CENG 424
Fall 2024
Homework 3

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December 9, 2023

Q1)

a)

For all cats, there exists a dog that is friends with that cat.

b)

There exists a cat that is friends with all dogs.

Q2)

a)

$$\exists x.(\forall y.p(x,y) \implies p(z,z)) \iff (\exists x.p(x,x) \implies \exists y.p(y,y))$$

The right hand side of the equation, $\exists x.p(x,x) \implies \exists y.p(y,y)$ is always true, i.e. valid. So, what determines the validity of the whole equation is the left hand side of the equation.

Now, consider a Herbrand universe containing two constants, a and b, and the following truth assignments:

$$p(a,a) = T$$
$$p(a,b) = T$$
$$p(b,a) = T$$
$$p(b,b) = F$$

Note that since z is not quantified, it is a free variable. So, we can assign z to a or b.

In this case, the left hand side of the equation becomes false. However, if we consider the following truth assignments:

$$p(a, a) = F$$
$$p(a, b) = F$$
$$p(b, a) = F$$
$$p(b, b) = F$$

The left hand side of the equation becomes true. So, the whole equation is contingent.

b)

$$(\forall x.(p(x) \vee q(x)) \implies (\exists y.p(y) \implies (p(x) \implies \forall y.p(y)))$$

In order to prove that this sentence is not unsatisfiable, it is enough to show that the left hand side of the equation, $\forall x.(p(x) \vee q(x))$, can be false (because $F \implies \phi$ is true no matter what ϕ is).

It would be possible in a Herbrand universe containing two constants, a and b, and the following truth assignments:

$$p(a) = F$$
$$p(b) = F$$
$$q(a) = F$$
$$q(b) = F$$

With these truth assignments, the left hand side of the equation becomes false. So, the equation is not unsatisfiable. To show that the equation is not valid, we can show that the equation is false for some truth assignments. In order for the equation to be false, the left hand side of the equation should be true and the right hand side of the equation $\exists y.p(y) \implies (p(x) \implies \forall y.p(y))$ should be false.

In order for the left hand side of the equation to be true, $\exists y.p(y)$ should be true and $p(x) \implies \forall y.p(y)$ should be false. It is achievable with the following truth assignments:

$$p(a) = T$$
$$p(b) = T$$
$$q(a) = F$$
$$q(b) = F$$

With these assignments, $\forall x.(p(x) \vee q(x) \equiv T$, $\exists y.p(y) \equiv T$ and $p(x) \implies \forall y.p(y) \equiv F$ since x in $p(x)$ is a free variable. So, the left hand side of the equation is true and the right hand side of the equation is false. So, the equation is not valid. Since the equation is not unsatisfiable and not valid, it is contingent.

c)

$$\exists y.(p(y) \implies \exists x.q(x,y)) \implies \neg\exists x.q(y,x)$$

In order to prove that this sentence is not unsatisfiable, it is enough to show that the left hand side of the equation, $\exists y.(p(y) \implies \exists x.q(x,y))$, can be false (because $F \implies \phi$ is true no matter what ϕ is).

It would be possible in a Herbrand universe containing two constants, a and b, and the following truth assignments:

$$\begin{aligned} p(a) &= T \\ p(b) &= T \\ q(a,a) &= F \\ q(a,b) &= F \\ q(b,a) &= F \\ q(b,b) &= F \end{aligned}$$

The assignments above satisfies that there is no y that makes $p(y) \implies \exists x.q(x,y)$ is true, because $p(y)$ is always true and $\exists x.q(x,y)$ is always false for all y. So, the left hand side of the equation can be false. The overall sentence, therefore, is not unsatisfiable.

In order to prove that the sentence is not valid, it is enough to show that the sentence is false for some truth assignments. In order for the sentence to be false, the left hand side of the equation should be true and the right hand side of the equation, $\neg\exists x.q(y,x)$, should be false.

For the right hand side to be false, no combination of x and y should make $q(y,x)$ true, in other words, there should be no x that makes $q(y,x)$ true for all y, because note that y is a free variable. For the left hand side to be true, there should be a y that makes $p(y) \implies \exists x.q(x,y)$ true. It is achievable by simply assigning F to $p(y)$ for all y. truth assignments:

$$\begin{aligned} p(a) &= F \\ p(b) &= F \\ q(a,a) &= F \\ q(a,b) &= F \\ q(b,a) &= F \\ q(b,b) &= F \end{aligned}$$

Q3)

$$
\begin{aligned}
&\forall x.(p(x) \implies q(x)) && \textit{premise} && (1)\\
&\neg\exists z.r(z) && \textit{premise} && (2)\\
&\exists y.p(y) \vee r(a) && \textit{premise} && (3)\\
&\neg\exists z.r(z) \implies \forall z.\neg r(z) && \textit{premise} && (4)\\
&p(d) \vee r(a) && EI, 3 && (5)\\
&\forall z.\neg r(z) && MP, 2, 4 && (6)\\
&\neg r(a) && UI, 6 && (7)\\
&(r(a) \vee p(d)) \iff (\neg r(a) \implies p(d)) && OQ && (8)\\
&((r(a) \vee p(d)) \iff (\neg r(a) \implies p(d))) \implies && && \\
&((r(a) \vee p(d)) \implies (\neg r(a) \implies p(d))) && EQ && (9)\\
&(r(a) \vee p(d)) \implies (\neg r(a) \implies p(d)) && MP, 8, 9 && (10)\\
&\neg r(a) \implies p(d) && MP, 5, 10 && (11)\\
&p(d) && MP, 7, 11 && (12)\\
&p(d) \implies q(d) && UI, 1 && (13)\\
&q(d) && MP, 12, 13 && (14)\\
&\exists z.q(z) && EI, 14 && (15)
\end{aligned}
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Q4)

$$
\begin{aligned}
&\forall y.A(a,y) \wedge \forall x.\forall y.((A(x,y)) \implies A(B(x),B(y))) && \textit{premises}\\
&\forall y.A(a,y) \wedge \forall x.\forall y.(\neg(A(x,y)) \vee A(B(x),B(y))) && I\\
&\forall y.A(a,y) \wedge \forall x.\forall y.(\neg A(x,y) \vee A(B(x),B(y))) && N\\
&\forall y.A(a,y) \wedge \forall x.\forall z.(\neg A(x,z) \vee A(B(x),B(z))) && S\\
&\forall y.A(a,y) \wedge \forall x.\forall z.(\neg A(x,z) \vee A(B(x),B(z))) && E\\
&A(a,y) \wedge \forall x.\forall z.(\neg A(x,z) \vee A(B(x),B(z))) && A\\
&A(a,y) \wedge \forall z.(\neg A(x,z) \vee A(B(x),B(z))) && \\
&A(a,y) \wedge (\neg A(x,z) \vee A(B(x),B(z))) && \\
&A(a,y) \wedge (\neg A(x,z) \vee A(B(x),B(z))) && D\\
&\{A(a,y)\} && O\\
&\{\neg A(x,z), A(B(x),B(z))\} &&
\end{aligned}
$$

$$
\begin{array}{rr}
\neg\exists z.(A(a,z) \wedge A(z,B(B(a)))) & \textit{negated goal} \\
\neg\exists z.(A(a,z) \wedge A(z,B(B(a)))) & I \\
\forall z.\neg(A(a,z) \wedge A(z,B(B(a)))) & N \\
\forall z.(\neg A(a,z) \vee \neg A(z,B(B(a)))) & \\
\forall z.(\neg A(a,z) \vee \neg A(z,B(B(a)))) & S \\
\forall z.(\neg A(a,z) \vee \neg A(z,B(B(a)))) & E \\
\neg A(a,z) \vee \neg A(z,B(B(a))) & A \\
\neg A(a,z) \vee \neg A(z,B(B(a))) & D \\
\{\neg A(a,z), \neg A(z,B(B(a)))\} & O
\end{array}
$$

$$
\begin{array}{rrr}
\{A(a,y)\} & \textit{premise} & (1) \\
\{\neg A(x,z_1), A(B(x),B(z_1))\} & \textit{premise} & (2) \\
\{\neg A(a,z_2), \neg A(z_2,B(B(a)))\} & \textit{negated goal} & (3) \\
\{A(B(a),B(y))\} & 1,2\ \{z_1 \leftarrow y, x \leftarrow a\} & (4) \\
\{\neg A(x,B(a)), \neg A(a,B(x))\} & 3,4\ \{z_1 \leftarrow B(a), z_2 \leftarrow B(x)\} & (5) \\
\{\neg A(a,B(B(a)))\} & 4,5\ \{x \leftarrow B(a), y \leftarrow a\} & (6) \\
\{\} & 1,6\ \{y \leftarrow B(B(a))\} & (7)
\end{array}
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