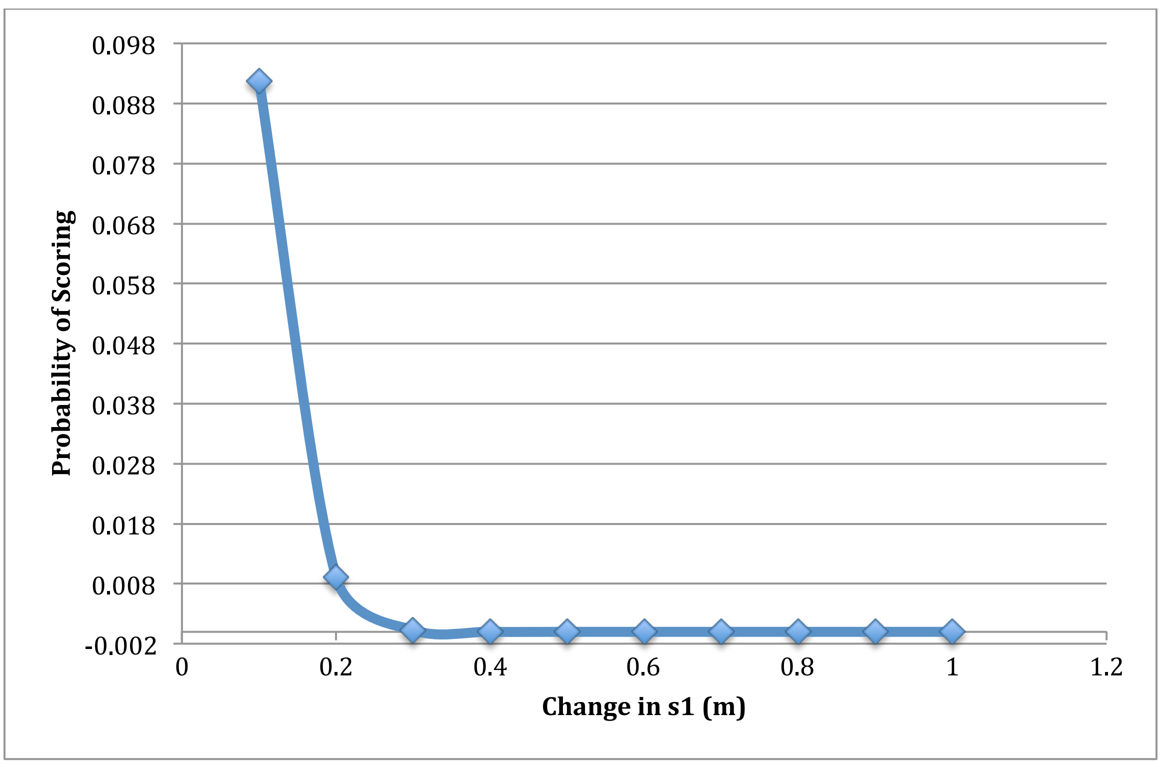

Figure 17: The relationship between s_1 and the probability of scoring

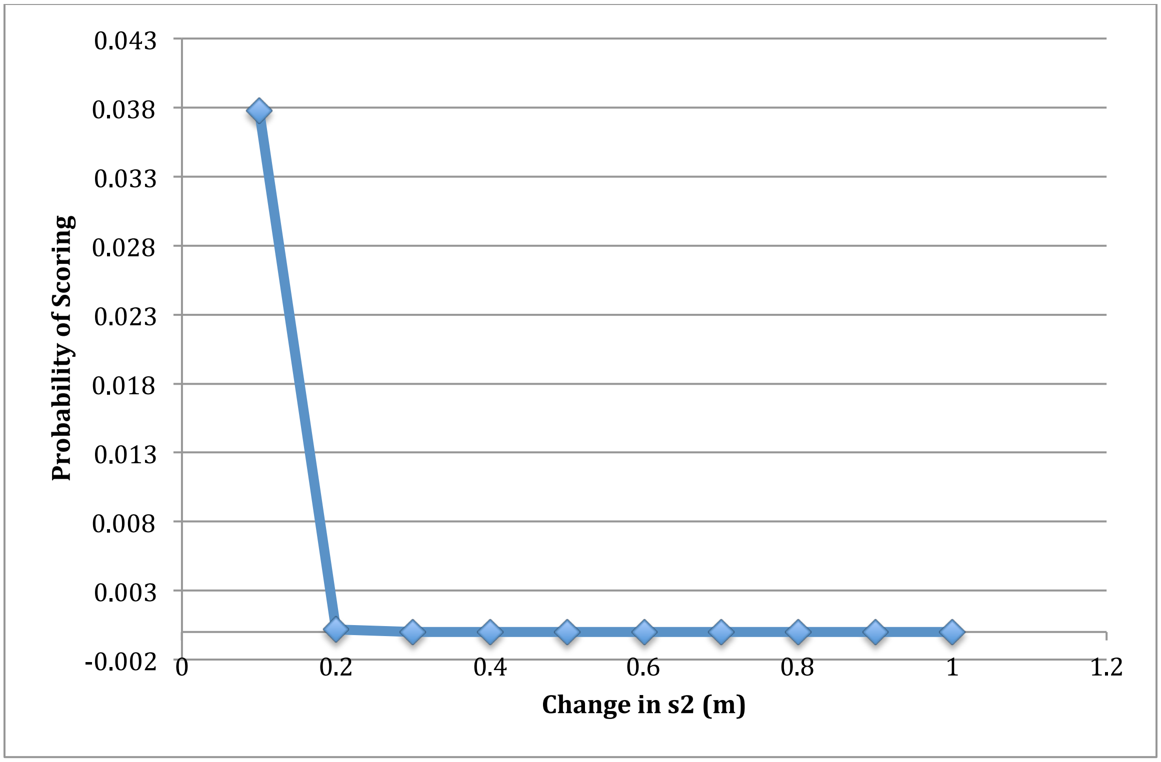

Figure 18: The relationship between s_2 and the probability of scoring

From the following graphs, a few things can be deduced:

1. All changes in distances except for d_1—the vertical distance from the centers of the cue ball to the billiard ball—have an exponential decrease in probability.
2. The increase in s_1 and s_2—the horizontal displacement between the centers of the cue ball and the billiard ball—seem to have the most significant drop in probability that approaches or reaches zero.

From this, we can now not only use the formula to understand the probability of scoring a billiard ball using a cue ball, but we can also understand the trend in how the change in vertical or horizontal position of the billiard ball or the cue ball affects the probability of scoring. However, whether these pieces of information can truly be used for actual billiard games will be discussed in the limitations.

Conclusion

After the calculations, the research question—How does the mathematics in physics help in understanding the probability of scoring in pocket billiards—can be answered. Using probability density functions and the understanding of two-body collisions, the probability of scoring a billiard ball can be expressed as the following formula:

$$Probability\ of\ Scoring$$

$$= \left(\frac{1}{4}\right)\left[\operatorname{erf}\left(\frac{\theta_{1_{max}}}{2\theta_p}\right) - \operatorname{erf}\left(\frac{\theta_{1_{min}}}{2\theta_p}\right)\right]\left\{\left[\operatorname{erf}\left(\frac{\sqrt{2}\arctan\frac{s_2 + r}{d_2}}{\theta_{1_{max}} - \theta_{1_{min}}}\right)\right]\right.$$

$$\left. - \left[\operatorname{erf}\left(\frac{\sqrt{2}\arctan\frac{s_2 - r}{d_2}}{\theta_{1_{max}} - \theta_{1_{min}}}\right)\right]\right\}$$

Using the formula, anyone can understand the difficulty of a scoring in pocket billiards, while also understand the change in probability when the position of the billiard ball or the cue ball is changed. These can certainly not only allow the research question to be answered, but it can also possibly have its application in actual billiard games by quantifying and visualising the probabilities of each player scoring different billiard balls based on their skill level. However, there are certain limitations that may reduce its practicality in a real life situation.

Limitations

Of course, there is already a limitation in this method due to the amount of assumptions made when calculating. This itself causes the formula to lose practicality since billiards are largely dependent on factors such as friction, which are utilized frequently when scoring billiard balls and positioning oneself closer to the winning condition. This means that in order to have a perfect formula for practical use, one must consider a whole group of conditions such as the angle at which the cue ball is hit by the cue. In addition, some players even chip the cue ball over when scoring the billiard ball, showing amount of additional math required for a complete formula ("Guinness" 00:01:46-00:02:46).

Secondly, the comparisons between the relationship between the position of the balls and the probability of scoring only shows the change in probability when only one variable among the four—d_1, d_2, s_1, s_2—is varied. This means that there is an uncertainty in the trend when multiple variables are changed at once.

However, the largest flaw in practicality of the formula is the amount of variables required for the calculation. In the formula, six different variables are present, at which four of them—the horizontal and vertical translations—need to be measured accurately during the game. Though billiards is not a very fast paced game, it is no doubt a challenge when it comes to measuring each and every translation of the billiard balls for calculating the probability of scoring a single billiard ball out of several different billiard balls.

Finally, the formula only finds the probability of scoring a single billiard ball directly through a single collision by the cue ball. In the actual game, there are times when players score by using the cue ball to score the billiard ball indirectly. This consists of either using the collision between multiple billiard balls or using a rebound from the sides of the table to score the biliard ball.

Real World Application

Pocket billiards is an example of a two-body collision. For that reason, it can be used as a method to visualize and hypothesize the outcome other two-body collisions such as the activities of atomic elements and particles. Unlike pocket billiards where there is small and coherent number of billiard balls, the number of sphere-shaped particles that scientists such as physicists and chemists try to research reach numbers over several ten-folds of what is shown on a pocket billiard table. Therefore, it can become problematic when it comes to calculating each individual particle to understand their activities in a certain environment. This is when scientists can utilize probability to convert a large amount of data all into a single number to evaluate their hypotheses similarly to “Schrödinger’s wave function”, which can provide the “probability of finding a particle” in a given region (Homer and Bowen-Jones 485-486). By extending the formulae made to calculate the probability of scoring billiard balls, other potential formulae or at least algorithms can be programmed to understand the outome of collisions on a higher level of difficulty.

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